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# Two remarks on non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$

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**Abstract.** We present some estimates on the geometry of the exceptional value sets of non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$ and their components.

**1. Introduction.** Recall that the *exceptional value set* $E_h$ of a polynomial mapping $h : \mathbb{C}^m \to \mathbb{C}^n$ is the smallest subset $E_h \subset \mathbb{C}^n$ such that the restriction $h : h^{-1}(\mathbb{C}^n \setminus E_h) \to \mathbb{C}^n \setminus E_h$ gives a locally trivial smooth fibration. The mysterious Jacobian conjecture (see [BCW] and [E]), posed by Keller in 1939 and still open even in the two-dimensional case, asserts that a polynomial map of $\mathbb{C}^n$ with non-zero constant Jacobian must be a polynomial bijection. Consider a polynomial map $f = (P, Q) : \mathbb{C}^2_{(x,y)} \to \mathbb{C}^2_{(u,v)}$ and define $J(P, Q) := P_x Q_y - P_y Q_x$. It is well known that if $J(P, Q) \equiv \text{const} \neq 0$ but $f$ is not bijective, then the sets $E_P$ and $E_Q$ must be non-empty finite sets and $E_f$ is a curve so that each of its irreducible components is a polynomially parameterized curve (the image of a polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$) (see, for example, [J1, J2]).

In this note we prove the following.

THEOREM 1. *Suppose $f = (P, Q)$ is a polynomial map with non-zero constant Jacobian. Then $u_0 \in E_P$ if and only if the line $u = u_0$ is tangent to an irreducible local branch of $E_f$.*

Here, by saying that a line $L$ is tangent to an irreducible local branch curve $\gamma$ we mean that the intersection multiplicity of $L$ and $\gamma$ at their common point is larger than 1, or equivalently, either the common point is a singular point of $\gamma$ or $L$ is the line tangent to $\gamma$ at this point.

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This theorem leads to the following fact that may be used to consider the non-existence of non-zero constant Jacobian polynomial maps with exceptional value curve of a given type.

THEOREM 2. *The exceptional value set of a polynomial map of* $\mathbb{C}^2$ *with non-zero constant Jacobian cannot be isomorphic to a curve composed of the images of polynomial maps of the form* $t \mapsto (t^k, q(t))$, $k \in \mathbb{N}$, $q(t) \in \mathbb{C}[t]$.

A special property of a curve with irreducible components given by parameterizations in Theorem 2 is that its irreducible local branches may be tangent to the line $u = 0$ only. Simply connected curves are simple examples of such curves. This can be easily deduced from Lin–Zaĭdenberg's theorem on simply connected curves (Theorem B in [ZL]).

COROLLARY 3. *The exceptional value set of a polynomial map of* $\mathbb{C}^2$ *with non-zero constant Jacobian cannot be a simply connected curve.*

Proofs of Theorems 1 and 2 will be given in Sections 3 and 4.

**2. Preliminaries.** In this section, we present some elementary facts which will be used in the proof of Theorem 1.

(i) We will work with finite fractional power series with parameter $\xi$ of the form

$$\varphi(x,\xi) = \sum_{k=1}^{K-1} a_k x^{n_k/m} + \xi x^{n_K/m}, \qquad 0 \neq a_k \in \mathbb{C},\ n_k \in \mathbb{Z},\ m \in \mathbb{N}, \tag{1}$$

with $n_1 > \ldots > n_K$ and $\gcd(\{n_k : k \le K\}) = 1$.

Let $h(x,y)$ be a non-constant primitive polynomial monic in $y$ with $\deg_y h = \deg h$. A series $\varphi(x,\xi)$ in (1) is called a *Newton–Puiseux type* of $h$ if

$$h(x,\varphi(x,\xi)) = h_\varphi(\xi) + \text{lower terms in } x, \qquad h_\varphi \in \mathbb{C}[\xi], \deg h_\varphi > 0.$$

Define $\operatorname{mult}(\varphi) := m$ and $\operatorname{ord}(\varphi) := n_K$.

The Newton–Puiseux types of $h(x,y)$ can be constructed from the Newton–Puiseux expansions at infinity of the curve $h(x,y) = 0$. In fact, if $y = y(x)$ is such an expansion, then there is a unique Newton–Puiseux type $\varphi$ of $h$ and a number $\xi_0 \in \mathbb{C}$ such that $y(x) = \varphi(x,\xi_0) + \text{lower terms in } x$.

For a Newton–Puiseux type $\varphi$ of $h$ the rational map $\Phi : \mathbb{C} \times \mathbb{C} \to \mathbb{CP}^2$ defined by

$$\Phi(t,\xi) := (t^{-\operatorname{mult}(\varphi)}, \varphi(t^{-\operatorname{mult}(\varphi)}, \xi)) \tag{2}$$

determines an unbranched $i_\varphi$-sheeted covering from $\mathbb{C}^* \times \mathbb{C}$ onto $\Phi(\mathbb{C}^* \times \mathbb{C}) \subset \mathbb{C}^2$, where $i_\varphi := \operatorname{mult}(\varphi)/\gcd\{n_k : k = 1, \ldots, K-1\}$. Hence, one can use the polynomial $H_\varphi(t,\xi) := h \circ \Phi(t,\xi)$ as a kind of "extension" of $h(x,y)$.

Lemma 4 (see [C, Theorem 2.4]). *Suppose $h(x,y)$ is a primitive polynomial monic in $y$. Then $c \in E_h$ if and only if $c$ is either a critical value of $h$ or a critical value of $h_\varphi(\xi)$ for any Newton–Puiseux type $\varphi$ of $h$.*

In fact, by Newton's theorem the polynomial $h(x,y)-c$ can be factorized as

$$h(x,y) - c = C\prod_u (y - u(x)), \tag{3}$$

where the product runs over all Newton–Puiseux expansions at infinity of the curve $h = c$. Substituting $y = \varphi(x,\xi)$ into this representation, one can see that a number $\alpha \in \mathbb{C}$ is a zero of $h_\varphi(\xi) - c = 0$ with multiplicity $n$ if and only if the level $h = c$ has exactly $n$ Newton–Puiseux expansions at infinity of the form $\varphi(x,\alpha) +$ lower terms in $x$. Thus, Puiseux data at infinity of the curves $h = c$ must be changed when $c$ is a critical value of $h_\varphi$.

(ii) Consider a polynomial map $f = (P,Q) : \mathbb{C}^2 \to \mathbb{C}^2$. By definition the exceptional value set $E_f$ is composed of the critical value set of $f$ and the so-called *non-proper value set* $A_f$ of $f$, the set of all values $a \in \mathbb{C}^2$ such that there exists a sequence $\mathbb{C}^2 \ni p_i \to \infty$ with $f(p_i) \to a$. By [J1], the non-proper value set $A_f$, if non-empty, is a curve composed of the images of some polynomial maps from $\mathbb{C}$ into $\mathbb{C}^2$.

A series $\varphi(x,\xi)$ in (1) is called a *dicritical series* of $f$ if

$$f(x,\varphi(x,\xi)) = f_\varphi(\xi) + \text{lower terms in } x, \qquad \deg f_\varphi > 0.$$

Lemma 5.

$$A_f = \bigcup_{\varphi \textit{ is a dicritical series of } f} f_\varphi(\mathbb{C}).$$

*Proof.* Let $\varphi$ be a dicritical series of $f$ and $\Phi(t,\xi)$ be as in (2). The map $\Phi$ sends $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^2$ and the line $\{0\} \times \mathbb{C}$ to the line at infinity of $\mathbb{CP}^2$. Then the polynomial map $F_\varphi(t,\xi) := f \circ \Phi(t,\xi)$ maps the line $\{0\} \times \mathbb{C}$ into $A_f \subset \mathbb{C}^2$. Therefore, $f_\varphi(\mathbb{C})$ is an irreducible component of $A_f$, since $\deg f_\varphi > 0$.

Conversely, assume that $\ell$ is an irreducible component of $A_f$. By definition we can choose a smooth point $(u_0,v_0)$ of $\ell$ and an irreducible branch at infinity $\gamma$ of the curve $P = u_0$ (or $Q = v_0$) whose image $f(\gamma)$ is a branch curve intersecting $\ell$ transversally at $(u_0,v_0)$. Let $\varphi(x,\xi)$ be the Newton–Puiseux type of $P$ corresponding to a Newton–Puiseux expansion at infinity of $\gamma$. Then by definitions we can verify that $\varphi$ is a dicritical series of $f$ and $f_\varphi(\mathbb{C}) = \ell$. ∎

**3. Proof of Theorem 1.** We consider a polynomial map $f = (P,Q) : \mathbb{C}^2 \to \mathbb{C}^2$ with non-zero constant Jacobian. Fix affine coordinates $(x,y)$ so that $P$ and $Q$ are monic in $y$. For a series $\varphi$ in (1) write

$$P(x,\varphi(x,\xi)) = p_\varphi(\xi)x^{a_\varphi/\mathrm{mult}(\varphi)} + \text{lower terms in } x,$$
$$Q(x,\varphi(x,\xi)) = q_\varphi(\xi)x^{b_\varphi/\mathrm{mult}(\varphi)} + \text{lower terms in } x,$$

where $p_\varphi \neq 0$ and $q_\varphi \neq 0$.

LEMMA 6. (i) *Let $\varphi$ be a Newton–Puiseux type of $P$. If $\varphi$ is not a dicritical series of $f$, then* $\deg p_\varphi(\xi) = 1$, $q_\varphi(\xi) \equiv \mathrm{const} \neq 0$ *and* $b_\varphi > 0$.

(ii) *A dicritical series of $f$ must be a Newton–Puiseux type of both $P$ and $Q$.*

*Proof.* (i) First, we will show that

$$(*) \qquad \deg p_\varphi = 1, \quad q_\varphi \equiv \mathrm{const} \neq 0.$$

Differentiating $F(t^{-\mathrm{mult}(\varphi)}, \varphi(t^{-\mathrm{mult}(\varphi)}, \xi))$, since $a_\varphi = 0$ and $b_\varphi \neq 0$ we have

$$\mathrm{mult}(\varphi)J(P,Q)t^{\mathrm{ord}(\varphi)-\mathrm{mult}(\varphi)-1} = -b_\varphi \dot{p}_\varphi q_\varphi t^{-b_\varphi-1} + \text{higher terms in } t.$$

Since $J(P,Q) \equiv \mathrm{const} \neq 0$ and $\deg p_\varphi > 0$, we get $(*)$.

Now, assume to the contrary that $b_\varphi < 0$. Then there exists a Newton–Puiseux root at infinity $u(x)$ of the curve $Q = 0$ such that $u(x) = \varphi(x,\xi_0) +$ lower terms in $x$. Let $\psi(x,\xi)$ be the Newton–Puiseux type of $Q$ corresponding to $u(x)$. Obviously, $a_\psi > a_\varphi = 0$ and hence $\psi$ is not a dicritical series of $f$. Furthermore, $\varphi(x,\xi) = \psi(x,\alpha) +$ lower terms in $x$ for a zero $\alpha$ of $p_\psi(\xi)$. This is impossible, since $p_\psi(\xi) \equiv \mathrm{const} \neq 0$ by applying $(*)$ to the Newton–Puiseux type $\psi$ of $Q$. Thus, we get $b_\varphi > 0$.

(ii) This is obtained from (i) and the definitions. ∎

*Proof of Theorem 1.* If $E_f = \emptyset$, then $f$ is bijective and $E_P = E_Q = \emptyset$. Hence, we need only consider the situation when $E_f \neq \emptyset$. In this situation $E_f = A_f$, since $f$ has no singularity.

First, suppose the line $u = u_0$ is tangent to an irreducible local branch curve of an irreducible component $\ell$ of $A_f$. By Lemma 5, there is a dicritical series $\varphi$ of $f$ such that $\ell$ is the image of $f_\varphi := (p_\varphi, q_\varphi)$. By Lemma 6(ii) the series $\varphi$ is a Newton–Puiseux type of both $P$ and $Q$. Since the line $u = u_0$ is tangent to $\ell$, $u_0$ must be a critical value of $p_\varphi$. Hence, by Lemma 4 the number $u_0$ is an exceptional value of $P$.

Conversely, let $u_0 \in \mathbb{C}$ and $L := \{u = u_0\}$. Assume that $L$ intersects transversally each of the irreducible local branches of $A_f$ located at points in $L \cap A_f$. We have to show that $u_0$ cannot be an exceptional value of $P$. In view of Lemmas 4 and 6 we only need to verify that $u_0$ is a regular value of $p_\varphi(\xi)$ for every dicritical series $\varphi$ of $f$.

Let $\varphi$ be such a series and $\ell := f_\varphi(\mathbb{C})$, which is an irreducible component of $A_f$ by Lemma 5. Let $(u_0, v_0) \in \ell$ and $\xi_0 \in \mathbb{C}$ with $f_\varphi(\xi_0) = (u_0, v_0)$. We have to show that $\dot{p}_\varphi(\xi_0) \neq 0$.

Consider the polynomial map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^2$ given by

$$F(t,\xi) := f \circ (t^{-\mathrm{mult}(\varphi)}, \varphi(t^{-\mathrm{mult}(\varphi)}, \xi)) = (p_\varphi, q_\varphi)(\xi) + \text{higher terms in } t.$$

For this map $F(\{0\} \times \mathbb{C}) = \ell$ and

$$\det DF(t,\xi) = -\mathrm{mult}(\varphi) J(P,Q) t^{\mathrm{ord}(\varphi)-\mathrm{mult}(\varphi)-1}.$$

Since $J(P,Q) \equiv \text{const} \neq 0$, $F$ has a singularity on the line $t = 0$ only.

Let $\gamma := f_\varphi(\{\xi : |\xi - \xi_0| < \varepsilon\}$ for some $\varepsilon > 0$ small enough. By assumption the line $L$ intersects $\gamma$ transversally at $(u_0, v_0)$. So, we can choose a sufficiently small neighborhood $U$ of $(u_0, v_0)$ so that $\gamma := \ell \cap U$ is a smooth branch curve parameterized by $v = v_0 + h(u - u_0)$ for a homeomorphic function $h$, $h(0) = 0$. Define new coordinates $(\overline{u}, \overline{v}) = (u - u_0, v - v_0 - h(u - u_0))$ in $U$ and $(\overline{t}, \overline{\xi}) = (t, \xi - \xi_0)$ in a sufficiently small neighborhood $V$ of $(0, \xi_0)$. Let $\overline{F} = (\overline{F}_1, \overline{F}_2)$ be the representation of $F$ in these coordinates. Then

$$\overline{F}_1(\overline{t}, \overline{\xi}) = p_\varphi(\overline{\xi}) - u_0 + \text{higher terms in } \overline{t}, \tag{4}$$

$\overline{F}(0,0) = (0,0)$, $\overline{F}(\{\overline{t} = 0\}) \subset \gamma = \{\overline{v} = 0\}$ and $\det D\overline{F}(\overline{t}, \overline{\xi}) \neq 0$ for $\overline{t} \neq 0$. Then, by examining the Newton diagrams of $\overline{F}_1$, $\overline{F}_2$ and $\det D\overline{F}$ we can verify that

$$\overline{F}(\overline{t}, \overline{\xi}) = (\overline{\xi} u_1(\overline{t}, \overline{\xi}) + \overline{t} u_2(\overline{t}, \overline{\xi}), \overline{t}^k v_1(\overline{t}, \overline{\xi})), \tag{5}$$

where $u_1$, $u_2$ and $v_1$ are homeomorphic functions defined in $V$, $u_1(0,0) \neq 0$ and $v_1(0,0) \neq 0$ (see, for example, [O, Lemma 4.1]). From (4) and (5) it follows that $\dot{p}_\varphi(\xi_0) \neq 0$. ■

Remark. From Lemmas 6(ii) and 5 one can easily see that the exceptional value set $E_f$ of a non-singular polynomial map $f$ cannot contain an irreducible component isomorphic to a line.

**4. Proof of Theorem 2.** Let $f = (P, Q)$ be the representation of $f$ in a coordinate system in which $E_f$ consists of the images of some polynomial maps of the form $t \mapsto (t^k, q(t))$, $k \in \mathbb{N}$. By applying Theorem 1 we find that $E_P \subset \{0\}$. As $f$ has no singularity, $P$ is a non-singular primitive polynomial. Then from Suzuki's equality

$$\sum_{c \in \mathbb{C}} (\chi_c - \chi) = 1 - \chi$$

(see [S]) we get $\chi_0 = 1$. Here, $\chi_c$ and $\chi$ indicate the Euler–Poincaré characteristics of the fiber $P = c$ and of the generic fiber of $P$, respectively. Since the curve $P = 0$ is smooth, it has one connected component $\ell$ diffeomorphic to $\mathbb{C}$. This component $\ell$ must be isomorphic to $\mathbb{C}$ by the Abhyankar–Moh Theorem [AM] and the restriction of $f$ to $\ell$ must be injective. Then, as observed by Gwoździewicz [G], $f$ must be bijective. This is impossible, since $E_f \neq \emptyset$. ■

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