Bifurcation theorems for nonlinear problems with lack of compactness

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Abstract. We deal with a bifurcation result for the Dirichlet problem

\[-\Delta_p u = \frac{\mu}{|x|^p} |u|^{p-2}u + \lambda f(x,u) \quad \text{a.e. in } \Omega,\]

\[u|_{\partial \Omega} = 0.\]

Starting from a weak lower semicontinuity result by E. Montefusco, which allows us to apply a general variational principle by B. Ricceri, we prove that, for \( \mu \) close to zero, there exists a positive number \( \lambda^*_\mu \) such that for every \( \lambda \in [0, \lambda^*_\mu] \), the above problem admits a nonzero weak solution \( u_\lambda \) in \( W^{1,p}_0(\Omega) \) satisfying \( \lim_{\lambda \to 0^+} \|u_\lambda\| = 0. \)

1. Introduction. In the present paper we are interested in the existence of solutions for the Dirichlet problem

\[(P_{\lambda,\mu}) \begin{cases} -\Delta_p u = \frac{\mu}{|x|^p} |u|^{p-2}u + \lambda f(x,u) & \text{a.e. in } \Omega, \\ u|_{\partial \Omega} = 0, \end{cases}\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) \((n \geq 2)\) containing the origin, \( 1 < p < N \), \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function with \( f(x,0) = 0 \), and \( \lambda, \mu \) are two parameters respectively positive and nonnegative.

The presence of the term \( \mu/|x|^p \) does not allow us to apply the classical variational approach. The Hardy inequality ensures that \( W^{1,p}_0(\Omega) \) is continuously, but not compactly embedded in \( L^p(\Omega) \) with respect to the weight \( |x|^{-p} \). Because of this lack of compactness we are not able to obtain the weak lower semicontinuity of the energy functional via the classical De Giorgi theorem.

The problem

\[(P_{\mu}) \begin{cases} -\Delta_p u = \frac{\mu}{|x|^p} |u|^{p-2}u + f(x) & \text{a.e. in } \Omega, \\ u|_{\partial \Omega} = 0, \end{cases}\]

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is studied in [3], where \( f \in W^{-1,p'}(\Omega) \) and \( \mu < H, \) \( H \) being the best constant in the Hardy inequality. In particular, starting from the coercivity and homogeneity of the energy functional, the authors of [3] are able to prove the required compactness property by finding a minimizing sequence converging to a global minimum.

The authors of [1] are interested in minima of the nondifferentiable functional

\[
J(u) = \int_\Omega j(x, \nabla u) \, dx - a \int_\Omega \frac{|u|^2}{|x|^2} \, dx - \int_\Omega f(x)u(x) \, dx,
\]

where \( f \in L^2(\Omega) \) and \( j(x, \xi) \) is a convex function with respect to \( \xi, \) satisfying

\[
\alpha|\xi|^2 \leq j(x, \xi) \leq \beta|\xi|^2
\]

for every \( \xi \in \mathbb{R}^n, \) a.e. in \( \Omega. \) With no information about the weak lower semicontinuity of the functional, they state the existence of a global minimum using a truncation approach.

We refer moreover to the recent papers [2] and [4] for a complete survey of the topic. The authors of [4] deal with the problem

\[
(P) \quad \begin{cases}
-\Delta_p u = \frac{\mu}{|x|^s} |u|^{q-2}u + \lambda |u|^{r-2}u & \text{a.e. in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where \( 0 \leq s \leq p < n, \) \( q \leq p^*(s) = (p-s)/(n-p)p. \) In particular in the case \( s = q = p, \) \( p < r < p^* \) they obtain infinitely many solutions, at least one of them being positive, for any \( \lambda > 0 \) and \( 0 < \mu < H. \) We notice that the main assumption in that paper is a bound on \( r, \) which is incompatible with our hypothesis on the nonlinearity at zero (see condition (3)).

Here we propose a novel approach to the subject. In particular, combining a weak lower semicontinuity result by E. Montefusco [5] with a recent variational principle by B. Ricceri [6] we establish a bifurcation theorem for problem \((P_{\lambda,\mu})\) just assuming a suitable behaviour of \( f(x, t) \) at zero. Moreover, we prove that if \( \lambda \) is sufficiently small, then the energy functional related to the problem is negative and decreasing on the solutions.

Finally, using the same technique, we obtain an analogous result for the problem

\[
(P_{\lambda,\mu})^* \quad \begin{cases}
-\Delta_p u = \mu \left( \int_\Omega |u|^{p^*} \right)^{p/p^*-1} |u|^{p^*-2}u + \lambda f(x, u) & \text{a.e. in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where \( n, \) \( p \) and \( f \) are as in problem \((P_{\lambda,\mu}), \) \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) and \( p^* = np/(n-p). \)
2. Preliminaries. Assume the following growth condition on $f$: there exist two positive constants $a$, $q$ with

$$q < \frac{n(p - 1) + p}{n - p}$$

and a nonnegative constant $b$ such that

$$|f(x, \xi)| \leq a|\xi|^q + b$$

for every $\xi \in \mathbb{R}$ and a.e. in $\Omega$.

Denote by $X$ the space $W^{1,p}_0(\Omega)$ endowed with the norm

$$\|u\| = \left( \int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}.$$ 

Let us recall the Hardy inequality

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx \leq \frac{1}{H} \int_\Omega |\nabla u(x)|^p \, dx,$$

where $\Omega$ is an open set in $\mathbb{R}^n$ containing the origin and $H$ is the best constant in the inclusion of $W^{1,p}_0(\Omega)$ in $L^p(\Omega)$ with weight $|x|^{-p}$. In particular, when $\Omega$ is a ball, $H = ((n - p)/p)^p$ (see [3]).

For each $u \in X$ and $\mu \in \mathbb{R}$ put

$$\mathcal{H}_\mu(u) = \frac{1}{p} \int_\Omega |\nabla u(x)|^p \, dx - \mu \int_\Omega \frac{|u(x)|^p}{|x|^p} \, dx,$$

$$\Phi(u) = -\int_\Omega \left( \int_0^{u(x)} f(x, \xi) \, d\xi \right) \, dx.$$

In [5] it is shown that $\mathcal{H}_\mu$ is a well defined and continuously Gateaux differentiable functional in $X$. Moreover, if $\mu \in [0, H]$, then $\mathcal{H}_\mu$ is weakly lower semicontinuous and coercive.

Standard arguments show that $\Phi$ is a well defined and continuously Gateaux differentiable functional whose Gateaux derivative is a compact operator from $X$ to $X^*$. A weak solution of problem $(P_{\lambda,\mu})$ is any $u \in X$ such that

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \, dx - \mu \int_\Omega \frac{|u(x)|^{p-2}}{|x|^p} u(x)v(x) \, dx$$

$$- \lambda \int_\Omega f(x, u(x))v(x) \, dx = 0 \quad \text{for all } v \in X.$$

For each $\lambda, \mu \in \mathbb{R}$, consider the functional $J_{\lambda,\mu} : X \to \mathbb{R}$ defined by

$$J_{\lambda,\mu}(u) = \mathcal{H}_\mu(u) + \lambda \Phi(u)$$

and observe that it is the energy functional related to $(P_{\lambda,\mu})$. 
Finally, for the reader’s convenience, we recall the main tool that we will use. It is due to B. Ricceri and it can be stated as follows:

**Theorem A** ([6, Theorem 2.5]). Let $X$ be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially weakly lower semicontinuous functionals. Assume also that $\Psi$ is (strongly) continuous and $\lim_{\|x\| \to +\infty} \Psi(x) = +\infty$. For each $\varphi > \inf_X \Psi$, put

$$
\varphi(\varphi) = \inf_{x \in \Psi^{-1}([-\infty, \varphi])} \frac{\Phi(x) - \inf_{\text{cl}_{w} \Psi^{-1}([-\infty, \varphi])} \Phi}{\varphi - \Psi(x)},
$$

where $\text{cl}_{w}$ is the closure in the weak topology. Then, for each $\varphi > \inf_X \Psi$ and each $\lambda > \varphi(\varphi)$, the restriction of the functional $\Phi + \lambda \Psi$ to $\Psi^{-1}([-\infty, \varphi])$ has a global minimum.

### 3. Main result

**Theorem 3.1.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $f(x, 0) = 0$, satisfying condition (1). Assume that there are a nonempty open set $D \subseteq \Omega$ and a set $B \subseteq D$ of positive measure such that

$$
\limsup_{\xi \to 0^+} \frac{\inf_{x \in B} \int_0^\xi f(x, t) \, dt}{|\xi|^p} = +\infty,
$$

$$
\liminf_{\xi \to 0^+} \frac{\inf_{x \in D} \int_0^\xi f(x, t) \, dt}{|\xi|^p} > -\infty.
$$

Then for every $\mu \in [0, H]$ there exists a positive number $\lambda^*_\mu$ such that for every $\lambda \in ]0, \lambda^*_\mu[$ problem $(P_{\lambda, \mu})$ admits a nonzero weak solution $u_\lambda$ in $W^{1,p}_0(\Omega)$. Moreover,

$$
\lim_{\lambda \to 0^+} \|u_\lambda\| = 0
$$

and the function $\lambda \mapsto J_{\lambda, \mu}(u_\lambda)$ is negative and decreasing in $]0, \lambda^*_\mu[$.

**Proof.** Fix $\mu \in [0, H]$. We want to apply Theorem A, where $X = W^{1,p}_0(\Omega)$, $\Psi = \mathcal{H}_\mu$ and $\Phi$ is the functional introduced in Section 2.

Since $\mu \in [0, H]$, we have already observed in Section 2 that $\Phi$ and $\Psi$ are two sequentially weakly lower semicontinuous and continuously Gateaux differentiable functionals. Moreover, $\Psi$ is coercive and clearly $\inf_{u \in X} \Psi(u) = 0$.

Let $\overline{\varphi} > 0$ be such that $\varphi(\overline{\varphi}) > 0$ and put $\lambda^*_\mu = 1/\varphi(\overline{\varphi})$. Thanks to Theorem A, for every $\lambda \in ]0, \lambda^*_\mu[$ there exists $u_\lambda \in \Psi^{-1}([-\infty, \overline{\varphi}])$ such that

$$
\Phi'(u_\lambda) + \frac{1}{\lambda} \Psi'(u_\lambda) = 0
$$

and, in particular, $u_\lambda$ is a global minimum of the restriction of $\Phi + \frac{1}{\lambda} \Psi$ to $\Psi^{-1}([-\infty, \overline{\varphi}])$. 


Fix $\lambda \in ]0, \lambda^*_\mu[; we will prove that $u_\lambda \neq 0$. To this end, let us prove that

(5) \quad \liminf_{\|u\|\to 0^+} \frac{\Phi(u)}{\Psi(u)} = -\infty.

Thanks to (3) we can fix a sequence $\{\xi_k\}$ in $\mathbb{R}^+$ converging to zero and two constants $\delta$ and $\Gamma$ with $\delta > 0$ such that

$$\lim_{k \to +\infty} \inf_{x \in B} \frac{\int_0^{\xi_k} f(x, t) \, dt}{|\xi_k|^p} = +\infty$$

and

$$\inf_{x \in D} \int_0^\xi f(x, t) \, dt \geq \Gamma |\xi|^p$$

for all $\xi \in [0, \delta]$. Next, fix a set $C \subset B$ of positive measure and a function $v \in X$ such that $v(x) \in [0, 1]$ for all $x \in \Omega$, $v(x) = 1$ for all $x \in C$, and $v(x) = 0$ for all $x \in \Omega \setminus D$. Let $Q > 0$, put

$$|||v|||^p = \int_{\Omega} \frac{|v(x)|^p}{|x|^p} \, dx$$

and consider a positive number $T$ with

$$Q < \frac{T \text{meas}(C) + \Gamma \int_{D \setminus C} |v(x)|^p \, dx}{\frac{1}{p} ||v||^p - \frac{\mu}{p} |||v|||^p}.$$

Then there is $\nu \in \mathbb{N}$ such that $\xi_k < \delta$ and

$$\inf_{x \in B} \int_0^{\xi_k} f(x, t) \, dt \geq T |\xi_k|^p$$

for all $k > \nu$. Now, for each $k > \nu$, one has

(6) \quad -\frac{\Phi(\xi_k v)}{\Psi(\xi_k v)} = \frac{\int_{C} (\int_0^{\xi_k} f(x, t) \, dt) \, dx + \int_{D \setminus C} \int_0^{\xi_k v(x)} f(x, t) \, dt \, dx}{\frac{1}{p} ||\xi_k v||^p - \frac{\mu}{p} ||\xi_k v||^p}

\geq \frac{T \text{meas}(C) + \Gamma \int_{D \setminus C} |v(x)|^p \, dx}{\frac{1}{p} ||v||^p - \frac{\mu}{p} |||v|||^p} > Q.

From (6), clearly (5) follows. Hence, there is a sequence $\{w_k\}$ in $X$ converging to zero such that for $k$ large enough we have $w_k \in \Psi^{-1}([-\infty, \overline{\nu}])$, and

$$\Phi(w_k) + \frac{1}{\lambda} \Psi(w_k) < 0.$$

Since $u_\lambda$ is a global minimum of the restriction of $\Phi + \frac{1}{\lambda} \Psi$ to $\Psi^{-1}([-\infty, \overline{\nu}])$, we can conclude that
\[
\Phi(u_\lambda) + \frac{1}{\lambda} \Psi(u_\lambda) < 0 = \Phi(0) + \frac{1}{\lambda} \Psi(0)
\]
so that \(u_\lambda \neq 0\).

Observing that the weak solutions of problem \((P_{\lambda,\mu})\) are exactly the critical points of the functional \(J_{\lambda,\mu}\) and that
\[
J_{\lambda,\mu} = \lambda \left( \Phi + \frac{1}{\lambda} \Psi \right),
\]
one finds that the first part of our theorem is completely proved.

Since \(\Psi\) is coercive and \(u_\lambda \in \Psi^{-1}([-\infty,0[)\) for every \(\lambda \in ]0,\lambda^*_\mu[\), there exists a positive number \(L\) such that
\[
\|u_\lambda\| \leq L
\]
for every \(\lambda \in ]0,\lambda^*_\mu[\). Therefore, since \(\Phi'\) is a compact operator, there exists a positive number \(M\) such that
\[
\left| \int_\Omega f(x,u_\lambda(x))u_\lambda(x) \, dx \right| \leq \|\Phi'(u_\lambda)\|_{X^*} \|u_\lambda\| \leq M \cdot L^2
\]
for every \(\lambda \in ]0,\lambda^*_\mu[\).

By (4), \(J'_{\lambda}(u_\lambda) = 0\) for every \(\lambda \in ]0,\lambda^*_\mu[\) and in particular \((J'_{\lambda}(u_\lambda))(u_\lambda) = 0\), that is,
\[
p \cdot \Psi(u_\lambda) = \|u_\lambda\|^p - \mu \|u_\lambda\||u_\lambda\|^p = \lambda \int_\Omega f(x,u_\lambda(x))u_\lambda(x) \, dx
\]
for every \(\lambda \in ]0,\lambda^*_\mu[\). Hence
\[
\lim_{\lambda \to 0^+} \Psi(u_\lambda) = 0.
\]

Finally, putting together (2) and (9) yields
\[
\frac{1}{p} \|u_\lambda\|^p \leq \Psi(u_\lambda) + \frac{\mu}{p \cdot H} \|u_\lambda\|^p
\]
for every \(\lambda \in ]0,\lambda^*_\mu[\), hence
\[
\|u_\lambda\|^p \leq \frac{p \cdot H}{H - \mu} \Psi(u_\lambda)
\]
for every \(\lambda \in ]0,\lambda^*_\mu[\) and
\[
\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.
\]

From (7) and (8) it follows that the function \(\lambda \mapsto J_{\lambda,\mu}(u_\lambda)\) is negative in \(]0,\lambda^*_\mu[\). Finally, if we fix \(\lambda_1, \lambda_2 \in ]0,\lambda^*_\mu[\) with \(\lambda_1 < \lambda_2\) and put
\[
\begin{align*}
m_{\lambda_1} &= \Phi(u_{\lambda_1}) + \frac{1}{\lambda_1} \Psi(u_{\lambda_1}) = \inf_{v \in \Psi^{-1}([-\infty,0[)} \left( \Phi(v) + \frac{1}{\lambda_1} \Psi(v) \right), \\
m_{\lambda_2} &= \Phi(u_{\lambda_2}) + \frac{1}{\lambda_2} \Psi(u_{\lambda_2}) = \inf_{v \in \Psi^{-1}([-\infty,0[)} \left( \Phi(v) + \frac{1}{\lambda_2} \Psi(v) \right),
\end{align*}
\]
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then

\[ J_{\lambda, \mu}(u_\lambda) = \lambda_1 m_\lambda > \lambda_2 m_\lambda \geq \lambda_2 m_{\lambda_2} = J_{\lambda_2, \mu}(u_{\lambda_2}). \]

Hence, \( \lambda \mapsto J_{\lambda, \mu}(u_\lambda) \) is decreasing in \( ]0, \lambda^*_\mu[ \) and the proof is complete.

**Remark 1.** Observe that Theorem 3.1 is a bifurcation result. In fact, since \( f(x, 0) = 0 \) it follows that 0 is a solution of \((P_{\lambda, \mu})\) for every \( \lambda, \mu \).

Hence, \( \lambda = 0 \) is a bifurcation point for problem \((P_{\lambda, \mu})\), in the sense that \((0, 0)\) belongs to the closure in \( W^{1, p}_0(\Omega) \times \mathbb{R} \) of the set

\[ \{ (u, \lambda) \in W^{1, p}_0(\Omega) \times ]0, +\infty[ : u \text{ is a weak solution of } (P_{\lambda, \mu}), \ u \neq 0 \}. \]

Anyway, also when \( f(x, 0) \neq 0 \) and \( f \), in addition to assumption \((P_{\lambda, \mu})\), satisfies the growth condition

\[ |f(x, \xi)| \leq a(1 + |\xi|^q) \]

for every \( \xi \in \mathbb{R} \) and a.e. in \( \Omega \), where \( a > 0 \) and \( q < \frac{n(p-1)+p}{n-p} \), the statements of Theorem 3.1 are still true.

Here is an example of application of Theorem 3.1.

**Example 1.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with \( n \geq 2 \), \( 1 < p < n \) and \( \alpha, \beta : \Omega \to \mathbb{R} \) two continuous and bounded functions. Assume that \( \sup_{\Omega} \alpha > 0 \) and that \( \beta \) is a positive function with \( \inf_{\Omega} \beta > 0 \). Then for each \( \mu \in [0, H] \) there exists a positive number \( \lambda^*_\mu \) such that the problem

\[ (\tilde{P}_{\lambda, \mu}) \quad \left\{ \begin{array}{ll}
-\Delta_p u = \frac{\mu}{|x|^p} |u|^{p-2} u + \lambda(\alpha(x)|u|^{r-2} u + \beta(x)|u|^{s-2} u) \quad & \text{a.e. in } \Omega, \\
 u|_{\partial\Omega} = 0, & 
\end{array} \right. \]

with \( 1 < r < p \) and \( p < s < p^* \) admits a nonzero weak solution \( u_\lambda \) in \( W^{1, p}_0(\Omega) \). Moreover, \( \lim_{\lambda \to 0+} \|u_\lambda\| = 0 \) and the energy functional related to problem \((\tilde{P}_{\lambda, \mu})\) is negative and decreasing in \( ]0, \lambda^*_\mu[ \).

To prove this, we can apply Theorem 3.1 with

\[ f(x, \xi) = \alpha(x)|\xi|^{r-2} \xi + \beta(x)|\xi|^{s-2} \xi \]

for every \( (x, \xi) \in \overline{\Omega} \times \mathbb{R} \).

It is easy to verify that condition (1) holds. Denote by \( B_\varrho \) the open ball centred at \( x_0 \) with radius \( \varrho \), and let \( \varrho \) be such that \( B_\varrho \subseteq \overline{\Omega} \) and \( \min_{\partial B_\varrho} \alpha > 0 \).

If we put \( D = B = B_\varrho \), a simple computation shows that

\[ \lim_{\xi \to 0^+} \inf_{x \in B} \frac{\int_0^\xi f(x, t) \, dt}{|\xi|^p} \geq \min_B \alpha \frac{1}{r} \lim_{\xi \to 0^+} \frac{1}{|\xi|^{p-r}} = +\infty. \]

Hence all the assumptions of Theorem 3.1 are verified and the conclusion follows.

**Remark 2.** We point out that the energy functional \( J_{\lambda, \mu} \) related to problem \((\tilde{P}_{\lambda, \mu})\) is not coercive. In particular, it is unbounded from below.
In fact, if we fix $v \in W^{1,p}_0(\Omega)$ and $\tau \in \mathbb{R}$, then

$$J_{\lambda,\mu}(\tau v) = \frac{\tau^p}{p} \|v\|^p - \frac{\mu \tau^p}{p} \|v\|^p - \lambda \left[ \frac{\tau^p}{r} \left( \int_{\Omega} \alpha(x) \, dx \right) \|v\|^p + \frac{\tau^s}{s} \left( \int_{\Omega} \beta(x) \, dx \right) \|v\|^s \right].$$

So, as $s > p > r$, it follows that $\lim_{\tau \to +\infty} J_{\lambda,\mu}(\tau v) = -\infty$.

Let now $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and consider, for each $\lambda, \mu \in \mathbb{R}$, the functional $I_{\lambda,\mu} : X \to \mathbb{R}$ defined by putting

$$I_{\lambda,\mu}(u) = \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p \, dx - \frac{\mu}{\lambda} \left( \int_{\Omega} |u(x)|^{p^*} \, dx \right)^{p/p^*} + \lambda \Phi(u)$$

for each $u \in X$.

Taking into account the Sobolev inequality

$$(10) \quad \left( \int_{\Omega} |u(x)|^{p^*} \, dx \right)^{1/p^*} \leq \frac{1}{S^{1/p}} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{1/p},$$

where $u \in X$ and $S$ is the best constant in the Sobolev inclusion (see [7]), and observing that $I_{\lambda,\mu}$ is the energy functional related to problem $(P_{\lambda,\mu}^*)$ introduced in the introduction, in analogy to Theorem 3.1, we can state the following

**Theorem 3.2.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the assumptions of Theorem 3.1. Then for every $\mu \in [0, S]$ there exists a positive number $\nu_\mu^*$ such that for every $\lambda \in [0, \nu_\mu^*]$ problem $(P_{\lambda,\mu}^*)$ admits a nonzero weak solution $u_\lambda$ in $W^{1,p}_0(\Omega)$. Moreover,

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$$

and the function $\lambda \mapsto I_{\lambda,\mu}(u_\lambda)$ is negative and decreasing in $[0, \nu_\mu^*]$.

**References**


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