ANNALES
POLONICI MATHEMATICI
82.1 (2003)

On the Euler characteristic of the real Milnor fibres of an analytic function

by Piotr Dudziński (Gdańsk)

Abstract. The paper is concerned with the relations between real and complex topological invariants of germs of real-analytic functions. We give a formula for the Euler characteristic of the real Milnor fibres of a real-analytic germ in terms of the Milnor numbers of appropriate functions.

Introduction. In [3], McCrory and Parusiński proved that if $f:\mathbb{R}^n,0\to\mathbb{R},0$ is a germ of an analytic function, then the difference (and sum) of the Euler characteristics mod 4 of the real Milnor fibres of f over $+\delta$ and $-\delta$ can be expressed in terms of the dimensions of the generalized eigenspaces of the algebraic monodromy.

In this paper we shall prove an analogous formula for $\mathcal{A}_d$-germs. The notion of an $\mathcal{A}_d$-germ was introduced by Szafraniec in [5] as a generalization of a germ defined by a weighted homogeneous polynomial (the papers [2, 5] are concerned with topological invariants of $\mathcal{A}_d$-germs and generalize Wall's result [7]). In the case of $\mathcal{A}_d$-germs, we obtain another description of the real Milnor fibres, without using the complex monodromy. Instead we prove that the sum of the Euler characteristics mod 4 of the real Milnor fibres over $\pm\delta$ can be expressed in terms of the Euler characteristics of the Milnor fibres of appropriate restrictions of $f_{\mathbb{C}}$, where $f_{\mathbb{C}}$ denotes the complexification of f (Theorem 2). These characteristics, in turn, can be effectively calculated if 0 is an isolated critical point of $f_{\mathbb{C}}$, although the formula also holds in the non-isolated case.

In the first section we study the action of the dihedral group on the Milnor fibre of an analytic function. Theorem 1 describes the relation between the real and complex invariants of a real-analytic germ and it is the main tool we use in the proof of Theorem 2.

2000 *Mathematics Subject Classification*: Primary 14P15.
Key words and phrases: Euler characteristic, Milnor fibre.

1. Action of the dihedral group on the Milnor fibre of an analytic function. Let $d, w_1, \ldots, w_n$ be positive integers. For every $\lambda \in \mathbb{C}$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we shall write $\lambda.z = (\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n)$. Write $d = p^u v$, where p, u, v are positive integers such that p is prime, v is odd and prime to p. We may assume that $w_k \equiv 0 \bmod p$ if and only if $k \le m = m(p)$ for some integer $m \le n$.

Assume that $m < n$, i.e. some w_k is not divisible by p. Set $\eta = \exp(\pi i/p^u)$ and $\varepsilon = \eta^2$. For $j = 0, 1, \ldots, p^u - 1$ and $z \in \mathbb{C}^n$ we define $j(z) = \varepsilon^j.z = (\varepsilon^{jw_1} z_1, \ldots, \varepsilon^{jw_n} z_n)$.

If $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ is a germ of a real-analytic function, denote by $f_{\mathbb{C}} : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ its complexification, by $F(f)$ the *Milnor fibre* of $f_{\mathbb{C}}$, and by $F_{\mathbb{R}}(f)$ the *real Milnor fibre* of f, i.e. $F(f) = f_{\mathbb{C}}^{-1}(\xi) \cap B_r^{2n}$ and $F_{\mathbb{R}}(f) = f^{-1}(\xi) \cap B_r^n$, where $0 < \xi \ll r \ll 1$ and B_r^{2n} (resp. B_r^n) denotes the ball of radius r centred at the origin in $\mathbb{R}^{2n}$ (resp. $\mathbb{R}^n$). Let $\widetilde{f}$ denote the restriction of f to $\mathbb{R}^m \times \{0\} \subset \mathbb{R}^n$. Recall that we do not assume that f and g have an isolated singularity at the origin.

THEOREM 1. *If $f, g : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ are analytic such that*

$$f_{\mathbb{C}}(\varepsilon^j.z) = f_{\mathbb{C}}(z), \qquad g_{\mathbb{C}}(\varepsilon^j.z) = g_{\mathbb{C}}(z) \tag{1}$$

for $z \in \mathbb{C}^n$, $j \in \mathbb{Z}$, and

$$f_{\mathbb{C}}(\eta^j.x) = \begin{cases} f(x) & \text{if } j \text{ is even,} \\ g(x) & \text{if } j \text{ is odd,} \end{cases} \tag{2}$$

for $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, then

$$\chi(F(f)) - \chi(F(\widetilde{f})) \equiv a_+\chi(F_{\mathbb{R}}(f)) + a_-\chi(F_{\mathbb{R}}(g)) - p(\widetilde{a}_+\chi(F_{\mathbb{R}}(\widetilde{f})) + \widetilde{a}_-\chi(F_{\mathbb{R}}(\widetilde{g}))) \bmod 2p^u$$

where a_+/a_- (resp. $\widetilde{a}_+/\widetilde{a}_-$) denote the number of even/odd integers j such that $0 \le j \le p^u - 1$ (resp. $0 \le j \le p^{u-1} - 1$).

Set $a = p^u$. Condition (1) implies that the group $\mathbb{Z}_a$ acts on $F(f)$ (and on $F(g)$). Since f is real-analytic, the complex conjugation also acts on $F(f)$. Let G be the dihedral group of order $2a$, i.e. the group generated by elements γ, β with the relations $\gamma^2 = 1$, $\beta^a = 1$, $\gamma\beta^j = \beta^{-j}\gamma$, $j \in \mathbb{Z}$. From the above, there is an action of G on $F(f)$ given by $\gamma(z) = \overline{z}$, $\beta(z) = \varepsilon.z$.

Define $A_j = \{z \in F(f) \mid \varepsilon^j.z = z\}$ for $j = 0, \ldots, a-1$. Observe that if $j = a/p = p^{u-1}$, then $\varepsilon^{jw_k} z_k = z_k \exp(2\pi w_k i/p)$; if $k > m$, that is, w_k is not divisible by p, then $\exp(2\pi w_k i/p) \ne 1$, and consequently $A_{a/p} = \{z \in F(f) \mid z_k = 0 \text{ for } k > m\}$.

LEMMA 1.

$$\bigcup_{j=1}^{a-1} A_j = A_{a/p}.$$

Proof. Assume that $z \in F(f)$ and $z = \varepsilon^j.z$ for some $1 \le j \le a-1$. This means that $z_k = \varepsilon^{jw_k} z_k$ for $k = 1, \ldots, n$. Assume that $z_k \neq 0$ for some k, i.e. jw_k is divisible by $a = p^u$. Since j is not divisible by a, it follows that w_k is divisible by p. Hence $z_k = 0$ for $k > m$ and $A_j \subset A_{a/p}$ for $1 \le j \le a-1$. ■

LEMMA 2. (i) $\chi(B_j) = \begin{cases} \chi(F_{\mathbb{R}}(f)) & \text{if } j \text{ is even,} \\ \chi(F_{\mathbb{R}}(g)) & \text{if } j \text{ is odd.} \end{cases}$

(ii) *If* $0 \le j < j' \le a-1$, *then*

$$B_j \cap B_{j'} = A_{a/p} \cap B_j \cap B_{j'}.$$

(iii) *If* $0 \le j \le a/p - 1$, $0 \le s \le p-1$ *and* $j' = j + sa/p$, *then*

$$B_j \cap A_{a/p} = B_{j'} \cap A_{a/p}.$$

Proof. (i) Suppose that $\varepsilon^j.\overline{z} = z$ for some $0 \le j \le a-1$. This means that $z_k = \varepsilon^{jw_k}\overline{z}_k$, $1 \le k \le n$, hence $z_k^2 = \varepsilon^{jw_k} z_k \overline{z}_k = (\eta^{jw_k}|z_k|)^2$. It follows that $z_k = \eta^{jw_k} x_k$, $x_k \in \mathbb{R}$. Set $x = (x_1, \ldots, x_n)$. Then, from condition (2),

$$f_{\mathbb{C}}(z) = \begin{cases} f(x) & \text{if } j \text{ is even,} \\ g(x) & \text{if } j \text{ is odd,} \end{cases}$$

and consequently

$$\chi(B_j) = \begin{cases} \chi(F_{\mathbb{R}}(f)) & \text{if } j \text{ is even,} \\ \chi(F_{\mathbb{R}}(g)) & \text{if } j \text{ is odd.} \end{cases}$$

(ii) Assume that $z \in B_j \cap B_{j'}$ and $k < m$. Then $z_k = \eta^{jw_k} x_k = \eta^{j'w_k} x'_k$ for some $x_k, x'_k \in \mathbb{R}$. Clearly $|x_k| = |x'_k|$. If $x_k \neq 0$, then $w_k(j'-j)$ is divisible by $a = p^u$. Since w_k is not divisible by p for $k > m$, it follows that $j' - j$ is divisible by a, which contradicts the assumption that $0 \le j < j' \le a-1$.

(iii) Suppose that $z \in B_{j'} \cap A_{a/p}$. Then $z_k = 0$ for $k > m$ and w_k is divisible by p for $k \le m$. Hence $z_k = \eta^{(j+sa/p)w_k} x_k = \pm\eta^{jw_k} x_k$, so $z \in B_j \cap A_{a/p}$. ■

Proof of Theorem 1. The dihedral group G of order $2a$ acts freely on $F(f) - \bigcup_{j=0}^{a-1} (A_j \cup B_j)$, hence

$$\chi(F(f)) \equiv \chi\Big(\bigcup_{j=0}^{a-1} (A_j \cup B_j)\Big) \bmod 2a.$$

According to Lemma 1, $\bigcup_{j=1}^{a-1} A_j = A_{a/p}$. For simplicity we will write B_a instead of $A_{a/p}$. Thus,

$$\chi(F(f)) \equiv \chi\Big(\bigcup_{j=0}^{a} B_j\Big) \bmod 2a.$$

Clearly,

$$\chi\Big(\bigcup_{j=0}^{a} B_j\Big) = \sum_{q=1}^{a+1} (-1)^{q-1} S_q,$$

where $S_q = \sum_J T_J$, $J = (j_1, \ldots, j_q)$, $0 \le j_1 < \ldots < j_q \le a$ and $T_J = \chi(B_{j_1} \cap \ldots \cap B_{j_q})$. We may write $S_q = \sum_H T_H + \sum_I T_I$, where $H = (h_1, \ldots, h_q)$, $0 \le h_1 < \ldots < h_q = a$, $I = (i_1, \ldots, i_q)$, $0 \le i_1 < \ldots < i_q < a$.

Thus, we have $S_{q+1} = \sum_{I_a} T_{I_a} + \sum_{I'} T_{I'}$, where $I_a = (i_1, \ldots, i_q, a)$ and $I' = (i'_1, \ldots, i'_{q+1})$, $0 \le i'_1 < \ldots < i'_{q+1} < a$.

If $q = a$, then

$$S_{q+1} = S_{a+1} = \chi\Big(\bigcap_{j=0}^{a} B_j\Big).$$

Due to Lemma 2(ii), $\sum_I T_I = \sum_{I_a} T_{I_a}$ and consequently

$$\chi\Big(\bigcup_{j=0}^{a} B_j\Big) = \sum_{j=0}^{a} \chi(B_j) - \sum_{h=0}^{a-1} \chi(B_h \cap B_a).$$

Applying Lemma 2(i), (iii) we obtain

$$\begin{aligned}\chi\Big(\bigcup_{j=0}^{a} B_j\Big) &= a_+\chi(F_{\mathbb{R}}(f)) + a_-\chi(F_{\mathbb{R}}(g)) + \chi(B_a) - \sum_{j=0}^{a-1} \chi(B_j \cap B_a)\\ &= a_+\chi(F_{\mathbb{R}}(f)) + a_-\chi(F_{\mathbb{R}}(g)) + \chi(B_a) - p \sum_{j=0}^{a/p-1} \chi(B_j \cap B_a).\end{aligned}$$

By the definition $\widetilde{f} : \mathbb{R}^m \to \mathbb{R}$. For $z' = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and $j = 0, 1, \ldots, a/p - 1$ we define $j(z') = (\varepsilon^{jw_1} z_1, \ldots, \varepsilon^{jw_m} z_m)$, $\widetilde{B}_j = \{z' \in F(\widetilde{f}) \mid j(\overline{z}') = z'\}$, $\widetilde{C}_j = \{z' \in F(\widetilde{g}) \mid j(\overline{z}') = z'\}$.

Using the same arguments as above one can prove that

$$\chi(\widetilde{B}_j) = \begin{cases} \chi(F_{\mathbb{R}}(\widetilde{f})) & \text{if } j \text{ is even},\\ \chi(F_{\mathbb{R}}(\widetilde{g})) & \text{if } j \text{ is odd}.\end{cases}$$

Clearly, $\chi(B_a) = \chi(F(\widetilde{f}))$, and $\chi(\widetilde{B}_j) = \chi(B_j \cap B_a)$ for $j = 0, 1, \ldots, a/p - 1$. Thus

$$\begin{aligned}\chi(F(f)) \equiv{}& a_+\chi(F_{\mathbb{R}}(f)) + a_-\chi(F_{\mathbb{R}}(g)) + \chi(F(\widetilde{f}))\\ &- p(\widetilde{a}_+\chi(F_{\mathbb{R}}(\widetilde{f})) + \widetilde{a}_-\chi(F_{\mathbb{R}}(\widetilde{g}))) \bmod 2a. \ \blacksquare\end{aligned}$$

2. $\mathcal{A}_d$-germs. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a germ of a real-analytic function.

DEFINITION. Let $d \ge 2$ be an integer. We shall say that f is an *$\mathcal{A}_d$-germ* if there are positive integers $w_1, \ldots, w_n$ such that if $f = \sum_\alpha a_\alpha x^\alpha$ and $a_\alpha \ne 0$ then $\alpha_1 w_1 + \ldots + \alpha_n w_n \equiv d \bmod 2d$.

EXAMPLES. (i) Each germ defined by a weighted homogeneous polynomial of degree d is an $\mathcal{A}_d$-germ.

(ii) The germ $f(x,y,z,t) = x^4 + x^{12} + y^2 + z^3t + z^4t^4$ is an $\mathcal{A}_8$-germ, where $w_1 = 2$, $w_2 = 4$, $w_3 = 1$, $w_4 = 5$.

Let F_+ and F_- denote the positive and negative real Milnor fibres of f, that is, $F_+ = f^{-1}(\delta) \cap B_r^n$, $F_- = f^{-1}(-\delta) \cap B_r^n$, where $0 < \delta \ll r \ll 1$. Clearly, $F_+ = F_{\mathbb{R}}(f)$ and $F_- = F_{\mathbb{R}}(-f)$. Let $\widetilde{F}_+$ (resp. $\widetilde{F}_-$) denote the positive (resp. negative) real Milnor fibre of $\widetilde{f}$ (of course $\widetilde{f}$ is an $\mathcal{A}_d$-germ).

THEOREM 2. *If f is an $\mathcal{A}_d$-germ, then*

$$(\chi(F_+) + \chi(F_-)) - (\chi(\widetilde{F}_+) + \chi(\widetilde{F}_-)) \equiv 2(\chi(F(f)) - \chi(F(\widetilde{f})))/a \bmod 4.$$

Proof. We have $f_{\mathbb{C}}(\varepsilon^j.z) = f_{\mathbb{C}}(z)$ for $z \in \mathbb{C}^n$, $j \in \mathbb{Z}$. Moreover, v is odd, hence if $x \in \mathbb{R}^n$, $j \in \mathbb{Z}$, then $f_{\mathbb{C}}(\eta^j.x) = (-1)^j f(x)$. This means that the germs f and $-f$ satisfy conditions (1) and (2) of Theorem 1. Thus,

$$\chi(F(f)) - \chi(F(\widetilde{f})) \equiv a_+\chi(F_+) + a_-\chi(F_-) - p(\widetilde{a}_+\chi(\widetilde{F}_+) + \widetilde{a}_-\chi(\widetilde{F}_-)) \bmod 2a.$$

If d is even then $p = 2$ and $a_+ = a_- = a/2$. If d is odd then the map $(x_1, \ldots, x_n) \mapsto ((-1)^{w_1}x_1, \ldots, (-1)^{w_n}x_n)$ maps F_+ homeomorphically onto F_-. Then $a\chi(F_+) = a\chi(F_-) = a(\chi(F_+) + \chi(F_-))/2$ (similarly for $\widetilde{F}$). Hence in both cases we obtain

$$a(\chi(F_+) + \chi(F_-))/2 - a(\chi(\widetilde{F}_+) + \chi(\widetilde{F}_-))/2 \equiv \chi(F(f)) - \chi(F(\widetilde{f}))) \bmod 2a.$$

It follows that

$$(\chi(F_+) + \chi(F_-)) - (\chi(\widetilde{F}_+) + \chi(\widetilde{F}_-)) \equiv 2(\chi(F(f)) - \chi(F(\widetilde{f})))/a \bmod 4. \quad \blacksquare$$

As mentioned above, $\widetilde{f}$ is an $\mathcal{A}_d$-germ, so Theorem 2 may be applied to $\widetilde{f}$, and so on. Repeated application of Theorem 2 enables us to express the number $\chi(F_+) + \chi(F_-) \bmod 4$ only in terms of the Euler characteristics of the Milnor fibres of appropriate restrictions (given by the weights w_i) of $f_{\mathbb{C}}$. In the case of an algebraically isolated singularity of f, i.e., when $0 \in \mathbb{C}^n$ is isolated in the set of critical points of $f_{\mathbb{C}}$, those characteristics can be calculated effectively from the Milnor numbers of $f_{\mathbb{C}}$, $\widetilde{f}_{\mathbb{C}}$, etc. Recall that the Milnor number of $f_{\mathbb{C}}$ equals the dimension of an appropriate local algebra ([4]). Moreover, if $f_{\mathbb{C}}$ has an isolated singularity, then also $\widetilde{f}_{\mathbb{C}}$ has one ([2]). When 0 is not an isolated critical point of $f_{\mathbb{C}}$, then one can use Varchenko's method ([6]), although it is less effective.

References

[1] A. Dimca and L. Paunescu, *Real singularities and dihedral representations*, Workshop on Real and Complex Singularities (São Carlos, 1996), Math. Contemp. 12 (1997), 67–82.

[2] P. Dudziński, *Topological invariants of germs of real-analytic functions*, Glasgow Math. J. 39 (1997), 85–89.
[3] C. McCrory and A. Parusiński, *Complex monodromy and the topology of real algebraic sets*, Compositio Math. 106 (1997), 211–233.
[4] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton Univ. Press, 1968.
[5] Z. Szafraniec, *On the topological invariants of germs of analytic functions*, Topology 26 (1987), 235–238.
[6] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. 37 (1976), 253–262.
[7] C. T. C. Wall, *Topological invariance of the Milnor number mod 2*, Topology 22 (1983), 345–350.

Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-925 Gdańsk, Poland
E-mail: pd@delta.math.univ.gda.pl

Reçu par la Rédaction le 24.4.2002
Révisé le 24.2.2003 (1338)