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Equilibria and strict equilibria of multivalued maps on noninvariant sets

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Abstract. This paper is concerned with existence of equilibrium of a set-valued map in a given compact subset of a finite-dimensional space. Previously known conditions ensuring existence of equilibrium imply that the set is either invariant or viable for the differential inclusion generated by the set-valued map. We obtain some equilibrium existence results with conditions which imply neither invariance nor viability of the given set. The problem of existence of strict equilibria is also discussed.

1. Introduction. The problem of finding an equilibrium of a multivalued map F in a set $K \subset \mathbb{R}^n$, i.e., a point $x \in K$ such that $0 \in F(x)$, is important in many topics of analysis. Let us only underline that this problem contains the fixed point problem.

Results on existence of such equilibria have already been obtained for instance

• in [10] for K convex compact and F upper semicontinuous with closed convex values,

• in [3] for K a compact L-retract and F upper semicontinuous with closed convex values.

(Note that these papers and their bibliography concerns spaces more general than $\mathbb{R}^n$.)

We emphasize that conditions used in [10], [3], [11] and [9] *imply* (*at least in* $\mathbb{R}^n$) *that* K *is viable* for the differential inclusion

$$\dot{x}(t) \in F(x(t)) \qquad \text{for a.e. } t \geq 0, \tag{1}$$

namely: For each $x_0 \in K$ there is at least one trajectory $x(\cdot)$ for (1) with $x(0) = x_0$ such that $x(t) \in K$ for all $t \geq 0$. Indeed, these conditions imply

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the tangential condition

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset \tag{2}$$

(with $T_K(x)$ the usual contingent cone), which is equivalent by the viability theorem (cf. [1]) to the viability of K for (1). This fact is even a key point in the existence of equilibria (cf. [9] or [1, Theorem 3.7.5]).

Here, we propose a different approach. We give various sufficient conditions for equilibrium existence and these conditions do not imply the viability of K.

In the present paper, we study the equilibrium problem by looking at topological properties of the subset K_s of the boundary of K where all trajectories of (1) leave K immediately. This set is used in the Ważewski topological principle (cf. [6], [5]). Our first main result is the following:

THEOREM 1.1. *Let $K \subset \mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ satisfy*:

(I) *K is a compact $C^{1,1}$ n-manifold with boundary*;
(II) *F is a continuous map with compact convex values and at most linear growth*;
(III) *$K_s(F)$ is closed and*, *if nonempty*, *it is a $C^{1,1}$ $(n-1)$-submanifold of ∂K with boundary*;
(IV) $\chi(K, K_s(F)) \neq 0$.

Then there is an equilibrium of F in K.

The key point of our proof is a reduction to an ordinary differential equation via a suitable single-valued approximation of F *well adapted* to K_s. The single-valued case, where the map generates a flow, has been studied in [15].

Also we consider topological properties of the complement of K and derive the existence on equilibrium. Note that [13] contains results on equilibrium existence when the complement of K has suitable viability properties.

Finally, we study the existence of a *strict equilibrium* in K, i.e., a point $x \in K$ such that $\{0\} = F(x)$, using selection techniques developed for equilibria. The second main result is

THEOREM 1.2. *Let $K \subset \mathbb{R}^n$ and F satisfy assumptions* (I), (III), (IV) *of Theorem* 1.1,

$$0 \notin \operatorname{Int}_{\operatorname{lin} F(x)} F(x) \quad \text{for every } x \in K \text{ with } |F(x)| > 0 \tag{3}$$

(*where* $|F(x)| := \sup\{|y| \mid y \in F(x)\}$ *and* $\operatorname{Int}_{\operatorname{lin} F(x)} F(x)$ *denotes the relative interior of $F(x)$ in the subspace* $\operatorname{lin} F(x) \subset \mathbb{R}^n$ *spanned by $F(x)$*), *and let F be Lipschitz with $0 \notin F(x)$ for every $x \in \partial K$. Then there exists a strict equilibrium of F in* $\operatorname{Int} K$.

The organization of the paper is as follows. After some preliminaries the third section is devoted to sufficient conditions for solving the equilibrium problem without viability. In the last section, the existence of strict equilibria is studied.

2. Preliminaries. Throughout the paper, by $\operatorname{Int} A$, $\operatorname{cl} A$ (or $\bar{A}$) and ∂A we denote respectively the interior, closure and boundary of a subset A of a metric space X. The open ball centred at x_0 with radius r is denoted by $B(x_0, r)$, and the unit ball $B(0,1)$ by B_1. We also write $|\cdot|$ for the Euclidean norm. By $d_M(x)$ (or $\operatorname{dist}(x, M)$) we denote the distance from a point x to a closed set M. By the distance between two sets $N, M \subset X$ we mean the number $\operatorname{dist}(N, M) := \inf\{d(x,y) \mid x \in N,\, y \in M\}$.

In (1) we assume that F is a *Marchaud map*, i.e., F is upper semicontinuous with compact convex values and at most linear growth. By a solution to the inclusion (1) we mean an absolutely continuous map $x : [0, \infty) \to \mathbb{R}^n$ satisfying (1) almost everywhere. For $x_0 \in \mathbb{R}^n$, we denote by $S_F(x_0)$ the set of all solutions to (1) with $x(0) = x_0$ (starting from x_0).

Definition 2.1. We say that a trajectory $x(\cdot)$ for the inclusion (1) is *viable in* K if $x(t) \in K$ for all $t \geq 0$. A set K is said to be *viable under* F if for each $x_0 \in K$ there is at least one trajectory $x(\cdot)$ for (1) which is viable in K and $x(0) = x_0$. The *viability kernel* of K for F (written $\operatorname{Viab}_F(K)$) is the largest closed subset of K viable under F (possibly empty, in general). Equivalently (see [1, Theorem 4.1.2]), $\operatorname{Viab}_F(K)$ is the subset of all initial states such that from each of them starts at least one solution viable in K. We say that the set K is *invariant* under F if all trajectories for (1) starting from K are viable in K.

Obviously, for single-valued locally Lipschitz right-hand sides the notions of viable and invariant sets coincide. Later on we shall use the notation $\sigma(F(x))$ for a Steiner point of a convex set $F(x)$ (see e.g. [2]). If F is Lipschitz, the map $x \mapsto \sigma(F(x))$ is a Lipschitz selection of F.

One defines

$$T_K(x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0^+} \operatorname{dist}(x + hv, K)/h = 0\},$$

the *Bouligand tangent cone* to K at x. Viability and invariance conditions can be characterized using Bouligand cones, as follows (see e.g. [1]):

Proposition 2.2. *A closed set K is viable under a Marchaud map F if and only if*

$$F(x) \cap T_K(x) \neq \emptyset \quad \text{for every } x \in K.$$

If F is Lipschitz, then a closed set K is invariant under F if and only if

$$F(x) \subset T_K(x) \quad \text{for every } x \in K.$$

We introduce the following notations for subsets of the boundary of K:

$$K_s(F) := \{x_0 \in \partial K \mid \forall x \in S_F(x_0) : x \text{ leaves } K \text{ immediately}\},$$
$$K_e(F) := \{x_0 \in \partial K \mid \exists x \in S_F(x_0) : x \text{ leaves } K \text{ immediately}\}.$$

When there is no ambiguity, we shall write briefly K_s and K_e.

Note that for F Lipschitz continuous, using the local invariance theorem (see e.g. [1, Theorem 5.3.4]), one can describe K_e by suitable tangent cones conditions. Indeed, if K_e is closed, we have

$$\partial K \setminus K_e = \{x \in \partial K \mid \exists U_x \subset \partial K, \text{ open}, x \in U_x, \forall z \in U_x : F(z) \subset T_K(z)\}.$$

Moreover, $K_e = K_s$ when F is Lipschitz and single-valued.

In the Appendix we also give a necessary and a sufficient condition for points in ∂K to belong to K_s, in terms of tangent cones. These conditions were first proved in [5] and announced in [6].

Recall that a subset M of $\mathbb{R}^n$ is said to be a *proximal retract* (see e.g. [16]), if there is an open neighbourhood V of M such that

$$\pi_M(x) := \{y \in M \mid |y - x| = \inf_{u \in M} |u - x|\} \quad \text{is a singleton} \tag{4}$$

for every $x \in U$. This means that π_M is a retraction from V onto M. One can prove that each $C^{1,1}$ manifold is a proximal retract.

A subset M of a metric space X is an *$\mathcal{L}$-retract* (of X) if there are an open neighbourhood U of M in X, a retraction $r : U \to M$ and a constant $L > 0$ such that

$$d(r(x), x) \le L \operatorname{dist}(x, M) \quad \text{for every } x \in U.$$

It is seen that each proximal retract is an $\mathcal{L}$-retract with constant 1.

We say that a set K is of *finite type* if the graded vector space $\{H_q(K)\}_{q\ge 0}$ is of finite type, i.e., $H_q(K) = 0$ for almost all $q \ge 0$, and $\dim H_q(K) < \infty$ for all $q \ge 0$. Here H denotes the Čech homology functor.

For each set of finite type the *Euler characteristic*

$$\chi(K) := \sum_{q=0}^{\infty} (-1)^q \dim H_q(K)$$

is defined (see e.g. [4]). If we have a pair (K, M) of spaces (M closed in K), with $M \subset K$ and such that both K and M are of finite type, we can define $\chi(K, M) := \chi(K) - \chi(M)$. Note that $\chi(K) = \lambda(\mathrm{id})$, the Lefschetz number of the identity map.

We will say that a compact space X is a *Lefschetz set* if for any continuous map $f : X \to X$, the condition $\lambda(f) \ne 0$ implies that there is a fixed point of f. This class of spaces is large. It contains compact absolute neighbourhood retracts and, more generally, compact approximate absolute neighbourhood retracts (see e.g. [14]).

3. Equilibria without viability

3.1. *Using properties of K_s.* In this subsection we deal with the situation where K may not be a viability domain. The boundary ∂K may contain simultaneously points at which a field is strictly inward and others where it is strictly outward.

The first result is a simple consequence of the following.

PROPOSITION 3.1 (adapted version of Theorem 4.1 of [15]). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz map and suppose that K and $K_s(f)$ satisfy assumptions* (I) *and* (III) *of Theorem* 1.1 *with $\chi(K, K_s(f)) \neq 0$. Then there is an equilibrium of f in K.*

As a consequence one obtains the following easy

COROLLARY 3.2. *Let $K \subset \mathbb{R}^n$ be a compact subset, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Lipschitz map. Assume that K_s and K are as in Proposition* 3.1 *and that $K_s = K_e$. Then there is an equilibrium of F in K.*

To prove the above corollary, take any Lipschitz selection f of F and apply Proposition 3.1.

Above we have assumed that F is so regular that it has Lipschitz selections for which, under $K_s = K_e$, we can use a single-valued approach. Weakening these two assumptions, i.e. lipschitzeanity of F and equality $K_s = K_e$, is the aim of our main Theorem 1.1. In the proof we will use the following crucial lemma.

LEMMA 3.3. *Let K and F satisfy* (I)–(III) *of Theorem* 1.1 *and suppose $0 \notin F(x)$ for every $x \in \partial K$. Then, for every $\varepsilon > 0$, there exists $f : \mathbb{R}^n \to \mathbb{R}^n$ such that*

(a) *$\langle f(x), \nu_x \rangle < 0$ for every $x \in \partial K \setminus K_s(F)$, where ν_x is an outward normal vector to K at x;*

(b) *for every $x \in K$, if $f(x) = 0$, then there is $y \in K \cap B(x, \varepsilon)$ such that $0 \in F(y)$;*

(c) *$K_s(f) = K_s(F)$;*

(d) *f is Lipschitz.*

Proof. We proceed in two steps.

STEP 1. We construct an open set U in $\mathbb{R}^n$ such that $K \setminus K_s(F) \subset U$ and $K_s(F) \cap U = \emptyset$, and a map $g : U \to \mathbb{R}^n$ such that

(A) $\langle g(x), \nu_x \rangle < 0$ for every $x \in \partial K \setminus K_s(F)$;

(B) g is C^∞ in U;

(C) $g(x) \in F(x) + d_{K_s(F)}(x) B_1$ for every $x \in K \setminus K_s(F)$;

(D) for every $x \in K \setminus K_s(F)$, if $0 \notin F(y)$ for each $y \in B(x,\varepsilon) \cap K$, then $g(x) \in F(x) + \frac{1}{2} d_{F(x)}(0) B_1$.

To do this, we first find, for every $x \in K \setminus K_s(F)$, a vector $v_x \in \mathbb{R}^n$ and an open neighbourhood U_x of x in $\mathbb{R}^n$ satisfying the following three conditions:

$$U_x \cap K_s(F) = \emptyset, \tag{5}$$

$$\forall y \in U_x \cap K, \quad v_x \in (F(y) + d_{K_s(F)}(y) B_1) \cap \big(F(y) + \tfrac{1}{2} d_{F(y)}(0) B_1\big), \tag{6}$$

$$\forall x \in \partial K \setminus K_s(F)\ \forall y \in U_x \cap \partial K, \quad \langle v_x, \nu_y \rangle < 0. \tag{7}$$

Case 1. Let $x \in \partial K \setminus K_s(F)$. Then there exists $w_x \in F(x) \cap T_K(x)$. Set $\eta_x := d_{K_s(F)}(x) > 0$, $\mu_x := d_{F(x)}(0) > 0$ and $\varepsilon_x := \frac{1}{8} \min\{\mu_x, \eta_x\}$. Take an open neighbourhood $U_x \subset B(x,\varepsilon)$ of x such that $d_{K_s(F)}(y) \geq \eta_x/2$, $d_{F(y)}(0) \geq \mu_x/2$ and $F(x) \subset F(y) + \varepsilon_x B_1$ for every $y \in U_x$.

Now take $v_x := w_x - t_x \nu_x$ ($t_x > 0$ sufficiently small) such that $v_x \in F(x) + \varepsilon_x B_1$. Then, for $y \in U_x \cap K$,

$$v_x \in F(y) + 2\varepsilon_x B_1 \subset F(y) + \tfrac{1}{4} \eta_x B_1 \subset F(y) + \tfrac{1}{2} d_{K_s(F)}(y) B_1. \tag{8}$$

Analogously,

$$v_x \in F(y) + 2\varepsilon_x B_1 \subset F(y) + \tfrac{1}{4} \mu_x B_1 \subset F(y) + \tfrac{1}{2} d_{F(y)}(0) B_1. \tag{9}$$

Moreover, we can take U_x so small that $\langle v_x, \nu_y \rangle < 0$ for every $y \in U_x \cap \partial K$, since $\langle v_x, \nu_x \rangle = \langle w_x, \nu_x \rangle - t_x < 0$.

Case 2. Let $x \in \operatorname{Int} K$ and $0 \notin F(x)$. As above, we can find $U_x \subset B(x,\varepsilon)$ with $U_x \cap \partial K = \emptyset$ and $v_x \in F(x)$ satisfying (6) and $0 \notin F(y)$ for any $y \in U_x$.

Case 3. For $x \in \operatorname{Int} K$ with $0 \in F(x)$, take $v_x = 0$ and choose $U_x \subset B(x,\varepsilon)$ such that $U_x \cap \partial K = \emptyset$ and $v_x \in F(y) + d_{K_s(F)}(y) B_1$ for every $y \in U_x$.

Choose a countable, locally finite covering $\{U_{x_i} \mid x_i \in K \setminus K_s(F)\}$ of $K \setminus K_s(F)$ and consider a smooth (C^∞) partition of unity $\{\lambda_i : U_{x_i} \to [0,1]\}$ subordinate to it. Define $U := \bigcup_{i=1}^\infty U_{x_i}$ and notice that $K \setminus K_s(F) \subset U$ and $U \cap K_s(F) = \emptyset$. For every $x \in U$, set $I(x) := \{i \in \mathbb{N} \mid \lambda_i(x) \neq 0\}$ and define

$$g(x) := \sum_{i \in I(x)} \lambda_i(x) v_{x_i}.$$

Since each λ_i is C^∞, (B) holds. To verify (A), take $x \in \partial K \setminus K_s(F)$ and notice that, by the definition of v_{x_i} and U_{x_i}, for each $i \in I(x)$, one has $x_i \in \partial K \setminus K_s(F)$ and

$$\langle g(x), \nu_x \rangle = \sum_{i \in I(x)} \lambda_i(x) \langle v_{x_i}, \nu_x \rangle < 0.$$

Now, for $x \in K \setminus K_s(F)$ and $i \in I(x)$, because the right-hand side of (6) is convex one deduces that

$$g(x) = \sum_{i \in I(x)} \lambda_i(x) v_{x_i} \in F(x) + d_{K_s(F)}(x) B_1,$$

and (C) is also satisfied. Finally, let $x \in K \setminus K_s(F)$ be such that $0 \notin F(y)$ for every $y \in B(x, \varepsilon)$. Then, by (9), $v_{x_i} \in F(x) + \frac{1}{2} d_{F(x)}(0) B_1$, and hence $g(x) \in F(x) + \frac{1}{2} d_{F(x)}(0) B_1$, verifying (D) and ending Step 1.

STEP 2. In order to construct the map f we first define, for $x \in K_s(F)$, a map $v_x(\cdot)$ and an open neighbourhood U_x of x in the following way.

Case 1. Let $x \in \operatorname{Int}_{\partial K} K_s(F)$. We know that $\min_{v \in F(x)} \langle v, \nu_x \rangle \geq 0$ and $0 \notin F(x)$. Fix $w_x \in F(x)$, so $\langle w_x, \nu_x \rangle \geq 0$. Write $\mu_x := d_{F(x)}(0)$ and take an open neighbourhood U_x of x in $\mathbb{R}^n$ such that $U_x \cap \partial_{\partial K} K_s(F) = \emptyset$, $U_x \cap \partial K \subset K_s(F)$ and $d_{F(y)}(0) \geq \mu_x/2$ for every $y \in U_x \cap K$.

Define $v_x := w_x + t_x \nu_x$, where $t_x > 0$ is so small that $v_x \in F(x) + \frac{1}{8} \mu_x B_1$, and consequently, inclusion (9) holds for every $y \in U_x \cap K$. Moreover, $\langle v_x, \nu_y \rangle > 0$ for $y \in U_x \cap K_s(F)$.

Case 2. Suppose that $x \in \partial_{\partial K} K_s(F)$. Then, since $x \in K_s(F)$ which is closed, we know by Lemma 5.2 in the Appendix that there is $v \in F(x) \cap T_K(x)$ such that $v \in T_{K_s(F)}(x)$. There is an open neighbourhood V_x of x in $\mathbb{R}^n$ and a $C^{1,1}$ diffeomorphism $\phi : V_x \to \phi(V_x) \subset \mathbb{R}^n$ such that $\phi(x) = 0$,

$$\phi(V_x \cap K) = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_1 \leq 0\} =: X,$$
$$\phi(V_x \cap K_s(F)) = \{(y_1, \dots, y_n) \in X \mid y_1 = 0 \text{ and } y_2 \geq 0\} =: X_s.$$

As a consequence, for $z \in V_x \cap K$,

$$\operatorname{Int} \phi'(z)(T_K(z)) = \{(y_1, \dots, y_n) \in X \mid y_1 < 0\}, \tag{10}$$

$$\operatorname{Int} \phi'(z)(T_{\mathbb{R}^n \setminus K}(z)) = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_1 > 0\}, \tag{11}$$

and for $z \in V_x \cap \partial_{\partial K} K_s(F)$,

$$\operatorname{Int}_{\partial X} \phi'(z)(T_{K_s(F)}(z)) = \{(y_1, \dots, y_n) \in X \mid y_1 = 0 \text{ and } y_2 > 0\}. \tag{12}$$

Define on $\phi(V_x)$ the map $G(y) := \phi'(\phi^{-1}(y)) F(\phi^{-1}(y))$.

Let $\zeta(y) := (y_2, 1, 0, \dots, 0)$ and take $w := \phi'(x)v$. Then $\langle w, e_1 \rangle = 0$ and $\langle w, e_2 \rangle \geq 0$, where e_1, e_2 are the unit vectors of the first two axes. Let $u_\theta(y) := w + \theta \zeta(y)$, where $\theta > 0$. Then

$$\begin{aligned} \langle u_\theta(y), e_1 \rangle &< 0 \quad \text{for } y \in \partial X \setminus X_s, \\ \langle u_\theta(y), e_1 \rangle &> 0 \quad \text{for } y \in \operatorname{Int}_{\partial X} X_s, \\ \langle u_\theta(y), e_2 \rangle &> 0 \quad \text{for } y \in \partial_{\partial X} X_s. \end{aligned} \tag{13}$$

Define, for $z \in V_x$, $v_\theta(z) := (\phi^{-1})'(\phi(z)) u_\theta(\phi(z))$. Using (13) and the fact that tangent cones correspond to tangent cones under the diffeomorphism

ϕ (see (10)–(12)), we obtain

$$\begin{aligned} &\langle v_\theta(z), \nu_z\rangle < 0 \quad \text{for } z \in \partial K \setminus K_s(F),\\ (14)\qquad &\langle v_\theta(z), \nu_z\rangle > 0 \quad \text{for } z \in \operatorname{Int}_{\partial K} K_s(F),\\ &\langle v_\theta(z), n_z\rangle > 0 \quad \text{for } z \in \partial_{\partial K} K_s(F), \end{aligned}$$

where n_z is an inward normal vector to $K_s(F)$ in $T_{\partial K}(z)$.

Similarly to Case 1, we can find an open neighbourhood $U_x \subset V_x$ of x and a small $\theta_x > 0$ such that, for $v_x(z) := v_{\theta_x}(z)$, we have additionally

$$v_x(z) \in F(z) + \tfrac{1}{2} d_{F(z)}(0)B_1 \quad \text{for every } z \in U_x \cap K.$$

We have constructed an open covering of K consisting of the U from Step 1 and $\{U_x\}_{x\in K_s(F)}$. Since $K_s(F)$ is compact, we can choose a finite subcovering $\{U_i\}_{i=0}^k$, where $U_0 := U$, and consider a smooth partition of unity $\{\beta_i\}$ subordinate to it.

Define

$$f(x) := \beta_0(x)g(x) + \sum_{i=1}^{k} \beta_i(x)v_{x_i}(x).$$

To finish the proof it is sufficient to verify conditions (a)–(d).

To check (a), let $x \in \partial K \setminus K_s(F)$ and $I(x) := \{i \in \{1,\ldots,k\} \mid \beta_i(x) \neq 0\}$. From (14) and (A) it follows that $\langle v_{x_i}(x), \nu_x\rangle < 0$ for $i \in I(x)$, and consequently, $\langle g(x), \nu_x\rangle < 0$. Therefore, $\langle f(x), \nu_x\rangle < 0$.

Now, suppose that $x \in K$ is such that $0 \notin F(y)$ for every $y \in B(x,\varepsilon)$. Since $v_{x_i}(x) \in F(x) + \frac{1}{2} d_{F(x)}(0)B_1$ for each $i \in I(x)$, and $g(x) \in F(x) + \frac{1}{2} d_{F(x)}(0)B_1$ (see (D)), one obtains $f(x) \in F(x) + \frac{1}{2} d_{F(x)}(0)B_1$, and hence $f(x) \neq 0$; condition (b) is satisfied.

To verify (c), notice that $K_s(f) \subset K_s(F)$ because of (a). Moreover, for $x \in \operatorname{Int}_{\partial K} K_s(F)$, $\beta_0(x) = 0$ and $\langle v_{x_i}, \nu_x\rangle > 0$ (see Step 2, Case 1). Thus, $\langle f(x), \nu_x\rangle > 0$.

If $x \in \partial_{\partial K} K_s(F)$, then $x_i \in \partial_{\partial K} K_s(F)$ for every $i \in I(x)$. Moreover, $\beta_0(x) = 0$, $\langle v_{x_i}(x), \nu_x\rangle = 0$ and $\langle v_{x_i}, n_x\rangle > 0$, which implies that $\langle f(x), n_x\rangle > 0$ while $\langle f(x), \nu_x\rangle = 0$. From Lemma 5.2(1) it follows that a trajectory for f starting from x leaves the set K immediately. Therefore $K_s(f) = K_s(F)$.

Finally, f is Lipschitz by the regularity of β_i and of the diffeomorphism ϕ; the proof is complete. ∎

Proof of Theorem 1.1. We can assume that $0 \notin F(x)$ for every $x \in \partial K$. Applying Lemma 3.3 we find a Lipschitz single-valued map f satisfying (a)–(d). Thanks to (IV), we can use Proposition 3.1 for f and obtain its equilibrium $x_0 \in K$. Property (b) implies that there is also an equilibrium of F in K near x_0. ∎

3.2. *Using properties of the complement of K.* Set $\widehat{K} := \overline{\mathbb{R}^n \setminus K}$ and recall that (see [17], [18])

$$T_K(x) \cup T_{\widehat{K}}(x) = \mathbb{R}^n, \quad T_K(x) \cap T_{\widehat{K}}(x) = T_{\partial K}(x) \quad \text{for } x \in \partial K. \tag{15}$$

The first result using $\widehat{K}$ to study equilibria on K, which we would like to present, has been proved in [13] and reads as follows.

PROPOSITION 3.4 ([13, Theorem 3.1]). *Let $K = \overline{\operatorname{Int} K} \subset \mathbb{R}^n$ be a compact subset of finite type with $\chi(K) \neq 0$, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Lipschitz map satisfying*

$$F(x) \subset T_{\widehat{K}}(x) \quad \textit{for all } x \in \partial K, \qquad \operatorname{Viab}_F(K) \cap \partial K = \emptyset. \tag{16}$$

Then F has an equilibrium in $\operatorname{Int} K$.

REMARK 3.5. Note that condition (16) is clearly satisfied if for all $x \in \partial K$, $F(x) \cap T_K(x) = \emptyset$ or, more generally, if $K_s = \partial K$.

Replacing the condition $F(x) \subset T_{\widehat{K}}(x)$ by $F(x) \cap \operatorname{cl}(\mathbb{R}^n \setminus T_{\widehat{K}}(x)) = \emptyset$ narrows the class of problems which may be considered. Indeed, there are situations (see [13]) appropriate for Proposition 3.4 and such that $\operatorname{cl}(\mathbb{R}^n \setminus T_{\widehat{K}}(x)) = \mathbb{R}^n$ at some points.

Proposition 3.4 together with the above remark allows us to deal with a large class of sets. In [13] there is an example of a set K which is not even an absolute neighbourhood retract and which, together with a Lipschitz single-valued map, satisfies the assumptions of Proposition 3.4.

The goal of this subsection is to weaken assumption (16) to be able to consider also situations where K and $\widehat{K}$ are not invariant under $-F$ and F, respectively. We study the behaviour of F with respect to $\widehat{K}$.

We start with the following.

LEMMA 3.6. *Let K be a Lefschetz set with $\chi(K) \neq 0$, $\Omega \subset \mathbb{R}^n$ an open neighbourhood of K and $f : \Omega \to \mathbb{R}^n$ a Lipschitz map such that K is invariant under f. Then f has an equilibrium in K.*

Proof. It is well known that for f and every $t \geq 0$, the Poincaré operator $P_t := e_t \circ S_f : K \to K$, where $e_t(x(\cdot)) := x(t)$, is continuous and homotopic to $\mathrm{id} : K \to K$ by the homotopy $H(x,s) := P_{st}(x)$. Thus

$$0 \neq \chi(K) = \lambda(\mathrm{id}) = \lambda(P_t),$$

and since K is a Lefschetz set, there is a fixed point x_t of P_t, which means that there is a t-periodic trajectory for f.

Taking a sequence $t_n \to 0$, one can easily prove that there is a stationary trajectory, which is equivalent to f having an equilibrium. ∎

We will also use the following simultaneous selection and approximation result.

LEMMA 3.7. *Let E be a normed space, X a metric space and $A \subset X$ a compact subset. Assume that $F : X \multimap E$ and $\Psi : A \multimap E$ are convex valued, F is u.s.c., Ψ is l.s.c. and the following condition is satisfied*:

(17) *for each $x \in A$ there are $y_x \in F(x) \cap \Psi(x)$ and an open neighbourhood $U(x) \subset X$ of x such that $y_x \in \Psi(z)$ for every $z \in U(x) \cap A$.*

Then, for every $\varepsilon > 0$, there exist an open neighbourhood Ω_ε of A in X and a Lipschitz map $f : \Omega_\varepsilon \to E$ such that:

(i) *f is an ε-approximation of F, i.e., $f(x) \in F(B(x,\varepsilon)) + \varepsilon B_1$ for every $x \in \Omega_\varepsilon$,*
(ii) *f is a selection of Ψ on A.*

Proof. The idea of the proof is taken from [3]. For a given $\varepsilon > 0$, consider the open covering of A in X,

$$U(x) := B(x, \varepsilon/2) \cap \{x' \in X \mid F(x') \subset F(x) + (\varepsilon/2)B_1\}, \quad x \in A.$$

Using (17) we can find, for every $x \in A$, a point $y_x \in F(x) \cap \Psi(x)$ and an open neighbourhood $V(x) \subset U(x)$ of x in X such that $y_x \in \Psi(z)$ for each $z \in V(x) \cap A$. Since A is compact, we find a finite open star-refinement $\mathcal{V} = \{V_1, \ldots, V_k\}$ of $\{V(x)\}_{x \in A}$, i.e., for every $i \in \{1, \ldots, k\}$, there is $\bar{x} \in A$ such that

$$\operatorname{st}(V_i, \mathcal{V}) := \bigcup \{V_j \in \mathcal{V} \mid V_j \cap V_i \neq \emptyset\} \subset V(\bar{x}).$$

Let $\{\lambda_i\}_{i=1}^k$ be a partition of unity subordinate to $\mathcal{V}$. Set $\Omega_\varepsilon := \bigcup_{i=1}^k V_i$. Define $f : \Omega_\varepsilon \to E$ by

$$f(x) := \sum_{i=1}^k \lambda_i(x) y_i,$$

where $x_i \in V_i \cap A$ and $y_i := y_{x_i}$. Of course, f is Lipschitz. Moreover, by the convexity of the values of Ψ, f is a selection of Ψ on A.

Let $x \in \bigcup_{i=1}^k V_i$. Since $\mathcal{V}$ is a star-refinement of $\{U(x)\}_{x \in A}$, there is $\bar{x} \in A$ such that $x, x_i \in U(\bar{x})$ for each $i \in \{1, \ldots, k\}$ with $x \in V_i$. Therefore, $y_i \in F(x_i) \subset F(\bar{x}) + (\varepsilon/2)B_1$, and since $F(\bar{x})$ is convex, $f(x) \in F(\bar{x}) + (\varepsilon/2)B_1$.

Hence, $f(x) \in F(B(x,\varepsilon)) + \varepsilon B_1$ and the proof is complete. ∎

The lemmas above allow us to prove the following result on equilibria.

PROPOSITION 3.8. *Let $K = \overline{\operatorname{Int} K} \subset \mathbb{R}^n$ be a Lefschetz set with $\chi(K) \neq 0$ such that the Clarke cone map $\Psi(\cdot) := C_{\widehat{K}}(\cdot)$ is l.s.c. on $\widehat{K}$. Assume that $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Marchaud map satisfying* (17) *on $\widehat{K} \cap \Omega$. Then F has an equilibrium in K.*

REMARK 3.9. It can be proved that these assumptions imply that K is viable under $-F$. However, here we relax the regularity assumption on K, compared with [3].

Proof. Let $\varepsilon > 0$ be given. By Lemma 3.7, there exist an open neighbourhood Ω_ε of ∂K in Ω and a Lipschitz map $f : \Omega_\varepsilon \to \mathbb{R}^n$ such that f is an ε-approximation of F on Ω_ε and a selection of Ψ on ∂K.

If $K \setminus \Omega_\varepsilon \neq \emptyset$, then for an open set V with $\partial K \subset V \subset \overline{V} \subset \Omega_\varepsilon$, we define an Urysohn function $v : \mathbb{R}^n \to [0,1]$ by

$$v(x) := \max\left\{0, 1 - \frac{1}{\operatorname{dist}(\overline{V}, K \setminus \Omega_\varepsilon)}\, d_{\overline{V}}(x)\right\}.$$

Now we can take any Lipschitz ε-approximation g of F on K, which exists according to [7], and join f to g by

$$f_\varepsilon(x) := v(x)f(x) + (1 - v(x))g(x).$$

In this way we have obtained a Lipschitz map which is equal to f on ∂K and $f_\varepsilon(x) \in \operatorname{conv} F(B(x,\varepsilon)) + \varepsilon B_1$ for every $x \in K \cup \Omega_\varepsilon$.

We show that K is invariant under $-f_\varepsilon$. First, let x be a solution for $-f_\varepsilon$ starting from an interior point of K. Suppose, on the contrary, that x leaves K at some $s > 0$. Then $y(\cdot) := x(s - \cdot)$ is a solution for f_ε starting from $x(s) \in \partial K$ and reaching the interior point of K at $s > 0$. But this is impossible since $\widehat{K}$ is invariant under f_ε. By continuity of the solution map S_{-f_ε} the set K is invariant under $-f_\varepsilon$.

Since ε was arbitrary, we can choose a sequence $\varepsilon_m \to 0$ and $f_m := f_{\varepsilon_m}$. Using Lemma 3.6, we can find for every $m \geq 1$ a point $x_m \in K$ with $0 = f_m(x_m)$. By standard arguments, since F has a closed graph, we pass to the limit as $m \to \infty$ obtaining an equilibrium of F in K. ■

COROLLARY 3.10. *Let $K = \overline{\operatorname{Int} K} \subset \mathbb{R}^n$ be a Lefschetz set with $\chi(K) \neq 0$ such that the map $\Psi(\cdot) := C_{\widehat{K}}(\cdot)$ is l.s.c. on $\widehat{K}$. Assume that $\Omega \subset \mathbb{R}^n$ is an open neighbourhood of K and $F : \Omega \multimap \mathbb{R}^n$ a Marchaud map satisfying*

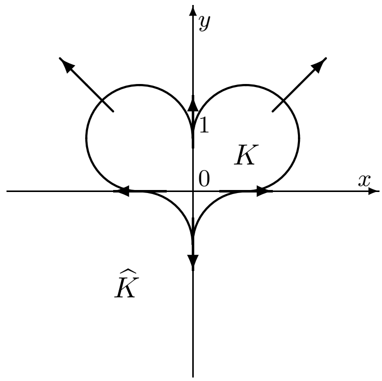
$$F(x) \cap \operatorname{Int} \Psi(x) \neq \emptyset \quad \text{for each } x \in \widehat{K} \cap \Omega. \tag{18}$$

Then F has an equilibrium in K.

Proof. It is easy to check that ocndition (18) implies (17). ■

EXAMPLE 3.11. Let

$$K_1 := \{(x,y) \in \mathbb{R}^2 \mid -1 \leq x \leq 0 \text{ and } \sqrt{1-(x+1)^2} - 1 \leq y \leq 1\},$$
$$K_2 := \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1 \text{ and } \sqrt{1-(x-1)^2} - 1 \leq y \leq 1\},$$
$$K := K_1 \cup K_2 \cup \operatorname{cl} B((1,1),1) \cup \operatorname{cl} B((-1,1),1).$$



Consider any Marchaud map $F : \mathbb{R}^2 \multimap \mathbb{R}^2$ with $x \in F(x)$ on ∂K. One can readily check that condition (17) is satisfied, as are the other assumptions of Theorem 3.8. Notice that neither K nor $\widehat{K}$ is an $\mathcal{L}$-retract, so the approach of [3] cannot be used.

Remark 3.12. Note that the class of sets such that the Clarke tangent cone map is l.s.c. contains the class of sleek sets, i.e., where the Bouligand cone map is l.s.c., so it contains all proximate retracts. One can easily see that this class is essentially larger. Note also that it is not included in the class of $\mathcal{L}$-retracts (the example above) and vice-versa (one can modify an example in [19, p. 219]).

Example 3.13. Consider the set K from Example 3.11 and the following problem:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), v(t)), \\ x(0) = x_0 \in K, \\ u, v \in U, \end{cases} \tag{19}$$

where $U = \operatorname{cl} B(0, r)$, $r < 1$. Assume that $f(x, u, v) = h(x, u) + g(x, v)$, where $h, g : \mathbb{R}^2 \to \mathbb{R}^2$ are continuous maps satisfying

$$h(x, u) = x + u \qquad \text{for every } x \in \partial K,$$
$$|g(x, v)| \le l|x|,\ l < r, \qquad \text{for each } x \in \partial K \text{ and } v \in U,$$

and $g(\{x\} \times U)$ is convex for every $x \in \mathbb{R}^n$.

Then there are $x_0 \in \mathbb{R}^n$ and controls u and v giving a stationary trajectory for (19). Indeed, one can check that $F(x) := f(\{x\} \times U \times U)$ satisfies the assumptions of Theorem 3.8.

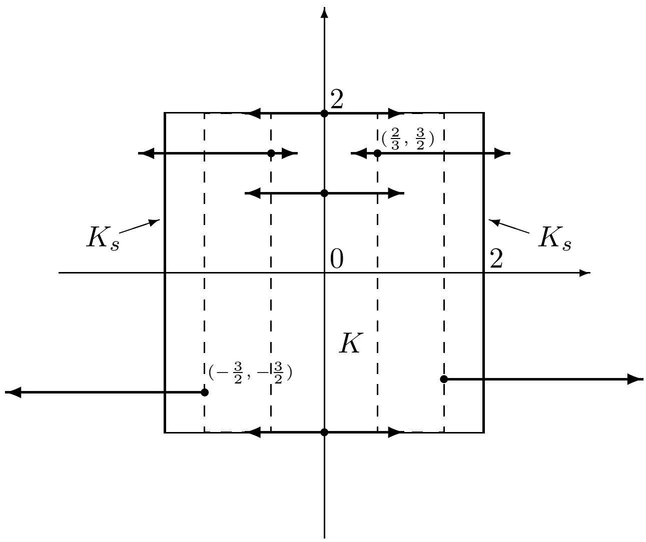
4. Strict equilibria. In this section we find several sufficient conditions for existence of strict equilibria in prescribed compact sets.

4.1. *Basic properties.* The notion of a strict equilibrium coincides with the one of ordinary equilibrium in the single-valued case while it brings important information for multivalued maps and differential inclusions. For instance, from each strict equilibrium of a Lipschitz map there starts only a stationary trajectory.

Some difficulties in finding strict equilibria are visible in the following example.

EXAMPLE 4.1. Consider the set $K := [-2, 2] \times [-2, 2]$ and the Lipschitz map $F : \mathbb{R}^2 \multimap \mathbb{R}^2$,

$$F(x, y) := [x - 1, x + 1] \times \{0\}.$$



It is easy to check that $K_s = K_e = (\{-2\} \times [-2, 2]) \cup (\{2\} \times [-2, 2])$, K_s is compact and disconnected with $\chi(K, K_s) = -1 \neq 0$. Notice that the behaviour of F on ∂K guarantees the existence of equilibria. Nevertheless, none of them is strict.

One of possible methods of finding strict equilibria is to study single-valued selections of the multivalued map, as in the following general observation.

REMARK 4.2. Let $K \subset X$ be a compact set and $F : K \multimap \mathbb{R}^n$ a multivalued map. Assume that there is a single-valued selection $f : K \to \mathbb{R}^n$ of F with

$$\forall x \in K : \quad f(x) = 0 \;\Rightarrow\; F(x) = \{0\}. \tag{20}$$

Then, obviously, each equilibrium of f is a strict equilibrium of F.

However, it is not obvious how to find, for a given multivalued map, a selection satisfying (20) and simultaneously having an equilibrium.

Below we give several conditions implying the existence of such a selection.

PROPOSITION 4.3. *Let $K \subset \mathbb{R}^n$ be a compact set, Ω an open neighbourhood of K in $\mathbb{R}^n$ and $F : \Omega \multimap \mathbb{R}^n$ a Lipschitz map satisfying* (3)*. Assume that the map $f := \sigma(F(\cdot))$, where $\sigma(F(x))$ is a Steiner point of $F(x)$, satisfies assumptions guaranteeing the existence of equilibria of f. Then there exists a strict equilibrium of F in K.*

Proof. Notice that f is a Lipschitz selection of F satisfying (20). ■

Below we give sufficient conditions for (3) to hold.

COROLLARY 4.4. *If instead of* (3) *we assume that*

$$F(x) \cap -F(x) \subset \{0\} \quad \text{for every } x \in K, \tag{21}$$

(*this can be expressed by saying that no value of F contains opposite directions*) *or*

(22) *there are a map $\gamma : K \to \partial B_1$ and a constant $c > 0$ such that $\langle \gamma(x), y\rangle \geq c|y|$ for every $x \in K$ and $y \in F(x)$*

(*that is*, *there is a guiding function for F*), *then there exists a strict equilibrium of F in K.*

We also list some sufficient conditions guaranteeing the existence of equilibria for all possible continuous (or Lipschitz) selections of F.

EXAMPLE 4.5. (1) Let K and F satisfy the assumptions of Proposition 3.4 or Corollary 3.2. Then each Lipschitz selection of F has an equilibrium in K.

(2) Let K be a closed ball in $\mathbb{R}^n$ and $F : K \multimap \mathbb{R}^n$ be of the form $F(x) = x - \varphi(x)$, where φ is a continuous, compact convex valued map satisfying

$$x \notin \lambda\varphi(x) \quad \text{for each } x \in \partial K \text{ and } 0 < \lambda < 1. \tag{23}$$

Then each continuous selection (existing due to Michael's selection theorem) has an equilibrium in K.

(3) Let $K = \overline{\operatorname{Int} K}$ be compact and $F : K \multimap \mathbb{R}^n$ a Lipschitz, compact convex valued map with $0 \notin F(\partial K)$. If $\operatorname{Deg}(F, \operatorname{Int} K) \neq 0$, then each continuous selection has an equilibrium in K.

The second statement follows easily from the Nonlinear Alternative applied to any continuous selection of φ (see e.g. [12]). The third one is an immediate consequence of a construction of the topological degree for compact convex valued u.s.c. maps (see e.g. [8]).

As a consequence, the following fixed point result can be obtained.

Corollary 4.6. *Let K be a closed ball in $\mathbb{R}^n$ and $\varphi : K \multimap \mathbb{R}^n$ a Lipschitz map with compact strictly convex values which satisfies* (23) *and*

$$x \notin \varphi(x) \setminus \operatorname{ext} \varphi(x) \quad \text{for every } x \in K, \tag{24}$$

where $\operatorname{ext} \varphi(x)$ *stands for the set of extremal points of* $\varphi(x)$. *Then there is a point $x \in K$ such that $\varphi(x) = \{x\}$.*

Proof. Note that the map $F(\cdot) := \cdot - \varphi(\cdot)$ satisfies (21) and use Example 4.5(2). ∎

4.2. *Strict equilibria on smooth sets.* In the present subsection, under some smoothness assumptions on K, we obtain strict equilibria of a Lipschitz map F without looking for a Lipschitz selection of F satisfying (20). Moreover, although the existence of a strict equilibrium implies the existence of equilibria of each selection, in the situation presented below ($K_s \neq K_e$) it would not be easy to check it for any of them.

This subsection is devoted to the proof of the second main result of our paper, Theorem 1.2. Note that the assumption $0 \notin F(x)$ on ∂K is essential. To see this, consider the following.

Example 4.7. Let $K := [0, 1] \subset \mathbb{R}$ and $F(x) = [-1, 0]$. Then $K_s = \emptyset$, so $\chi(K, K_s) = 1 \neq 0$. Obviously, conditions (I), (III) and (3) are satisfied while there is no strict equilibrium of F in K.

In what follows we proceed analogously to Subsection 3.1.

Proof of Theorem 1.2. We modify the proof of Lemma 3.3 and find first an open set $U \subset \mathbb{R}^n$ with $K \setminus K_s(F) \subset U$, $K_s(F) \cap U = \emptyset$ and a Lipschitz map $g : U \to \mathbb{R}^n$ satisfying (A), (C), (D) and

(E) for every $x \in \operatorname{Int} K$, if $g(x) = 0$, then $\sigma(F(x)) = 0$.

To do this, we repeat Step 1, Case 1 and Case 2 of the proof of Lemma 3.3, obtaining $v_x(y) := v_x$ for $y \in U_x$ in both cases.

In Case 3, that is, for $x \in \operatorname{Int} K$ with $0 \in F(x)$, take an open neighbourhood $U_x \subset B(x, \varepsilon)$ in $\mathbb{R}^n$ such that $U_x \cap \partial K = \emptyset$, and for each $y \in U_x$, put $v_x(y) := \sigma(F(y))$. Then, obviously, $v_x(y) \in F(y) \subset F(y) + d_{K_s(F)}(y) B_1$ for every $y \in U_x$.

Choosing a countable, locally finite covering $\{U_{x_i} \mid x_i \in K \setminus K_s(F)\}$ of $K \setminus K_s(F)$ and a subordinate smooth partition of unity $\{\lambda_i\}$, we can define $g : U = \bigcup_{i=1}^{\infty} U_{x_i} \to \mathbb{R}^n$ by

$$g(x) := \sum_{i \in I(x)} \lambda_i(x) v_{x_i}(x).$$

Since g is Lipschitz and satisfies (A), (C), (D), we have to show only (E).

We claim that for every $x \in \operatorname{Int} K$, if $g(x) = 0$, then $0 \in F(x_i)$ for each $i \in I(x)$ and $x_i \in \operatorname{Int} K$.

Indeed, otherwise there is $i \in I(x)$ with $0 \notin F(x_i)$. From the construction of U_{x_i} it follows that $0 \notin F(x)$. Therefore,

$$0 = g(x) \in F(x) + \tfrac{1}{2} d_{F(x)}(0) \not\ni 0;$$

a contradiction. Since there are no equilibria of F on ∂K, we have $x_i \in \operatorname{Int} K$.

Since $0 \in F(x_i)$ and $x_i \in \operatorname{Int} K$, by the construction of $v_{x_i}(\cdot)$ one has $v_{x_i} = \sigma(F(x))$ for every $i \in I(x)$. Hence, $g(x) = \sigma(F(x))$.

Now, following Step 2 of the proof of Lemma 3.3, we construct a suitable map f satisfying (a)–(d). Moreover, from the construction of U_{x_i} it follows that if $f(x) = 0$, then $\beta_i(x) = 0$ for every $i \geq 1$, which implies that

$$0 = f(x) = g(x) = \sigma(F(x)). \tag{25}$$

Applying Proposition 3.1 for f we find an equilibrium $x \in \operatorname{Int} K$ of f. By (25) and (3), it is a strict equilibrium of F; the proof is complete. ∎

5. Appendix. Below we give conditions, in terms of tangent cones, allowing us to check when a point $x \in \partial K$ belongs to K_s. As a consequence, we obtain a sufficient condition for K_s to be closed. We will use the following notation (cf. [18]):

$$K_o := \{x \in \partial K \mid F(x) \cap T_K(x) = \emptyset\}, \qquad D_K(x) := \mathbb{R}^n \setminus T_{\overline{\mathbb{R}^n \setminus K}}(x).$$

The relation between K_s and K_o is given in the following.

Lemma 5.1. *For any Marchaud map*,

$$K_o \subset K_s \subset \overline{K}_o.$$

Proof. If a point belongs to K_o, then any solution starting from this point leaves K immediately, otherwise the tangential condition (2) would be satisfied. So, $K_o \subset K_s$.

Let us now show that $K_s \subset \overline{K}_o$. The set $K \setminus \overline{K}_o$ is locally compact, and the tangential condition (2) is everywhere satisfied from the very definition of K_o. The viability theorem (cf. [1, Theorem 3.3.2]) states that for any point of $K \setminus \overline{K}_o$, there exists at least one solution starting from this point, which remains in K on some $[0, \tau], \tau > 0$, i.e., the point is not in K_s. This proves that if a point does not belong to $\overline{K}_o$, then it does not belong to K_s, i.e., $K_s \subset \overline{K}_o$. ∎

Unfortunately, the set K_o is seldom closed. We will now study what happens for points of $\overline{K}_o \setminus K_o$. A proof of the lemma below is given for the convenience of the reader because it has not been published anywhere yet.

Lemma 5.2 ([5, Proposition 3.1] or [6, Proposition 2.1]). *Let K be closed and F be a Marchaud map locally Lipschitz around $x \in \overline{K}_o \setminus K_o$.*

(1) *If* $F(x) \cap (D_K(x) \cup T_{\partial K \setminus K_o}(x)) = \emptyset$, *then* $x \in K_s$.
(2) *If* $F(x) \cap D_K(x) \neq \emptyset$ *or* $F(x) \cap T_{K_o}(x) = \emptyset$, *then* $x \notin K_s$.

Proof. To prove (1) assume for a while that $x \notin K_s$, i.e., that there is $x(\cdot) \in S_F(x)$ viable in K on $[0, \varepsilon]$ with $\varepsilon > 0$.

If there exists a sequence $t_m > 0$ converging to 0 such that $x(t_m) \in \partial K$ for any m, then, up to a subsequence, and because $x(t_m) \notin K_o$ from the viability theorem, we have

$$\frac{x(t_m) - x}{t_m} \to v \in F(x) \cap T_{\partial K \setminus K_o}(x),$$

which is impossible.

So, we can find $\alpha > 0$ such that $x(s) \in \operatorname{Int} K$ for $0 < s \leq \alpha$. Recall that $x \notin K_o$ means that $F(x) \cap T_K(x) \neq \emptyset$. On the other hand, from the assumption, $F(x) \cap D_K(x) = \emptyset$. So, by (15), there is $v \in F(x) \cap T_{\partial K}(x)$. Since $F(x) \cap T_{\partial K \setminus K_o}(x) = \emptyset$, one can find $h_m \to 0$ and $v_m \to v$ such that $x + h_m v_m \in K_o$. According to Filippov's theorem (see e.g. [1]), there exist $x_m(\cdot) \in S_F(x + h_m v_m)$ converging to $x(\cdot)$ in the topology of uniform convergence. Moreover, $x_m(\cdot)$ leaves K immediately for any m because $x + h_m v_m \in K_o \subset K_s$. Since $x(\alpha) \in \operatorname{Int} K$, also $x_m(\alpha) \in \operatorname{Int} K$ for m large enough, and there is $t_m \in (0, \alpha)$ such that $x_m(\cdot)$ is viable on $[t_m, \alpha]$. We choose t_m such that $x_m(t_m) \in \partial K$, and in fact $x_m(t_m) \in \partial K \setminus K_o$. Since $x(s) \in \operatorname{Int} K$ for $0 < s \leq \alpha$, the sequence t_m converges to 0.

Let us now remark that from the convexity of $F(x)$, up to a subsequence,

$$\frac{x_m(t_m) - (x + h_m v_m)}{t_m} \to w \in F(x). \tag{26}$$

Indeed, for any $\varepsilon > 0$, there exists a neighbourhood V of x on which $F(y) \subset F(x) + \varepsilon B_1$, because F is upper semicontinuous. For m large enough, $x_m(\cdot)$ remains in V on $[0, t_m]$, and we have

$$x_m(t_m) - (x + h_m v_m) \in t_m(F(x) + \varepsilon B_1),$$

which proves (26).

Now we show that $\frac{x_m(t_m) - x}{t_m + h_m}$ converges, up to a subsequence, to a point of $F(x) \cap T_{\partial K \setminus K_o}(x)$, which contradicts our assumption.

To this end, notice that

$$\frac{x_m(t_m) - x}{h_m + t_m} = \frac{t_m}{t_m + h_m} \, \frac{x_m(t_m) - (x + h_m v_m)}{t_m} + \frac{h_m}{t_m + h_m} \, v_m$$

which converges, up to a subsequence, to $\lambda w + (1 - \lambda) v$, where $\lambda \in [0, 1]$. We conclude thanks to the convexity of $F(x)$.

To prove (2), we have to show that there is $x(\cdot) \in S_F(x)$ which remains in K on some $[0, \varepsilon]$ with $\varepsilon > 0$.

If $F(x) \cap D_K(x) \neq \emptyset$, Filippov's Theorem shows that there exists a solution in $S_F(x)$ locally viable in K.

If not, since $x \notin K_o$, the set $F(x) \cap T_{\partial K}(x)$ is not empty. Since $F(x) \cap T_{K_o}(x) = \emptyset$, there exists $v \in F(x) \cap T_{\partial K \setminus K_o}(x)$. Let $v_m \to v$ and $h_m \to 0$ be such that $x + h_m v_m \in \partial K \setminus \overline{K}_s$ (recall that $\overline{K}_s = \overline{K}_o$). Therefore there exist $x_m(\cdot) \in S_F(x + h_m v_m)$ viable in K on $[0, t_m]$, where $t_m \in (0, \infty]$. We choose $x_m(\cdot)$ and t_m in such a way that

$$\begin{aligned} t_m &= \sup\{\inf\{t > 0 \mid x(t) \notin K\} \mid x(\cdot) \in S_F(x + h_m v_m)\} \\ &= \inf\{t > 0 \mid x_m(t) \notin K\}. \end{aligned}$$

Since F is a Marchaud map, such t_m and x_m exist (see [1, p. 135]). It is seen that if $t_m < \infty$, then $x_m(t_m) \in K_s$.

Assume that a subsequence of $\{t_m\}$, called $\{t_m\}$ again, converges to 0. Then, as in the first part of the proof, we can show that a subsequence of $\frac{x_m(t_m)-x}{t_m+h_m}$ converges to an element of $F(x) \cap T_{K_s}(x)$, i.e., to an element of $F(x) \cap T_{K_o}(x)$. This is impossible.

So, there exists $\varepsilon > 0$ such that each $x_m(\cdot)$ is viable in K on $[0, \varepsilon]$. A subsequence of $\{x_m(\cdot)\}$ converges to an element $x(\cdot) \in S_F(x)$ which remains in K on $[0, \varepsilon]$, i.e., x does not belong to K_s. This ends the proof. ∎

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