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Weighted pseudo almost automorphic functions with applications to abstract dynamic equations on time scales

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Abstract. We propose a concept of weighted pseudo almost automorphic functions on almost periodic time scales and study some important properties of weighted pseudo almost automorphic functions on almost periodic time scales. As applications, we obtain the conditions for the existence of weighted pseudo almost automorphic mild solutions to a class of semilinear dynamic equations on almost periodic time scales.

1. Introduction. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his 1988 Ph.D. thesis [Hi] in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory (see [BoP1, BoP2]). A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but it also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\}$ or $\mathbb{T} = \overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}$ (which has important applications in quantum theory) and can be applied to different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$, the space of harmonic numbers. Many papers have been published on the theory of dynamic equations on time scales [CasP, ErPR, LiZ, LiuX, SAO, SL, LiCZ, LiW1, LiW2]. Therefore, dealing with problems of differential equations on time scales becomes very important and meaningful in the research field of dynamic systems.

Almost automorphic functions are more general than almost periodic functions and they were introduced by Bochner. For more details about this topic we refer to the recent book [N1], where the author gave an

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important overview of the theory of almost automorphic functions and their applications to differential equations. Almost automorphic solutions in the context of differential equations had been studied by several authors [EzN, GN, N2]. N'Guérékata with colaborators [EzFN], and Xiao with colaborators [LiaNXZ, XLZ] established existence and uniqueness theorems of pseudo almost automorphic solutions to some semilinear abstract differential equations. Recently, N'Guérékata with colaborators [BlMN] introduced the concept of weighted pseudo almost automorphic, which generalizes the one of weighted pseudo almost periodicity [HaE, ZX], and they proved some interesting properties of the space of weighted pseudo almost automorphic functions like the completeness and the composition theorem, which have many applications in the context of differential equations.

In fact, both continuous and discrete systems are important in implementation and applications. But it is troublesome to study the existence and stability of weighted almost automorphic solutions for continuous and discrete systems, separately. Therefore, it is meaningful to study that issue on time scales, which can unify the continuous and discrete situations.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results. In Section 3, the concept of weighted pseudo almost automorphic functions is introduced on almost periodic time scales and some basic properties are investigated. Based on these results, in Section 4, as applications, we study the existence of weighted pseudo almost automorphic mild solutions to a class of abstract semilinear dynamic equations on almost periodic time scales. To the best of our knowledge, no similar results have appeared in the related literature.

2. Preliminaries. Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The *forward* and *backward jump operators* $\sigma, \varrho : \mathbb{T} \to \mathbb{T}$ and the *graininess* $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \varrho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called *left-dense* if $t > \inf \mathbb{T}$ and $\varrho(t) = t$, *left-scattered* if $\varrho(t) < t$, *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and *right-scattered* if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Definition 2.1 ([BoP1, BoP2, P]). Let $\mathbb{X}$ be a Banach space. A function $f : \mathbb{H} \to \mathbb{X}$ is called (*strongly*) *continuous* at $t_0 \in \mathbb{H} \subseteq \mathbb{T}$ if for any $\varepsilon > 0$, there exists $U(t_0, \delta)$ such that for any $s \in U(t_0, \delta)$,

$$\|f(s) - f(t_0)\| < \varepsilon.$$

f is called (strongly) continuous on $\mathbb{H}$ provided that it is (strongly) continuous at each $t \in \mathbb{H}$.

DEFINITION 2.2 ([BoP1, BoP2, P]). Let $D \subset \mathbb{T}$ be an open set, $f : D \to \mathbb{X}$ and let $t \in \mathbb{T}^\kappa$. Then we define $B : \mathbb{T} \to \mathbb{X}$ (provided it exists) via the property that given any $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap D$ for some $\delta > 0$) such that

$$\|[f(\sigma(t)) - f(s)] - [\sigma(t) - s]B\| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

Then we say that f is Δ-*differentiable* at t, B is called the Δ-*derivative* of f at t, and we write $B = f^\Delta(t)$.

DEFINITION 2.3 ([BoP1, BoP2, P]). A function $f : \mathbb{T} \to \mathbb{X}$ is called *regulated* provided its right-hand limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-hand limits exist (finite) at all left-dense points in $\mathbb{T}$.

DEFINITION 2.4 ([BoP1, BoP2, P]). A function $f : \mathbb{T} \to \mathbb{X}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-hand limits exist (finite) at all left-dense point in $\mathbb{T}$.

Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces. We denote by $B(\mathbb{X}, \mathbb{Y})$ the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$. This is simply denoted as $B(\mathbb{X})$ when $\mathbb{X} = \mathbb{Y}$.

A function $p : \mathbb{T} \to B(\mathbb{X})$ is called *regressive* provided $\mathrm{Id} + \mu(t)p(t)$ is invertible for all $t \in \mathbb{T}^k$, where Id is the identity operator. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to B(\mathbb{X})$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, B(\mathbb{X}))$.

REMARK 2.1. An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive provided $\mathrm{Id} + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$, and the class of all regressive and rd-continuous functions is denoted, similarly to the above case, by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

DEFINITION 2.5 ([BoP1, BoP2, P]). Let $y : \mathbb{T} \to \mathbb{X}$ be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the *delta integral* by

$$\int_a^t y(s)\,\Delta s = Y(t) - Y(a), \quad \text{where } t, a \in \mathbb{T}.$$

DEFINITION 2.6 ([BoP1, BoP2]). If $r : \mathbb{T} \to \mathbb{R}$ is a regressive function, then the *generalized exponential function* e_r is defined by

$$e_r(t, s) = \exp\Big\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\,\Delta\tau\Big\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \mathrm{Log}(1 + hz)/h & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

DEFINITION 2.7 ([BoP1, BoP2]). Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions. Define

$$p \oplus q = p + q + \mu p\, q, \qquad \ominus p = -\frac{p}{1+\mu p}, \qquad p \ominus q = p \oplus (\ominus q).$$

DEFINITION 2.8 ([BoP2]). For every $x, y \in \mathbb{R}$, write $[x, y) = \{t \in \mathbb{R} : x \le t < y\}$, and define a countably additive measure m_1 on the set

$$\mathcal{F}_1 = \{[\tilde{a}, \tilde{b}) \cap \mathbb{T} : \tilde{a}, \tilde{b} \in \mathbb{T},\ \tilde{a} \le \tilde{b}\}$$

that assigns to each interval $[\tilde{a}, \tilde{b}) \cap \mathbb{T}$ its length, that is,

$$m_1([\tilde{a}, \tilde{b})) = \tilde{b} - \tilde{a}.$$

The interval $[\tilde{a}, \tilde{a})$ is understood as the empty set. The measure m_1 generates the outer measure m_1^* on $\mathcal{P}(\mathbb{T})$, defined for each $E \in \mathcal{P}(\mathbb{T})$ as

$$m_1^*(E) = \begin{cases} \inf\limits_{\tilde{\mathcal{R}}} \Big\{ \sum\limits_{i \in I_{\tilde{\mathcal{R}}}} (\tilde{b}_i - \tilde{a}_i) \Big\} \in \mathbb{R}^+, & b \notin E, \\ +\infty, & b \in E, \end{cases}$$

with

$$\tilde{\mathcal{R}} = \Big\{ \{[\tilde{a}_i, \tilde{b}_i) \cap \mathbb{T} \in \mathcal{F}_1\}_{i \in I_{\tilde{\mathcal{R}}}} : I_{\tilde{\mathcal{R}}} \subset N, E \subset \bigcup_{i \in I_{\tilde{\mathcal{R}}}} ([a_i, b_i) \cap \mathbb{T}) \Big\}.$$

A set $A \subset \mathbb{T}$ is said to be Δ-*measurable* if

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A))$$

for all subsets E of $\mathbb{T}$. Define

$$\mathcal{M}(m_1^*) = \{A \subset \mathbb{T} : A \text{ is } \Delta\text{-measurable}\}.$$

The Lebesgue Δ-measure, denoted by μ_Δ, is the restriction of m_1^* to $\mathcal{M}(m_1^*)$.

For more details about Δ-measurability, we refer the reader to Chapter 5 in [BoP2] and its further development in [CabV].

LEMMA 2.1 ([BoP1, BoP2]). *Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*
(iii) $e_p(t, s) = 1/e_p(s, t) = e_{\ominus p}(s, t)$*;*
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$*;*
(v) $(e_{\ominus p}(t, s))^\Delta = (\ominus p)(t)e_{\ominus p}(t, s)$.

LEMMA 2.2 ([BoP1, BoP2]). *If $a, b, c \in \mathbb{T}$ and $p \in \mathcal{R}$, then*

$$\int_a^b p(t)e_p(c, \sigma(t))\, \Delta t = e_p(c, a) - e_p(c, b).$$

3. Weighted pseudo almost automorphic functions on time scales

DEFINITION 3.1 ([LiW1]). A time scale $\mathbb{T}$ is called *almost periodic* if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq \{0\}. \tag{3.1}$$

REMARK 3.1. In [LiW2], the following definition is given:

Let $\mathscr{C}$ be a collection of subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called *almost periodic with respect to* $\mathscr{C}$ if

$$\mathscr{C}^* = \Big\{\pm\tau \in \bigcap_{c \in \mathscr{C}} c : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\Big\} \neq \emptyset,$$

and $\mathscr{C}^*$ is called the *smallest almost periodic* set of $\mathbb{T}$.

Obviously, if $\mathscr{C} = \{\mathbb{R}\}$, then $\mathscr{C}^* = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\} = \Pi$, and if we require the ε-translation number set of f in $\mathscr{C}^*$ (see [LiW2, Definition 2.6]) to be a relatively dense set in $\mathbb{T}$, then $\mathscr{C}^*$ is $\mathbb{R}$ or $\tau\mathbb{Z}$, $\tau \in (0, \infty)$, that is, the set Π.

In the following, we always assume that $\mathbb{T}$ is an almost periodic time scale.

DEFINITION 3.2.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function. We say that f is *almost automorphic* if from every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n \to \infty} f(t + \tau_n)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n \to \infty} g(t - \tau_n) = f(t)$$

for each $t \in \mathbb{T}$. Denote by $\mathrm{AA}(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be *almost automorphic* if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $\mathrm{AA}(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

REMARK 3.2. In Definition 3.2, when the convergence is uniform for $t \in \mathbb{T}$, one can show that f is almost periodic. Hence, one can easily see that if $\mathbb{X} = \mathbb{R}^n$, Definition 3.2 is more general than Definitions 3.9 and 3.10 in [LiW1].

Let $\mathbb{X}$ be a Banach space endowed with the norm $\|\cdot\|_{\mathbb{X}}$. $\mathrm{BC}(\mathbb{T}, \mathbb{X})$ is the space of bounded continuous functions from $\mathbb{T}$ to $\mathbb{X}$ equipped with the supremum norm defined by

$$\|u\|_\infty = \sup_{t \in \mathbb{T}} \|u(t)\|_{\mathbb{X}}.$$

Let U be the set of all functions $\rho : \mathbb{T} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{T}$.

For a given $r \in [0, \infty) \cap \Pi$, set

$$m(r, \rho) := \int_{t_0 - r}^{t_0 + r} \rho(s)\,\Delta s$$

for each $\rho \in U$. Define

$$U_\infty := \Big\{\rho \in U : \lim_{r\to\infty} m(r, \rho) = \infty\Big\},$$
$$U_B := \Big\{\rho \in U_\infty : \rho \text{ is bounded and } \inf_{s\in\mathbb{T}} \rho(s) > 0\Big\}.$$

It is clear that $U_B \subset U_\infty \subset U$. Now for $\rho \in U_\infty$ define

$$\mathrm{PAA}_0(\mathbb{T}, \rho) := \Big\{f \in \mathrm{BC}(\mathbb{T}, \mathbb{X}) : \lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{t_0 - r}^{t_0 + r} \|f(s)\|\rho(s)\,\Delta s = 0,\ t_0 \in \mathbb{T},\ r \in \Pi\Big\}.$$

Similarly, we define $\mathrm{PAA}_0(\mathbb{T} \times \mathbb{X}, \rho)$ as the collection of all functions $F : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is continuous with respect to its two arguments and $F(\cdot, y)$ is bounded for each $y \in \mathbb{X}$, and

$$\lim_{r\to\infty} \frac{1}{m(r, \rho)} \int_{t_0 - r}^{t_0 + r} \|F(s, y)\|\rho(s)\,\Delta s = 0$$

uniformly in $y \in \mathbb{X}$, where $r \in \Pi$.

We are now ready to introduce the sets $\mathrm{WPAA}(\mathbb{T}, \rho)$ and $\mathrm{WPAA}(\mathbb{T}\times\mathbb{X}, \rho)$ of weighted pseudo almost automorphic functions:

$$\mathrm{WPAA}(\mathbb{T}, \rho) = \{f = g + \phi \in \mathrm{BC}(\mathbb{T}, \mathbb{X}) : g \in \mathrm{AA}(\mathbb{T}, \mathbb{X}) \text{ and } \phi \in \mathrm{PAA}_0(\mathbb{T}, \rho)\},$$
$$\mathrm{WPAA}(\mathbb{T} \times \mathbb{X}, \rho) = \{f = g + \phi \in \mathrm{BC}(\mathbb{T} \times \mathbb{X}, \mathbb{X}) : g \in \mathrm{AA}(\mathbb{T} \times \mathbb{X}, \mathbb{X}) \text{ and } \phi \in \mathrm{PAA}_0(\mathbb{T} \times \mathbb{X}, \rho)\}.$$

From the definition, one can easily show

LEMMA 3.1. *If* $f = g + \phi$ *with* $g \in \mathrm{AA}(\mathbb{T}, \mathbb{X})$, *and* $\phi \in \mathrm{PAA}_0(\mathbb{T}, \rho)$ *where* $\rho \in U_B$, *then* $g(\mathbb{T}) \subset \overline{f(\mathbb{T})}$.

THEOREM 3.1. *Assume that* $\mathrm{PAA}_0(\mathbb{T}, \rho)$ *is translation invariant. Then the decomposition of a weighted pseudo almost automorphic function according to* $\mathrm{AA} \oplus \mathrm{PAA}_0$ *is unique for any* $\rho \in U_B$.

Proof. Assume that $f = g_1 + \phi_1$ and $f = g_2 + \phi_2$. Then $0 = (g_1 - g_2) + (\phi_1 - \phi_2)$. Since $g_1 - g_2 \in \mathrm{AA}(\mathbb{T}, \mathbb{X})$, and $\phi_1 - \phi_2 \in \mathrm{PAA}_0(\mathbb{T}, \rho)$, in view of

Lemma 3.1, we deduce that $g_1 - g_2 = 0$. Consequently, $\phi_1 - \phi_2 = 0$, that is, $\phi_1 = \phi_2$. ∎

THEOREM 3.2. *Assume that* $\mathrm{PAA}_0(\mathbb{T},\rho)$ *is translation invariant and* $\rho \in U_B$, *then* $(\mathrm{WPAA}(\mathbb{T},\rho), \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{WPAA}(\mathbb{T},\rho)$. We can write uniquely $f_n = g_n + \phi_n$. Using Lemma 3.1, we see that $\|g_p - g_q\|_\infty \le \|f_p - f_q\|_\infty$, from which we deduce that $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{AA}(\mathbb{T},\mathbb{X})$. So, $\phi_n = f_n - g_n$ is a Cauchy sequence in $\mathrm{PAA}_0(\mathbb{T},\rho)$. We deduce that $g_n \to g \in \mathrm{AA}(\mathbb{T},\mathbb{X})$, $\phi_n \to \phi \in \mathrm{PAA}_0(\mathbb{T},\rho)$, and finally $f_n \to g + \phi \in \mathrm{WPAA}(\mathbb{T},\rho)$. ∎

DEFINITION 3.3. Let $\rho_1, \rho_2 \in U_\infty$. One says that ρ_1 equivalent to ρ_2, written $\rho_1 \sim \rho_2$, if $\rho_1/\rho_2 \in U_B$.

Let $\rho_1, \rho_2, \rho_3 \in U_\infty$. It is clear that $\rho_1 \prec \rho_1$ (reflexivity); if $\rho_1 \prec \rho_2$, then $\rho_2 \prec \rho_1$ (symmetry); and if $\rho_1 \prec \rho_2$ and $\rho_2 \prec \rho_3$, then $\rho_1 \prec \rho_3$ (transitivity). So, $\prec$ is an equivalence relation on U_∞.

THEOREM 3.3. *Let* $\rho_1, \rho_2 \in U_\infty$. *If* $\rho_1 \sim \rho_2$, *then* $\mathrm{WPAA}(\mathbb{T},\rho_1) = \mathrm{WPAA}(\mathbb{T},\rho_2)$.

Proof. Assume that $\rho_1 \sim \rho_2$. There exist $a, b > 0$ such that $a\rho_1 \le \rho_2 \le b\rho_1$. So,

$$am(r,\rho_1) \le m(r,\rho_2) \le bm(r,\rho_1),$$

where $r \in \Pi$, and

$$\begin{aligned}\frac{a}{b}\,\frac{1}{m(r,\rho_1)} \int_{t_0-r}^{t_0+r} \|\phi(s)\|\rho_1(s)\,\Delta s &\le \frac{1}{m(r,\rho_2)} \int_{t_0-r}^{t_0+r} \|\phi(s)\|\rho_2(s)\Delta s\\ &\le \frac{b}{a}\,\frac{1}{m(r,\rho_1)} \int_{t_0-r}^{t_0+r} \|\phi(s)\|\rho_1(s)\,\Delta s. \ \blacksquare\end{aligned}$$

LEMMA 3.2. *Let* $f \in \mathrm{BC}(\mathbb{T},\mathbb{X})$. *Then* $f \in \mathrm{PAA}_0(\mathbb{T},\rho)$ *where* $\rho \in U_B$ *if and only if for every* $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(f)) = 0,$$

where $r \in \Pi$ *and* $M_{r,\varepsilon}(f) := \{t \in [t_0 - r, t_0 + r] \cap \mathbb{T} : \|f(t)\| \ge \varepsilon\}$.

Proof. (a) *Necessity.* For contradiction, suppose that there exists $\varepsilon_0 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon_0}(f)) \ne 0.$$

Then there exists $\delta > 0$ such that for every $n \in \mathbb{N}$, $\frac{1}{m(r_n,\rho)}\mu_\Delta(M_{r_n,\varepsilon_0}(f)) \geq \delta$ for some $r_n > n$, where $r_n \in \Pi$. So we get

$$\begin{aligned}\frac{1}{m(r_n,\rho)}&\int_{t_0-r_n}^{t_0+r_n} \|f(s)\|\rho(s)\,\Delta s\\ &= \frac{1}{m(r_n,\rho)}\int_{M_{r_n,\varepsilon_0}(f)} \|f(s)\|\rho(s)\,\Delta s\\ &\quad + \frac{1}{m(r_n,\rho)}\int_{([t_0-r,t_0+r]\cap\mathbb{T})\setminus M_{r_n,\varepsilon_0}(f)} \|f(s)\|\rho(s)\,\Delta s\\ &\geq \frac{1}{m(r_n,\rho)}\int_{M_{r_n,\varepsilon_0}(f)} \|f(s)\|\rho(s)\,\Delta s\\ &\geq \frac{\varepsilon_0}{m(r_n,\rho)}\int_{M_{r_n,\varepsilon_0}(f)} \|f(s)\|\rho(s)\,\Delta s \geq \varepsilon_0\delta\gamma,\end{aligned}$$

where $\gamma = \inf_{s\in\mathbb{T}}\rho(s)$. This contradicts the assumption.

(b) *Sufficiency.* Assume that $\lim_{r\to\infty}\frac{1}{m(r,\rho)}\mu_\Delta(M_{r,\varepsilon}(f)) = 0$. Then for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for every $r > r_0$,

$$\frac{1}{m(r,\rho)}\mu_\Delta(M_{r,\varepsilon}(f)) < \frac{\varepsilon}{KM},$$

where $M := \sup_{t\in\mathbb{T}}\|f(t)\| < \infty$ and $K := \sup_{t\in\mathbb{T}}\rho(t) < \infty$.

Now, we have

$$\begin{aligned}\frac{1}{m(r,\rho)}&\int_{t_0-r}^{t_0+r} \|f(s)\|\,\Delta s\\ &= \frac{1}{m(r,\rho)}\Big(\int_{M_{r,\varepsilon}(f)} \|f(s)\|\rho(s)\,\Delta s + \int_{([t_0-r,t_0+r]\cap\mathbb{T})\setminus M_{r,\varepsilon}(f)} \|f(s)\|\rho(s)\,\Delta s\Big)\\ &\leq \frac{MK}{m(r,\rho)}\mu_\Delta(M_{r,k\varepsilon}(f)) + \frac{\varepsilon}{m(r,\rho)}\int_{([t_0-r,t_0+r]\cap\mathbb{T})\setminus M_{r,\varepsilon}(f)} \rho(s)\,\Delta s \leq 2\varepsilon.\end{aligned}$$

Therefore, $\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{t_0-r}^{t_0+r}\|f(s)\|\rho(s)\,\Delta s = 0$, that is, $f \in \mathrm{PAA}_0(\mathbb{T},\rho)$. ∎

LEMMA 3.3. *If* $g(t,x) \in \mathrm{AA}(\mathbb{T}\times\mathbb{X},\mathbb{X})$ *and* $\alpha(t) \in \mathrm{AA}(\mathbb{T},\mathbb{X})$, *then* $G(t) := g(t,\alpha(t)) \in \mathrm{AA}(\mathbb{T},\mathbb{X})$.

Proof. As $g(t,x) \in \mathrm{AA}(\mathbb{T}\times\mathbb{X},\mathbb{X})$, from every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$ we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g^*(t,x) := \lim_{n\to\infty} g(t+\tau_n,x)$$

is well defined for each $t \in \mathbb{T}$. In view of assumption (i) and $\alpha \in \mathrm{AA}(\mathbb{T},\mathbb{X})$, one can extract $\{\tau_n'\}_{n=1}^{\infty} \subset \{\tau_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} g(t+\tau_n', \alpha(t+\tau_n')) = \lim_{n\to\infty} g(t+\tau_n', \alpha^*(t)) = g^*(t, \alpha^*(t)).$$

Hence, $G(t) \in \mathrm{AA}(\mathbb{T}, \mathbb{X})$. ∎

We make the following assumptions:

(H1) $f(t,x)$ is uniformly continuous in any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{T}$.

(H2) $g(t,x)$ is uniformly continuous in any bounded subset $K \subset \mathbb{X}$ uniformly in $t \in \mathbb{T}$.

THEOREM 3.4. *Let* $f = g + \phi \in \mathrm{WPAA}(\mathbb{T}\times\mathbb{X}, \rho)$ *where* $g \in \mathrm{AA}(\mathbb{T}\times\mathbb{X}, \mathbb{X})$, $\phi \in \mathrm{PAA}_0(\mathbb{T}\times\mathbb{X}, \rho)$, $\rho \in U_\infty$ *and assume that* (H1) *and* (H2) *are satisfied. Then* $L(\cdot) := f(\cdot, h(\cdot)) \in \mathrm{WPAA}(\mathbb{T}, \rho)$ *if* $h \in \mathrm{WPAA}(\mathbb{T}, \rho)$.

Proof. We have $f = g + \phi$ where $g \in \mathrm{AA}(\mathbb{T}\times\mathbb{X}, \mathbb{X})$ and $\phi \in \mathrm{PAA}_0(\mathbb{T}\times\mathbb{X}, \rho)$ and $h = \mu_0 + \nu_0$ where $\mu_0 \in \mathrm{AA}(\mathbb{T}, \mathbb{X})$ and $\nu_0 \in \mathrm{PAA}_0(\mathbb{T}, \rho)$.

Now let us write

$$\begin{aligned} L(\cdot) &= g(\cdot, \mu_0(\cdot)) + f(\cdot, h(\cdot)) - g(\cdot, \mu_0(\cdot)) \\ &= g(\cdot, \mu_0(\cdot)) + f(\cdot, h(\cdot)) - f(\cdot, \mu_0(\cdot)) + \phi(\cdot, \mu_0(\cdot)). \end{aligned}$$

By Lemma 3.3, $g(\cdot, \mu_0(\cdot)) \in \mathrm{AA}(\mathbb{T}, \mathbb{X})$. Consider now the function

$$\Psi(\cdot) := f(\cdot, h(\cdot)) - f(\cdot, \mu_0(\cdot)).$$

Clearly $\Psi \in \mathrm{BC}(\mathbb{T}, \mathbb{X})$. For Ψ to be in $\mathrm{PAA}_0(\mathbb{T}, \rho)$, it is sufficient to show that

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\Psi)) = 0.$$

By Lemma 3.1, $\mu(\mathbb{T}) \subset \overline{h(\mathbb{T})}$ is a bounded set. Using assumption (H1) with $K = \overline{h(\mathbb{T})}$, we see that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in K,\ \|x-y\| < \delta \ \Rightarrow\ \|f(t,x) - f(t,y)\| < \varepsilon,\ t \in \mathbb{T}.$$

Thus we can obtain

$$\begin{aligned} &\frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\Psi(t))) \\ &\qquad = \frac{1}{m(r,\rho)} \mu_\Delta\big(M_{r,\varepsilon}(f(t,h(t)) - f(t,\mu_0(t)))\big) \\ &\qquad \le \frac{1}{m(r,\rho)} \mu_\Delta\big(M_{r,\delta}(h(t) - \mu_0(t))\big) = \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\delta}(\nu_0(t))). \end{aligned}$$

Now since $\nu_0 \in \mathrm{PAA}_0(\mathbb{T}, \rho)$, Lemma 3.2 yields $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\nu_0(t))) = 0$. Consequently, $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\Psi)(t)) = 0$. Thus, $\Psi \in \mathrm{PAA}_0(\mathbb{T}, \mathbb{X})$.

Finally, we need to show $\phi(t, \mu_0(t)) \in \mathrm{PAA}_0(\mathbb{T}, \rho)$. Note that $\phi(t, \mu_0(t))$ is uniformly continuous on $[t_0-r, t_0+r]\cap\mathbb{T}$, and $\mu_0([t_0-r, t_0+r]\cap\mathbb{T})$ is compact since μ_0 is continuous on $\mathbb{T}$ as an almost automorphic function. Thus given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu_0([t_0 - r, t_0 + r] \cap \mathbb{T}) \subset \bigcup_{k=1}^{m} B_k$ where $B_k = \{x \in \mathbb{X} : \|x - x_k\| < \delta\}$ for some $x_k \in \mu_0([t_0 - r, t_0 + r] \cap \mathbb{T})$, and

$$\|\phi(t, \mu_0(t)) - \phi(t, x_k)\| < \varepsilon/2, \ \mu_0(t) \in B_k, \ t \in [t_0 - r, t_0 + r] \cap \mathbb{T}. \tag{3.2}$$

It is easy to see that the set $U_k := \{t \in [t_0 - r, t_0 + r] \cap \mathbb{T} : \mu_0(t) \in B_k\}$ is open in $[t_0 - r, t_0 + r] \cap \mathbb{T}$, and $[t_0 - r, t_0 + r] \cap \mathbb{T} = \bigcup_{k=1}^{m} U_k$. Define

$$V_1 = U_1, \quad V_k = U_k \setminus \bigcup_{i=1}^{k-1} U_i, \quad 2 \le k \le m.$$

Then it is clear that $V_i \cap V_j = \emptyset$ if $i \neq j$, $1 \le i, j \le m$. So we get

$$\begin{aligned}
&\{t \in [t_0 - r, t_0 + r] \cap \mathbb{T} : \|\phi(t, \mu_0(t))\| \ge \varepsilon/2\} \\
&\qquad \subset \bigcup_{k=1}^{m} \{t \in V_k : \|\phi(t, \mu_0(t)) - \phi(t, x_k)\| + \|\phi(t, x_k)\| \ge \varepsilon\} \\
&\qquad \subset \bigcup_{k=1}^{m} (\{t \in V_k : \|\phi(t, \mu_0(t)) - \phi(t, x_k)\| \ge \varepsilon/2\} \\
&\qquad\quad \cup \{t \in V_k : \|\phi(t, x_k)\| \ge \varepsilon/2\}).
\end{aligned}$$

In view of (3.2), it follows that

$$\{t \in V_k : \|\phi(t, \mu_0(t)) - \phi(t, x_k)\| \ge \varepsilon/2\} = \emptyset, \quad k = 1, \ldots, m.$$

Thus we get

$$\frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\phi(t, \alpha(t)))) \le \sum_{k=1}^{m} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon}(\phi(t, x_k))).$$

Since $\phi(t, x) \in \mathrm{PAA}_0(\mathbb{T} \times \mathbb{X}, \rho)$ and $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon/2}(\phi(t, x_k)))$ $= 0$, it follows that $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \mu_\Delta(M_{r,\varepsilon/2}(\phi(t, \mu_0(t)))) = 0$, i.e., $\phi(t, \mu_0(t))$ $\in \mathrm{PAA}_0(\mathbb{T}, \rho)$. ∎

Theorem 3.4 has the following consequence:

COROLLARY 3.1. *Let $f = g + \phi \in \mathrm{WPAA}(\mathbb{T}, \rho)$ where $\rho \in U_\infty$. Assume that both f and g are Lipschitzian in $x \in \mathbb{X}$ uniformly in $t \in \mathbb{T}$. Then $L(\cdot) := f(\cdot, h(\cdot)) \in \mathrm{WPAA}(\mathbb{T}, \rho)$ if $h \in \mathrm{WPAA}(\mathbb{T}, \rho)$.*

4. Applications. Let $\mathbb{T}$ be an almost periodic time scale, and consider the linear dynamic equation

$$x^\Delta = A(t)x, \tag{4.1}$$

where $A(t)$ $(t \in \mathbb{T})$ is a linear operator in the Banach space $\mathbb{X}$.

DEFINITION 4.1 ([P]). $T(t,s) : \mathbb{T} \times \mathbb{T} \to B(\mathbb{X})$ is called the *linear evolution operator* associated to (4.1) if it satisfies the following conditions:

(1) $T(s,s) = \mathrm{Id}$, the identity operator in $\mathbb{X}$;
(2) $T(t,s)T(s,r) = T(t,r)$;
(3) the mapping $(t,s) \mapsto T(t,s)x$ is continuous for any fixed $x \in \mathbb{X}$.

DEFINITION 4.2. An evolution system $T(t,s)$ is called *exponentially stable* if there exist $K_0 \geq 1$ and $\omega > 0$ such that

$$\|T(t,s)\|_{B(\mathbb{X})} \leq K_0 e_{\ominus\omega}(t,\sigma(s)), \quad t \geq s.$$

REMARK 4.1. By Definition 4.2, if an evolution system $T(t,s)$ is exponentially stable, then there exist projections $P(t), Q(t) : \mathbb{T} \to B(\mathbb{X})$ for each $t \in \mathbb{T}$ such that $P(t) + Q(t) = \mathrm{Id}$,

$$\|Q(t)T(t,s)P(s)\|_{B(\mathbb{X})} \leq K_0 e_{\ominus\omega}(t,\sigma(s)), \quad t \geq s,$$

since

$$\|Q(t)T(t,s)P(s)\| \leq \|T(t,s)\|_{B(\mathbb{X})} \leq K_0 e_{\ominus\omega}(t,\sigma(s)), \quad t \geq s.$$

Consider the abstract differential equation

$$x^{\Delta}(t) = A(t)x(t) + f(t,x(t)), \quad t \in \mathbb{T}, \tag{4.2}$$

with the following assumptions:

(H_1) The family $\{A(t) : t \in \mathbb{T}\}$ of operators in $\mathbb{X}$ generates an exponentially stable evolution system $\{T(t,s) : t \geq s\}$, i.e., there exist $K_0 \geq 1$ and $\delta > 0$ such that

$$\|T(t,s)\|_{B(\mathbb{X})} \leq K_0 e_{\ominus\omega}(t,\sigma(s)), \quad t \geq s.$$

(H_2) $f = g + \phi \in \mathrm{WPAA}(\mathbb{T},\rho)$ where $\rho \in U_\infty$.
(H_3) $\|f(t,x) - f(t,y)\| \leq L_f\|x-y\|$, $\forall x,y \in \mathbb{X}$.
(H_4) $\|g(t,x) - g(t,y)\| \leq L_g\|x-y\|$, $\forall x,y \in \mathbb{X}$.

DEFINITION 4.3. A *mild solution* to (4.2) is a continuous function $x(t) : \mathbb{T} \to \mathbb{X}$ satisfying

$$x(t) = T(t,c)x(c) + \int_c^t T(t,s)f(s,x(s))\,\Delta s \quad \text{for all } t \geq c \text{ and all } c \in \mathbb{T}.$$

To investigate the existence and uniqueness of a weighted pseudo almost automorphic solution to (4.2), we need the following two lemmas:

LEMMA 4.1. *Let $v \in \mathrm{AA}(\mathbb{T},\mathbb{X})$ and suppose (H_1) is satisfied. If $u : \mathbb{T} \to \mathbb{X}$ is defined by*

$$u(t) = \int_{-\infty}^{t} T(t,s)v(s)\,\Delta s, \quad t \geq s,$$

then $u(\cdot) \in \mathrm{AA}(\mathbb{T},\mathbb{X})$.

Proof. Obviously, $u(t)$ is a continuous function. Let $\{s_n\}_{n=1}^{\infty} \subset \Pi$. Since v is almost automorphic, there exists a subsequence $\{\tau_n\}_{n=1}^{\infty} \subset \{s_n\}_{n=1}^{\infty}$ such that $h(t) := \lim_{n\to\infty} v(t+\tau_n)$ is well defined for each $t \in \mathbb{T}$.

Now, we consider

$$\begin{aligned} u(t+\tau_n) &= \int_{-\infty}^{t+\tau_n} T(t+\tau_n, s)v(s)\,\Delta s = \int_{-\infty}^{t} T(t+\tau_n, s+\tau_n)v(s+\tau_n)\,\Delta s \\ &= \int_{-\infty}^{t} T(t+\tau_n, s+\tau_n)v_n(s)\,\Delta s, \end{aligned}$$

where $v_n(s) = v(s+\tau_n)$, $n = 1, 2, \ldots$. Also we have

$$\begin{aligned} \|u(t+\tau_n)\| &\le \int_{-\infty}^{t} \|T(t+\tau_n, s+\tau_n)v_n(s)\|\,\Delta s \\ &\le \int_{-\infty}^{t} K_0 e_{\ominus\omega}(t, \sigma(s))\|v_n(s)\|\,\Delta s \\ &\le K_0\|v\| \int_{-\infty}^{t} e_{\ominus\omega}(t, \sigma(s))\,\Delta s \\ &= \frac{K_0\|v\|}{\ominus\omega}[e_{\ominus\omega}(t, -\infty) - e_{\ominus\omega}(t, t)] = \frac{K_0\|v\|(1+\bar{\mu}\omega)}{\omega}, \end{aligned}$$

where $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$.

It is easy to see that $v_n(s) \to h(s)$ as $n \to \infty$ for each $s \in \mathbb{T}$ fixed and any $t \ge s$, and we get

$$\lim_{n\to\infty} u(t+\tau_n) = \int_{-\infty}^{t} T(t, s)h(s)\,\Delta s,$$

by Lebesgue's dominated convergence theorem. Analogously to the above proof, we can obtain

$$\lim_{n\to\infty} \int_{-\infty}^{t-\tau_n} T(t-\tau_n, s)h(s)\,\Delta s = u(t).$$

This shows that $u(t)$ is an almost automorphic function. ∎

LEMMA 4.2. *Let $f = g + \phi \in \mathrm{WPAA}(\mathbb{T}, \rho)$ where $\rho \in U_\infty$ and $\{T(t, s) : t \ge s\}$ is exponentially stable. Then*

$$F(t) := \int_{-\infty}^{t} T(t, s)f(s)\,\Delta s \in \mathrm{WPAA}(\mathbb{T}, \rho).$$

Proof. Let $F(t) = G(t) + \Phi(t)$, where

$$G(t) := \int_{-\infty}^{t} T(t,s)g(s)\,\Delta s \quad \text{and} \quad \Phi(t) := \int_{-\infty}^{t} T(t,s)\phi(s)\,\Delta s.$$

Then $G(\cdot) \in \mathrm{AA}(\mathbb{T},\mathbb{X})$ by Lemma 4.1. Now let us show that $\Phi(\cdot) \in \mathrm{PAA}_0(\mathbb{T},\rho)$. It follows from Theorem 2.15 in [BoG] that

$$\begin{aligned}
\frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \|\Phi(s)\|\,\Delta s &= \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \Big\| \int_{-\infty}^{s} T(s,\theta)\phi(\theta)\,\Delta\theta \Big\|\,\Delta s \\
&\le \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \Delta s \int_{-\infty}^{s} K_0 e_{\ominus\omega}(s,\sigma(\theta))\|\phi(\theta)\|\,\Delta\theta \\
&= \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \Delta s \Big(\Big(\int_{-\infty}^{t_0-r} + \int_{t_0-r}^{s}\Big) K_0 e_{\ominus\omega}(s,\sigma(\theta))\|\phi(\theta)\|\,\Delta\theta\Big) \\
&= \frac{1}{m(r,\rho)} \int_{-\infty}^{t_0-r} \|\phi(\theta)\|\,\Delta\theta \int_{t_0-r}^{t_0+r} K_0 e_{\ominus\omega}(s,\sigma(\theta))\,\Delta s \\
&\quad + \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|\,\Delta\theta \int_{\theta}^{t_0+r} K_0 e_{\ominus\omega}(s,\sigma(\theta))\,\Delta s =: I_1 + I_2.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &= \frac{1}{m(r,\rho)} \int_{-\infty}^{t_0-r} \|\phi(\theta)\|\,\Delta\theta \int_{t_0-r}^{t_0+r} K_0 e_{\ominus\omega}(s,\sigma(\theta))\,\Delta s \\
&= \frac{1}{m(r,\rho)} \int_{-\infty}^{t_0-r} \|\phi(\theta)\|\,\Delta\theta \int_{t_0-r}^{t_0+r} \frac{K_0}{1+\mu(s)\ominus\omega} e_{\ominus\omega}(\sigma(s),\sigma(\theta))\,\Delta s \\
&\le \frac{1}{m(r,\rho)} K_0(1+\bar{\mu}\omega) \int_{-\infty}^{t_0-r} \|\phi(\theta)\|\,\Delta\theta \int_{t_0-r}^{t_0+r} e_{\omega}(\sigma(\theta),\sigma(s))\,\Delta s \\
&= \frac{1}{m(r,\rho)} \frac{K_0(1+\bar{\mu}\omega)}{\omega} \int_{-\infty}^{t_0-r} \|\phi(\theta)\|[e_\omega(\sigma(\theta),t_0-r) - e_\omega(\sigma(\theta),t_0+r)]\,\Delta\theta \\
&\le \frac{1}{m(r,\rho)} \frac{K_0(1+\bar{\mu}\omega)}{\omega} \|\phi\| \Big(\int_{-\infty}^{t_0-r} e_{\ominus\omega}(t_0-r,\sigma(\theta))\,\Delta\theta \\
&\quad - \int_{-\infty}^{t_0-r} e_{\ominus\omega}(t_0+r,\sigma(\theta))\,\Delta\theta \Big)
\end{aligned}$$

$$= \frac{1}{m(r,\rho)} \frac{K_0(1+\bar{\mu}\omega)}{\omega} \frac{1}{\ominus\omega}(e_{\ominus\omega}(t_0-r,-\infty) - e_{\ominus\omega}(t_0-r,t_0-r) - e_{\ominus\omega}(t_0+r,-\infty) + e_{\ominus\omega}(t_0+r,t_0-r)) \to 0 \quad \text{as } r \to \infty,$$

and

$$\begin{aligned} I_2 &= \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|\,\Delta\theta \int_{\theta}^{t_0+r} K_0 e_{\ominus\omega}(s,\sigma(\theta))\,\Delta s \\ &= \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|\,\Delta\theta \int_{\theta}^{t_0+r} \frac{K_0}{1+\mu(s)\ominus\omega} e_{\ominus\omega}(\sigma(s),\sigma(\theta))\,\Delta s \\ &\le \frac{1}{m(r,\rho)} K_0(1+\bar{\mu}\omega) \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|\,\Delta\theta \int_{\theta}^{t_0+r} e_{\omega}(\sigma(\theta),\sigma(s))\,\Delta s \\ &= \frac{1}{m(r,\rho)} \frac{K_0(1+\bar{\mu}\omega)}{\omega} \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|[e_{\omega}(\sigma(\theta),\theta) - e_{\omega}(\sigma(\theta),t_0+r)]\,\Delta\theta \\ &\le \frac{1}{m(r,\rho)} \frac{K_0(1+\bar{\mu}\omega)^2}{\omega} \int_{t_0-r}^{t_0+r} \|\phi(\theta)\|\,\Delta\theta. \end{aligned}$$

Since $\phi \in \mathrm{PAA}_0(\mathbb{T},\rho)$, we have $\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{t_0-r}^{t_0+r} \|\phi(s)\|\,\Delta s = 0$. Hence $\lim_{r\to\infty} I_2 = 0$. ∎

THEOREM 4.1. *Under assumptions* (H_1)–(H_4) *above. Equation* (4.2) *has a unique mild solution in* $\mathrm{WPAA}(\mathbb{T},\rho)$ *provided* $K_0L_f(1+\bar{\mu}\omega)/\omega < 1$, *where* $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$.

Proof. Consider the nonlinear operator Γ given by

$$(\Gamma x)(t) := \int_{-\infty}^{t} T(t,s) f(s,x(s))\,\Delta s.$$

By Lemma 4.2, we see that Γ maps $\mathrm{WPAA}(\mathbb{T},\rho)$ into $\mathrm{WPAA}(\mathbb{T},\rho)$.

Now if $x,y \in \mathrm{WPAA}(\mathbb{T},\rho)$, we have

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\| &= \Big\| \int_{-\infty}^{t} T(t,s)(f(s,x(s)) - f(s,y(s)))\,\Delta s \Big\| \\ &\le K_0 L_f \int_{-\infty}^{t} e_{\ominus\omega}(t,\sigma(s)) \|x(s)-y(s)\|\,\Delta s \\ &\le -\frac{K_0L_f}{\ominus\omega}\|x-y\|_\infty \le \frac{K_0L_f(1+\bar{\mu}\omega)}{\omega}\|x-y\|_\infty, \qquad \forall t \in \mathbb{T}. \end{aligned}$$

Thus

$$\|\Gamma x - \Gamma y\|_\infty \le \frac{K_0 L_f(1+\bar{\mu}\omega)}{\omega}\|x-y\|_\infty.$$

The conclusion follows by the contraction principle. ∎

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