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Decomposition into special cubes and its applications to quasi-subanalytic geometry

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Abstract. The main purpose of this paper is to present a natural method of decomposition into special cubes and to demonstrate how it makes it possible to efficiently achieve many well-known fundamental results from quasianalytic geometry as, for instance, Gabrielov's complement theorem, o-minimality or quasianalytic cell decomposition.

This paper deals with certain families of quasianalytic Q-functions as well as the corresponding categories Q of quasianalytic Q-manifolds and Q-mappings. Transformation to normal crossings by blowing up applies to such Q-functions (as discovered by Bierstone–Milman [2, 3] and Rolin–Speissegger–Wilkie [13]), and thence to Q-semianalytic sets. This gives rise to the geometry of Q-subanalytic sets, which are a natural generalization of the classical subanalytic sets.

Our main purpose is to present a decomposition of a relatively compact Q-semianalytic set into a finite union of special cubes, and of a relatively compact Q-subanalytic set into a finite number of immersion cubes. The former decomposition combines transformation to normal crossings by local blowing up (developed in [1, 3]) and a suitable partitioning; together with the method of fiber cutting, it yields the latter decomposition. Decomposition into special cubes will also become a basic tool in our subsequent paper [11] concerning quantifier elimination and the preparation theorem in quasianalytic geometry.

We apply decomposition into immersion cubes in our proof of Gabrielov's complement theorem for the case of Q-subanalytic sets. These two results both imply that the expansion $\mathcal{R}_Q$ of the real field by restricted quasianalytic Q-functions is an o-minimal polynomially bounded structure with exponent

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field $\mathbb{Q}$, which admits smooth quasianalytic cell decomposition (cf. [13] and also [12]).

Let us begin by fixing a family $\mathcal{Q} = (\mathcal{Q}_n)_{n\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^n$. For each open subset $U \subset \mathbb{R}^n$, $\mathcal{Q}(U) = \mathcal{Q}_n(U)$ is thus a subalgebra of the algebra $\mathcal{C}_n^\infty(U)$ of real smooth functions on U. By a *Q-function* we mean any function $f \in \mathcal{Q}(U)$. Similarly,

$$f = (f_1, \ldots, f_k) : U \to \mathbb{R}^k$$

is called a *Q-mapping* if so are its components $f_1, \ldots, f_k$. Following Bierstone–Milman [3], we impose the following six conditions on this family of sheaves:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;
2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);
3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \to V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^n$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \to U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;
4. $\mathcal{Q}$ is closed under differentiation;
5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;
6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\hat{f}_a = 0$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

Remarks. 1) By means of Q-mappings, one can build, in the ordinary manner, the category Q of Q-manifolds and Q-mappings, which is a subcategory of that of smooth manifolds and smooth mappings. Similarly, Q-analytic, Q-semianalytic and Q-subanalytic sets can be defined by means of quasianalytic Q-functions.

2) Condition 3 above implies that the implicit function theorem holds in the category Q, and that $\mathcal{Q}$ is closed under reciprocal, i.e. if $f \in \mathcal{Q}(U)$ vanishes nowhere in U, then $1/f \in \mathcal{Q}(U)$.

3) Bierstone–Milman [2, 3] have proven that the category Q admits even a canonical transformation to normal crossings and a canonical desingularization by blowing up.

The basic tool needed for our decomposition into special cubes is transformation to normal crossings by local blowing up (cf. [1, 3]), recalled below.

Let M be a Q-manifold, $\mathcal{I}$ a Q-sheaf of principal ideals on M and $K \subset M$ a compact subset of M. Then there exist a neighbourhood W of K and a surjective Q-mapping $\sigma : \widetilde{W} \to W$ such that:

(i) *σ is a composite of finitely many Q-mappings, each of which is either a blowing-up with smooth center or a surjection of the form $\coprod U_j \to \bigcup U_j$, where $(U_j)_j$ is a finite covering of the target space by coordinate charts and $\coprod$ means disjoint union;*

(ii) *The final transform $\widetilde{\mathcal{I}}$ of the divisor $\mathcal{I}$ is the zero divisor* (1) *and the final exceptional divisors simultaneously have only normal crossings.*

Let M be a Q-manifold and S a relatively compact subset of M. Then S is called a *special cube* of dimension d (associated with φ) if there exists a Q-mapping φ from the vicinity of $[-1,1]^d$ into M such that the restriction of φ to $(-1,1)^d$ is a diffeomorphism onto S. We say that S is *compatible* with Q-functions $f_1, \ldots, f_r : M \to \mathbb{R}$ if each f_i has a constant sign ($-1, 0$ or 1) on S. We can now state our key result.

Theorem on Covering with Special Cubes. *If $f_1, \ldots, f_p : M \to \mathbb{R}$ are Q-functions and $K \subset M$ is a compact subset of M, then some neighbourhood of K can be covered by a finite number of special cubes $S_1, \ldots, S_s$ that are compatible with $f_1, \ldots, f_p$.*

The proof is by induction on the dimension m of the ambient manifold M. Supposing that M is of dimension m and that the theorem is true for ambient manifolds of dimension $< m$, we first prove

Claim. *Let a be a point on a Q-manifold M of dimension m, $g_1, \ldots, g_r$ be Q-functions on M and $\sigma : \widetilde{M} \to M$ be a blowing-up with smooth center $C \subset M$. Suppose we can cover a neighbourhood U of the fiber $\sigma^{-1}(a)$ with finitely many special cubes T_j compatible with the pull-backs $g_1 \circ \sigma, \ldots, g_r \circ \sigma$ of the initial functions and with the exceptional hypersurface H of the blowing-up. Then a neighbourhood of the point a is a finite union of special cubes compatible with $g_1, \ldots, g_r$.*

Indeed, the image $\sigma(U)$ of any neighbourhood U of $\sigma^{-1}(a)$ is a neighbourhood of a, since the mapping σ is proper and thus closed. Each special cube T_j is either disjoint from the exceptional hypersurface H, or contained in it. The images under σ of the cubes of the first kind are special cubes compatible with $g_1, \ldots, g_r$, which cover the set $\sigma(U) \setminus C$. But it follows from the induction hypothesis that a neighbourhood a on the manifold C is a finite union of special cubes compatible with the restrictions to C of $g_1, \ldots, g_r$, as desired. ∎

Since the theorem is local with respect to the points of a given compact subset of the ambient manifold (i.e. the problem amounts to showing that each point of this compact set has a neighbourhood covered by a finite number of special cubes compatible with given Q-functions), the above claim yields the further line of reasoning.

We shall apply transformation to normal crossings to the divisor $\mathcal{I}_0 = \mathcal{I}$ generated by $g_1 \cdot \ldots \cdot g_r$. At the first stage of blowing up, we get a new divisor $\mathcal{I}_1$ by adding to the pull-back of $\mathcal{I}_0$ the exceptional hypersurface. The process can be continued, i.e. $\mathcal{I}_{k+1}$ is the sum of the pull-back of $\mathcal{I}_k$ under the successive local blowing-up σ_{k+1} and the exceptional hypersurface of σ_{k+1}. Eventually, we achieve a divisor $\mathcal{I}_l$ which has only normal crossings. Hence, on this final stage, every compact subset has a neighbourhood covered by finitely many special cubes T_j compatible with $\mathcal{I}_l$. In view of the claim, we are now allowed to proceed backwards so that the theorem follows. ■

Remark. Observe that the special cubes S_j of the covering under consideration and the inverse mappings $\psi_j : S_j \to (-1,1)^{d_j}$ of the associated Q-diffeomorphisms φ_j are described by terms in the language of restricted Q-analytic functions augmented by the name of the reciprocal function $1/x$. This refinement will be crucial for our subsequent paper [11] concerning quantifier elimination and the preparation theorem in quasianalytic geometry.

We can reformulate the above theorem as follows.

Theorem on Decomposition into Special Cubes. *Every relatively compact Q-semianalytic subset $E \subset M$ is a finite union of special cubes.* ■

Corollary. *Every relatively compact Q-subanalytic subset $E \subset M$ has finitely many connected components which are also Q-subanalytic.* ■

After Łojasiewicz [9], by the dimension $\dim E$ of a subset $E \subset M$ of a manifold M we mean

$$\dim E := \max\{\dim \Gamma : \Gamma \text{ is a submanifold of } M \text{ contained in } E\}.$$

Although this notion does not enjoy all properties of ordinary dimension, it is convenient when dealing with subsets of manifolds. In particular, a routine Baire argument shows that the dimension of a countable union of sets coincides with the maximum of their dimensions. Also, it follows from the constant rank theorem that the image of a submanifold of dimension d under a smooth mapping is a set of dimension $\leq d$.

A relatively compact subset C of a Q-manifold M is called an *immersion cube* of dimension d if there exists a Q-mapping φ from the vicinity of $[-1,1]^d$ into M such that the restriction of φ to $(-1,1)^d$ is an immersion onto C.

Fiber Cutting Theorem. *If $F \subset M$ is a relatively compact Q-subanalytic subset of dimension d, then F is a finite union of immersion cubes $C_1, \ldots, C_s$ and of a Q-subanalytic subset V of dimension $< d$:*

$$F = C_1 \cup \cdots \cup C_s \cup V.$$

The proof of this theorem combines both decomposition into special cubes described above and fiber cutting described e.g. in [4, 5, 1, 7]. We sketch the line of reasoning. Observe first that there exists a relatively compact Q-semianalytic subset of $M \times \mathbb{R}^n$ such that $F = \pi(E)$, where $\pi : M \times \mathbb{R}^n \to M$ is the canonical projection. We can present the set E as a finite union of special cubes $S_i \subset M \times \mathbb{R}^n$ on each of which the projection π has constant rank d, and of a Q-semianalytic subset E' on which π has rank $< d$. Then

$$F = \bigcup_i \pi(S_i) \cup W$$

with the Q-subanalytic subset $W = \pi(E')$ of dimension $< d$. The classical method of fiber cutting (making use—after a suitable refinement of the cubes—of a carpeting function which is positive on the cube and vanishes on its frontier) allows us to replace the sets S_i of dimension $> d$ with some Q-semianalytic subsets

$$E_i' \subset S_i \subset M \times \mathbb{R}^n \quad \text{with} \quad \dim E_i' < \dim S_i.$$

We now repeat this process with each set E_i', and so on.

Eventually, we find finitely many special cubes $T_j \subset E \subset M \times \mathbb{R}^n$ of dimension d and a Q-subanalytic subset $V \subset F$ of dimension $< d$ such that

$$F = \bigcup_j \pi(T_j) \cup V$$

and that the projection π has constant rank d on each of the sets T_j. Then the sets $C_j := \pi(T_j)$ are the desired immersion cubes. ■

Corollary 1 (decomposition into immersion cubes). *Every relatively compact Q-subanalytic subset $F \subset M$ is a finite union of immersion cubes.*

This follows directly from the fiber cutting theorem by induction with respect to $\dim F$. ■

Corollary 2. *Let $f(x) : (a, b) \to \mathbb{R}$ be a bounded function with Q-subanalytic graph, defined on an interval (a, b), $a, b \in \mathbb{R}$. Then there are points*

$$a_0 = a < a_1 < \cdots < a_{n-1} < a_n = b$$

such that the graph of f over each subinterval (a_{i-1}, a_i), $i = 1, \ldots, n$, has a parametrization $x = \varphi_i(t)$, $y = \psi_i(t)$ with $t \in (0, 1)$, where φ_i, ψ_i are Q-functions in the vicinity of the interval $[-1, 1]$, φ is strictly increasing and ψ is either strictly monotone or constant. ■

Corollary 3. *If $f : (0, \varepsilon) \to \mathbb{R}$ ($\varepsilon > 0$) is a bounded function with Q-subanalytic graph, then $f(x)$ is asymptotic at 0 to a rational power cx^r ($r \geq 0$, $c \in \mathbb{R}$), i.e.*

$$\lim_{x \to 0^+} \frac{f(x)}{cx^r} = 1. \ ■$$

Corollary 4. *Every relatively compact Q-subanalytic subset F of $\mathbb{R}^m$ is a finite union of immersion cubes C which satisfy the following condition: if $\varphi : (-1,1)^d \to C$ is an immersion cube of dimension d, then there exists a linear subspace V of $\mathbb{R}^m$ of dimension d such that the orthogonal projection $p : \mathbb{R}^m \to V$ is an immersion from C into V.*

An immersion cube C that satisfies the above additional condition will be called a *special immersion cube*.

We argue by induction with respect to $\dim F$. If C is an immersion cube of dimension d in a decomposition of F, one can find a linear subspace V of $\mathbb{R}^m$ of dimension d such that $p : C \to V$ has generic rank d. The set

$$E := \{t \in (-1,1)^d : \mathrm{rank}_t(p \circ \varphi) < d\}$$

is a closed Q-analytic subset of $(-1,1)^d$ of dimension $< d$. Then the set $\varphi(E)$ can be covered by special immersion cubes by induction hypothesis. Finally, if $\{S_i\}$ is a decomposition of the complement $(-1,1)^d \setminus E$ into special cubes, then $\varphi(S_i)$ are special immersion cubes which cover the complement $C \setminus \varphi(E)$. This completes the proof. ∎

The refined decomposition from Corollary 4 will be needed in our proof of the well-known complement theorem for Q-subanalytic sets.

Complement Theorem. *Let M be a Q-manifold. If $F \subset M$ is a Q-subanalytic subset of M, so is its complement $M \setminus F$.*

The proof is by induction on the dimension m of the ambient manifold M. We shall consider two cases: $\dim F =: d < m$ and $\dim F = m$.

Since the problem is local, we may assume that F is a relatively compact subset in $\mathbb{R}^m$, and next, by Corollary 2, that F is a special immersion cube. We keep the notation of Corollary 2.

In the first case, put $q = p \circ \varphi$ and $T = (-1,1)^d$; the set $U = p(F) = q(T)$ is obviously an open Q-subanalytic subset in $\mathbb{R}^d$. Clearly, the restriction

$$\mathrm{res}\, q : T \setminus q^{-1}(q(\partial T)) \to U \setminus q(\partial T)$$

is a proper mapping; here $\partial T := \overline{T} \setminus T$ denotes the frontier of T. Consequently, being a local homeomorphism, $\mathrm{res}\, q$ is a topological covering. It has therefore a constant number of points in all fibres over each connected component of the set $U \setminus q(\partial T)$.

By the induction hypothesis applied to the ambient manifold $\mathbb{R}^d$ of dimension $< m$, the complement $U \setminus q(\partial T)$ is a Q-subanalytic subset in $\mathbb{R}^d$, and thus it has finitely many connected components. Hence the number of points in all fibres of the restriction under consideration is bounded by an integer n. As the set $q(\partial T)$ is of dimension $< d$, $q(\partial T) \cap U$ is a nowhere-dense subset of U, and consequently the number of points in all fibres of the restriction $\mathrm{res}\, q : T \to U$ is bounded by n too. A fortiori the number of points in all fibres of the restriction $\operatorname{res} p : F \to U$ is bounded by n. Clearly, the sets

$$U_k := \{u \in U : \sharp\, p^{-1}(u) \cap F \geq k\}, \qquad k = 1, \ldots, n,$$

are Q-subanalytic subsets in $\mathbb{R}^d$, whence, again by the induction hypothesis, so are the sets

$$V_k := \{u \in U : \sharp\, p^{-1}(u) \cap F = k\}, \qquad k = 1, \ldots, n.$$

We leave it to the reader to verify that in the circumstances the complement $\mathbb{R}^m \setminus F$ is a Q-subanalytic subset of $\mathbb{R}^m$ as well.

In the second case, φ is a local homeomorphism of $T = (-1,1)^m$ onto $U = \varphi(T) \subset \mathbb{R}^m$. Due to the first case we have just considered, the complement $\mathbb{R}^m \setminus \varphi(\partial T)$ is a Q-subanalytic subset in $\mathbb{R}^m$. Next, observe that $\mathbb{R}^m \setminus \varphi(\overline{T})$ is an open and closed subset of $\mathbb{R}^m \setminus \varphi(\partial T)$, because $\varphi(T)$ is open and $\varphi(\overline{T})$ is closed. Hence $\mathbb{R}^m \setminus \varphi(\overline{T})$, as the union of certain connected components of the Q-subanalytic set $\mathbb{R}^m \setminus \partial T$, is a Q-subanalytic subset in $\mathbb{R}^m$ too. Again due to the first case, the set

$$\varphi(\partial T) \setminus (\varphi(T) \cap \varphi(\partial T))$$

is a Q-subanalytic subset in $\mathbb{R}^m$, whence so is the complement

$$\mathbb{R}^m \setminus \varphi(T) = (\mathbb{R}^m \setminus \varphi(\overline{T})) \cup (\varphi(\partial T) \setminus (\varphi(\partial T) \cap \varphi(T)).$$

This completes the proof. ■

We conclude that if $F \subset M$ is a Q-subanalytic subset of M, so are its closure $\overline{F}$ and frontier ∂F. Consider now the expansion $\mathcal{R}_Q$ of the real field $\mathbb{R}$ by restricted Q-functions, i.e. functions of the form

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in [-1,1]^m, \\ 0 & \text{otherwise,} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^m$. Then the complement theorem may be rephrased as follows.

COROLLARY 1. *The structure $\mathcal{R}_Q$ is model complete and o-minimal.* ■

REMARK. Let Φ be an arbitrary semialgebraic diffeomorphism of $\mathbb{R}^m$ onto $(-1,1)^m$. The above may be summarized by the following observation:

A set $E \subset \mathbb{R}^m$ is definable in the structure $\mathcal{R}_Q$ iff $\Phi(E)$ is a (relatively compact) Q-subanalytic subset of $\mathbb{R}^m$. In other words, the definable subsets in the expansion $\mathcal{R}_Q$ of the real field coincide with those subsets of $\mathbb{R}^m$ that are Q-subanalytic in any semialgebraic compactification of $\mathbb{R}^m$.

COROLLARY 2. *The o-minimal structure $\mathcal{R}_Q$ is polynomially bounded with field of exponents $\mathbb{Q}$.*

This follows directly from Corollary 3 to the fiber cutting theorem. ■

By a *Q-cell* we mean a Q-subanalytic cell defined by means of Q-functions. Yet another consequence of the complement theorem and decomposition into immersion cubes is the fundamental well-known result below (cf. [13]).

Quasianalytic Cell Decomposition Theorem. *Consider definable sets $E_1, \ldots, E_k \subset \mathbb{R}^m$ and a definable function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^m$. Then*

- (I_m) *There is a decomposition of $\mathbb{R}^m$ into finitely many Q-cells which partitions each of the sets $E_1, \ldots, E_k$.*
- (II_m) *There is a decomposition of $\mathbb{R}^m$ into finitely many Q-cells which partitions the set E and is such that the restriction of f to each of those Q-cells is a Q-function.*

In order to prove this, we shall use a typical induction argument with respect to m (see e.g. [6, Chap. 3]). Notice first that the proof of (I_m) uses both (I_{m-1}) and (II_{m-1}), and is standard (*loc. cit.*).

Next, applying any semialgebraic diffeomorphism Φ from the foregoing remark, one may assume that the graph of f is a subset of $(-1,1)^{m+1}$. Then, due to decomposition into immersion cubes applied to the graph of f and by (I_m), one can partition $\mathbb{R}^m$ into finitely many Q-cells C_i such that the restriction of f to each C_i is a Q-function (or an empty set), so that we obtain (II_m). This is the basic, non-standard step of induction for cell decomposition in the quasianalytic setting. ∎

Corollary. *Every relatively compact Q-subanalytic subset F of $\mathbb{R}^m$ can be partitioned into finitely many Q-cells.* ∎

Open problems. 1) We do not know whether every relatively compact Q-subanalytic subset F of a Q-manifold M is a finite union of special cubes.

2) Does the family of smooth definable functions in the structure $\mathcal{R}_Q$ coincide with the family of definable Q-functions?

3) Does every o-minimal polynomially bounded expansion of the real field $\mathbb{R}$ admit smooth quasianalytic cell decomposition?

4) Gabrielov's method [7] of truncating Taylor series can be transferred to the quasianalytic setting. Therefore, if $E \subset M$ is a Q-semianalytic subset of a Q-manifold M, so are the closure $\overline{E}$ and the frontier ∂E. We do not know whether a connected component of E is Q-semianalytic as well.

One can consider the opposite situation: given a polynomially bounded expansion $\mathcal{R}$ of the real field $\mathbb{R}$, the global smooth $\mathcal{R}$-definable functions form a family R of quasianalytic functions. Let $\mathcal{R}'$ be the o-minimal substructure generated by those global smooth functions from R; R-semianalytic sets are those from the boolean algebra generated by the sets of the form

$$\{x \in \mathbb{R}^N : f(x) = 0\}$$

with $f(x)$ being a global smooth R-function on $\mathbb{R}^N$, and R-subanalytic sets are projections of R-semianalytic sets.

We proved in [10] that the ring of global smooth definable functions is topologically noetherian. Nevertheless, the question whether the complement theorem holds for R-subanalytic sets or, equivalently, whether the structure $\mathcal{R}'$ is model-complete, seems to be much more difficult and is yet unsolved.

Remark. We should mention that an affirmative answer to the foregoing problem, given in O. Le Gal's thesis [8], contained an essential gap. The corrected, published version of his paper provides only a partial solution, as it imposes an additional strong condition of global character on the differential algebra of definable functions.

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