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Symmetry problems 2

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Abstract. Some symmetry problems are formulated and solved. New simple proofs are given for some symmetry problems studied earlier. One of the results is as follows: if a single-layer potential of a surface, homeomorphic to a sphere, with a constant charge density, is equal to $c/|x|$ for all sufficiently large $|x|$, where $c > 0$ is a constant, then the surface is a sphere.

1. Introduction. Symmetry problems are of interest both theoretically and in applications.

A well-known, and still unsolved, symmetry problem is the Pompeiu problem (see [3], [4]). It consists in proving the following:

If $D \subset \mathbb{R}^n$, $n \geq 2$, is homeomorphic to a ball, and the boundary S of D is sufficiently smooth ($S \in C^{1,\lambda}$, $\lambda > 0$, is sufficient), and if the problem

$$(1)\qquad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \qquad u|_S = c, \qquad u_N|_S = 0, \qquad k^2 = \text{const} > 0,$$

has a solution, then S is a sphere.

A similar problem (*Schiffer's conjecture*) is also unsolved:

If the problem

$$(2)\qquad (\nabla^2 + k^2)u = 0 \quad \text{in } D, \qquad u|_S = 0, \qquad u_N|_S = c \neq 0, \qquad k^2 = \text{const} > 0,$$

has a solution, then S is a sphere.

In [5] it is proved that if

$$(3)\qquad \int_D \frac{dy}{4\pi|x-y|} = \frac{c}{|x|}, \qquad \forall x \in B_R' = \{x \in \mathbb{R}^3 : |x| > R\},\ c = \text{const} > 0,$$

then D is a ball.

Here and below we assume that $D \subset \mathbb{R}^3$ is a bounded domain homeomorphic to a ball, with a sufficiently smooth boundary S (S being Lipschitz suffices), $B_R = \{x : |x| \leq R\}$, and $B_R \supset D$. By $\mathcal{H}$ we denote the set of all

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harmonic functions in a domain which contains D. By $|D|$ and $|S|$ we denote the volume of D and the surface area of S, respectively.

Our goal is to give a simple proof of the three symmetry-type results formulated in Theorem 1 in Section 2.

In [7] the following result is obtained:

If

$$\Delta u = 1 \quad \text{in } D, \qquad u|_S = 0, \qquad u_N|_S = \mu = \text{const} > 0, \tag{4}$$

then S is a sphere.

This result is obtained by A. D. Aleksandrov's "moving plane" argument, and is equivalent to the following:

If

$$\frac{1}{|D|}\int_D h(x)\,dx = \frac{1}{|S|}\int_S h(s)\,ds, \qquad \forall h \in \mathcal{H}, \tag{5}$$

then S is a sphere.

The equivalence of (4) and (5) can be proved as follows.

Suppose (4) holds. Multiply (4) by an arbitrary $h \in \mathcal{H}$, integrate by parts and get

$$\int_D h(x)\,dx = \mu \int_S h(s)\,ds. \tag{6}$$

If $h = 1$ in (6), then one gets $\mu = |D|/|S|$, so (6) is identical to (5).

Suppose (5) holds. Then (6) holds. Let v solve the problem $\Delta v = 1$ in D, $v|_S = 0$. This v exists and is unique. Using (6), the equation $\Delta h = 0$ in D, and the Green's formula, one gets

$$\mu \int_S h(s)\,ds = \int_D h(x)\,dx = \int_D h(x)\Delta v\,dx = \int_S h(s) v_N\,ds. \tag{7}$$

Thus,

$$\int_S h(s)[v_N - \mu]\,ds = 0, \qquad \forall h \in \mathcal{H}. \tag{8}$$

The set of restrictions to S of all harmonic functions in D is dense in $L^2(S)$ (see, e.g., [5]). Thus, (8) implies $v_N|_S = \mu$. Therefore, (4) holds.

2. Results and proofs. Our main results are formulated in the following theorem:

THEOREM 1. *Let $D \subset \mathbb{R}^3$ be a bounded domain homeomorphic to a ball, S be its Lipschitz boundary, $D' := \mathbb{R}^3 \setminus D$. If any one of the following assumptions holds, then S is a sphere*:

1. *We have*

$$u(x) := \int_S \frac{ds}{4\pi|x-s|} = \frac{c}{|x|}, \quad \forall x \in B'_R, \quad c = \text{const}, \tag{9}$$

where $B'_R := \{x : |x| > R\}$, $D \subset B_R$, $B_R := \mathbb{R}^3 \setminus B'_R$.

2. *We have*

$$\frac{1}{|S|}\int_S h(s)\,ds = h(0), \quad \forall h \in \mathcal{H}. \tag{10}$$

3. *There exists a solution to the problem*

$$\Delta_y u = \delta(y) \quad \text{in } D, \quad u|_S = 0, \quad u_N|_S = c_1 = \text{const}, \tag{11}$$

where $\delta(y)$ *is the delta-function.*

In (10), 0 *is the origin*, $0 \in D$, $|S|$ *is the surface area of* S, *and* $\mathcal{H}$ *is the set of all harmonic functions in a domain containing* D.

Proof. 1. Assume (9). Then $c = |S|/(4\pi)$ as one can see by letting $|x| \to \infty$. If (9) holds for all $x \in B'_R$ then, by the unique continuation property for harmonic functions, (9) holds for all $x \in D'$. Let N_s be the unit normal to S at the point $s \in S$, pointing into D'. The known jump formula for the normal derivative of a single-layer potential ([2, p. 14]) yields

$$u^+_{N_{s_0}} = u^-_{N_{s_0}} + 1, \quad u^-_{N_{s_0}} = -\frac{|S|}{4\pi}\frac{N_{s_0}\cdot s_0}{|s_0|^3}, \quad s_0 \in S. \tag{12}$$

If S is not a sphere, then there exists an $s_0 \in S$ with $|s_0| \le |s|$ for $s \in S$. The ball $B_{|s_0|}$ of radius $|s_0|$, centered at the origin, is contained in D. At the point s_0 the normal N_{s_0} to S is directed along the vector s_0, so

$$u^-_{N_{s_0}} = -\frac{|S|}{4\pi|s_0|^2} < -1, \tag{13}$$

because $|S| > 4\pi|s_0|^2$ by the isoperimetric inequality ([1]). This and formula (12) imply

$$u^+_{N_{s_0}} < 0. \tag{14}$$

On the other hand,

$$u(s) = \frac{1}{4\pi|s|} \le \frac{1}{4\pi|s_0|}. \tag{15}$$

So the function $u(x)$, harmonic and continuous in D, attains its maximum on S at the point s_0, because $u|_S = \frac{1}{4\pi|s|}\big|_S$. Therefore, by the maximum principle,

$$u(x) \le u(s_0), \quad \forall x \in D.$$

In particular, $u(s_0) - u(s_0 - \epsilon N_{s_0}) \geq 0$ for all sufficiently small $\epsilon > 0$. Consequently, $u_{N_{s_0}} \geq 0$. This contradicts (14), and the contradiction proves that S is a sphere.

2. Assume (10). Let $h(y) = \frac{1}{4\pi|x-y|}$, $x \in D'$, $y \in D$. This function is harmonic in D. Thus, (10) yields (9):

$$\int_S \frac{ds}{4\pi|x-s|} = \frac{|S|}{4\pi|x|} = \frac{c}{|x|}, \quad \forall x \in D', \quad c := \frac{|S|}{4\pi}. \tag{16}$$

We have already proved that (16) implies that S is a sphere.

3. Assume (11). Multiply (11) by $1/(4\pi|x-y|)$, $x \in D'$, integrate over D, and then integrate by parts to get

$$c_1 \int_S \frac{ds}{4\pi|x-s|} = \frac{1}{4\pi|x|}, \quad \forall x \in D'. \tag{17}$$

By the result proved in assertion 1, this implies that S is a sphere. ∎

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