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On growth and zeros of differences of some meromorphic functions

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Abstract. Let f be a transcendental meromorphic function and

$$g_k(z) = f(z+c_1) + \cdots + f(z+c_k) - kf(z),$$
$$G(z) = \frac{f(z+c_1) + f(z+c_2) + \cdots + f(z+c_k) - kf(z)}{f(z)}.$$

A number of results are obtained concerning zeros and fixed points of the difference $g_k(z)$ and the divided difference $G(z)$.

1. Introduction and main results. Recently, there has been an increasing interest in studying difference equations in the complex plane. Halburd and Korhonen [HK1, HK2] established a version of Nevanlinna theory based on difference operators. Bergweiler and Langley [BL] investigated the existence of zeros of $\triangle f$ and $\frac{\triangle f(z)}{f(z)}$, and obtained several profound and significant results, which may be viewed as discrete analogues of the following theorem on the zeros of f'.

THEOREM A ([BE, ELR, H]). *Let f be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} T(r,f)/r = 0.$$

Then f' has infinitely many zeros.

If f satisfies the hypotheses of Theorem A, by Hurwitz's theorem we know that if z_0 is a zero of $f'(z)$ then $\triangle_c f(z) = f(z+c) - f(z)$ has a zero near z_0 for all sufficiently small $c \in \mathbb{C}\backslash\{0\}$. Hence it is natural to ask whether $\triangle_c f(z)$ must have infinitely many zeros or not. Bergweiler and Langley [BL] answered this problem, and obtained the following theorems.

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Theorem B ([BL]). *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let f be a transcendental entire function with*

$$\rho(f) \leq \rho < 1/2 + \delta_0 < 1.$$

Then

$$G(z) = \frac{f(z+1) - f(z)}{f(z)}$$

has infinitely many zeros.

Here $\rho(f)$ denotes the order of growth of the meromorphic function $f(z)$. In what follows $\lambda(f)$ and $\lambda(1/f)$ denote the exponents of convergence of the zeros and poles of $f(z)$, respectively. In this paper, we shall assume that the reader is familiar with the basic concepts of Nevanlinna theory (see [H1, YY]).

Theorem C ([BL]). *Let f be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $\triangle_c f(z) = f(z+c) - f(z)$ has infinitely many zeros.*

Chen and Shon [CS1] considered zeros and fixed points of differences and divided differences of entire functions with $\rho(f) = 1$ and obtained the following theorem.

Theorem D ([CS1]). *Let $c \in \mathbb{C} \setminus \{0\}$ and let f be a transcendental entire function with $\rho(f) = \rho = 1$ that has infinitely many zeros with $\lambda(f) = \lambda < 1$. Then $\triangle_c f(z) = f(z+c) - f(z)$ has infinitely many zeros and infinitely many fixed points.*

Recently, Chen and Shon [CS2] considered the following three problems:

(i) What conditions will guarantee that the difference $f(z+c) - f(z)$ has infinitely many zeros for a meromorphic function f?

(ii) What is the exponent of convergence of zeros of the difference $f(z+c) - f(z)$ if it has infinitely many zeros?

(iii) What can we say about the zeros of

$$f(z+c) - f(z) - l(z) \quad \text{and} \quad \frac{f(z+c) - f(z)}{f(z)} - l(z),$$

where $l(z)$ is a polynomial?

For question (i), the following theorem shows that the conditions that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles will guarantee that the difference $f(z+c) - f(z)$ has infinitely many zeros, without any hypothesis on c.

Theorem E ([CS2]). *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a meromorphic function of order of growth $\rho(f) = \rho \leq 1$. Suppose that f satisfies $\lambda(1/f) <$*

$\lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then

$$\triangle_c f(z) = f(z+c) - f(z)$$

has infinitely many zeros and satisfies $\lambda(\triangle_c f) = \lambda(f)$.

Concerning question (ii), Theorem E also shows that if $f(z+c) - f(z)$ has infinitely many zeros, then $\lambda(f(z+c) - f(z)) = \lambda(f)$

As for question (iii), the following two theorems show that

$$f(z+c) - f(z) - l(z) \text{ and } \frac{f(z+c) - f(z)}{f(z)} - l(z)$$

have infinitely many zeros, respectively.

THEOREM F ([CS2]). *Let c and $f(z)$ satisfy the conditions of Theorem E. Suppose that $l(z)$ is a polynomial. Then $\triangle_c f(z) - l(z)$ has infinitely many zeros and satisfies $\lambda(\triangle_c f - l) = \rho(f)$.*

THEOREM G ([CS2]). *Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and f a transcendental meromorphic function of order of growth $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental moromorphic function with $\rho(h) < 1$. Suppose that $l(z)$ is a nonconstant polynomial. Then*

$$G_1(z) = \frac{f(z+c) - f(z)}{f(z)} - l(z)$$

has infinitely many zeros.

The aim of the paper is to generalize Theorems E–G. In [CS2], Chen and Shon consider the zeros of the differences $\triangle_c f(z)$ under the assumption $\rho(f) \leq 1$. We study the zeros of the sum $g_k(z) = \triangle_{c_1} f(z) + \cdots + \triangle_{c_k} f(z)$ under the assumption $\rho(f) < \infty$. In particular, we study the densities of the zeros of $g_k(z) - l(z)$ and of $G_k(z) = \frac{f(z+c_1)+\cdots+f(z+c_k)-kf(z)}{f(z)} - l(z)$. We prove the following three theorems.

THEOREM 1.1. *Let $f(z)$ be a finite order meromorphic function with $\lambda(1/f) < \lambda(f) < 1$. Let $c_1, \dots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$, let $g_k(z) = f(z+c_1) + \cdots + f(z+c_k) - kf(z)$, and suppose $g_k(z) \not\equiv 0$. Then:*

(i) *If $\rho(f) = \rho < 1$, we have $\lambda(g_k) = \lambda(f)$.*
(ii) *If $1 \leq \rho(f) = \rho < \infty$, we have $\lambda(g_k) \geq \lambda(f)$.*

THEOREM 1.2. *Let f, c_j $(j = 1, \dots, k), g_k(z)$ satisfy the conditions of Theorem 1.1. Suppose that $l(z)$ is a nonconstant polynomial, and let $g_k(z, L) := g_k(z) - l(z)$. Then:*

(i) *If $\rho(f) < 1$, we have $\lambda(g_k(z, L)) = \rho(f)$.*
(ii) *If $1 \leq \rho(f) < \infty$, we have $\lambda(g_k(z, L)) \geq 1$.*

THEOREM 1.3. *Let f be a transcendental meromorphic function of order of growth $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\rho(h) < 1$. Let $c_1, \dots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \dots + c_k \neq 0$. Suppose that $l(z)$ is a nonconstant polynomial. Then*

$$G_k(z) = \frac{f(z+c_1) + \dots + f(z+c_k) - kf(z)}{f(z)} - l(z) \tag{1.3}$$

has infinitely many zeros.

REMARK. In the special case when $l(z) = z$, one obtains results on fixed points.

2. Some lemmas. In order to prove our theorems, we need the following lemmas and notions.

Following Hayman [H2, pp. 75–76], we define an *ε-set* to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If E is an ε-set then the set of $r \geq 1$ for which the circle $S(0, r) = \{z \in \mathbb{C} : |z| = r\}$ meets E has finite logarithmic measure, and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Bergweiler and Langley [BL] have shown that differences of meromorphic functions of order less than one behave asymptotically like their derivatives in the complex plane.

LEMMA 2.1 ([BL]). *Let f be transcendental and meromorphic of order less than 1 in the plane. Let $h > 0$. Then there exists an ε-set E such that*

$$f(z+c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leq h$.

The following lemma due to Bergweiler and Langley [BL] gives an asymptotic identity involving a meromorphic function of order less than one, its derivative and its shift.

LEMMA 2.2 ([BL]). *Let f be a function transcendental and meromorphic in the plane of order less than 1. Let $h > 0$. Then there exists an ε-set E such that*

$$\frac{f'(z+c)}{f(z+c)} \to 0, \qquad \frac{f(z+c)}{f(z)} \to 1 \qquad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E,$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large z not in E the function f has no zeros or poles in $|\varsigma - z| \leq h$.

In Lemma 2.1 of [BL], Bergweiler and Langley prove that $\triangle f(z) = f(z+c) - f(z)$ and $\frac{\triangle f(z)}{f}$ are both transcendental. The following lemma is a generalization of Lemma 2.1 of [BL] and states that $g_k(z) = \triangle_{c_1} f(z) + \cdots + \triangle_{c_k} f(z)$ and $G(z) = g_k(z)/f(z)$ are also transcendental.

LEMMA 2.3. *Let f be a transcendental meromorphic function with $\rho(f) = \rho < 1$. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$. Then $g_k(z)$ and $G(z) = g_k(z)/f(z)$ are both transcendental.*

Proof. Without loss of generality, it may be assumed that $c_1 = 1$ and $\operatorname{Re} c_2 = \min\{\operatorname{Re} c_i : i = 2, \ldots, k\}$. Assume that $g_k(z)$ is a rational function. Then

$$f(z+1) + f(z+c_2) + \cdots + f(z+c_k) = R(z) + kf(z), \tag{2.1}$$

where $R(z)$ is a rational function. Suppose that $A = \{x_j + iy_j : j = 1, \ldots, s\}$ consists of all poles of $R(z)$.

Set

$$M = 2\max\{|x_j| + |y_j| + 1 + \cdots + |c_k| : j = 1, \ldots, s\}$$

and

$$\begin{aligned} D_1 &= \{z : \operatorname{Re} z > M\}, & D_2 &= \{z : \operatorname{Re} z < -M\}, \\ D_3 &= \{z : \operatorname{Im} z > M\}, & D_4 &= \{z : \operatorname{Im} z < -M\}. \end{aligned}$$

Now we prove that $f(z)$ has at most finitely many poles. Suppose, contrary to the assertion, that $f(z)$ has infinitely many poles. Then there is at least one D_j such that $f(z)$ has infinitely many poles in D_j.

If $f(z)$ has infinitely many poles in D_1, let z_0 be one. If $\operatorname{Re} c_2 \geq 0$, then for each $m_i \in \mathbb{N}$, $i = 1, \ldots, k$, $z_{m_1,\ldots,m_k} = z_0 + m_1 + m_2c_2 + \cdots + m_kc_k \in D_1$ and $R(z_{m_1,\ldots,m_k}) \neq \infty$. By (2.1), we find that $f(z)$ has an infinite sequence of poles of the form

$$\{z_{m_1,\ldots,m_k} = z_0 + m_1 + m_2c_2 + \cdots + m_kc_k : m_i \in \mathbb{N}\ (1 \leq i \leq k)\}.$$

Moreover, it can be seen from (2.1) that for each pole in this sequence there is another pole within a distance of $1 + \cdots + |c_k|$, and so $\lambda(1/f) \geq 1$, a contradiction.

If $\operatorname{Re} c_2 < 0$, and there exist some c_j $(2 \leq j \leq k)$ such that $c_j = c_2$; without loss of generality, we may suppose that $c_2 = \cdots = c_t$ $(2 \leq t \leq k)$. Then we can rewrite (2.1) as

$$\begin{aligned} f(z+1-c_2) + f(z+c_{t+1}-c_2) + \cdots + f(z+c_k-c_2) - kf(z-c_2) \\ = R(z-c_2) - (t-1)f(z). \end{aligned} \tag{2.2}$$

For each $m_i \in \mathbb{N}$, $i = 1, 2, t+1, \ldots, k$,

$$\begin{aligned} z^*_{m_1,m_2,m_{t+1},\ldots,m_k} = z_0 &+ m_1(1-c_2) - m_2c_2 \\ &+ m_{t+1}(c_{t+1}-c_2) + \cdots + m_k(c_k-c_2) \in D_1 \end{aligned}$$

and $R(z^*_{m_1,m_2,m_{t+1},\cdots,m_k}) \neq \infty$. From (2.2), we find that $f(z)$ has an infinite sequence of poles of the form

$$\{z^*_{m_1,m_2,m_{t+1},\cdots,m_k} = z_0 + m_1(1-c_2) - m_2c_2 + m_{t+1}(c_{t+1}-c_2) + \cdots + m_k(c_k - c_2),\ m_i \in \mathbb{N}\ (i = 1, 2, t+1, \ldots, k)\}.$$

So $\lambda(1/f) \geq 1$, a contradiction.

If f has infinitely many poles in D_2 (or D_3, or D_4), using a similar method, we obtain $\lambda(1/f) \geq 1$, a contradiction. Hence f has at most finitely many poles.

Thus, there exists a rational function R_1 such that $h(z) = f(z) - R_1(z)$ is a transcendental entire function. By (2.1), we have

$$h(z+c_1) + \cdots + h(z+c_k) = kh(z) + P(z), \tag{2.3}$$

where $P(z) = R(z) + kR_1(z) - R_1(z+c_1) - \cdots - R_k(z+c_k)$. Since $h(z+c_j)$ $(j = 1, \ldots, k)$ and $h(z)$ are entire functions, we infer that $P(z)$ is a polynomial. By Lemma 2.1, there exists an ε-set E such that

$$h(z+c_j) - h(z) = c_j h'(z)(1+o(1))\ (j = 1, \ldots, k) \text{ as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{2.4}$$

If $P(z) \equiv 0$, by (2.3) and (2.4), as $z \to \infty$ in $\mathbb{C} \setminus E$, we have

$$(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = 0,$$

and since $c_1 + \cdots + c_k \neq 0$, we obtain $h'(z) = 0$ (as $z \notin E$). This is impossible. Hence $P(z) \not\equiv 0$. Set $\deg P = l \geq 0$; then $P(z) = cz^l(1+o(1))$, where $c\ (\neq 0)$ is a constant. By (2.3) and (2.4), as $z \to \infty$ in $\mathbb{C} \setminus E$, we get

$$(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = cz^l(1 + o(1)),$$

which contradicts the fact that $h'(z)$ is transcendental.

Next, we assume that $G(z)$ is a rational function. Then

$$\frac{f(z+c_1) + \cdots + f(z+c_k) - kf(z)}{f(z)} = \theta(z),$$

where $\theta(z)$ is a rational function, By Lemma 2.1, there exists an ε-set E such that

$$\frac{(c_1 + \cdots + c_k)f'(z)(1 + o(1))}{f(z)} = \theta(z) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E; \tag{2.5}$$

however, since $f(z)$ is transcendental and has either infinitely many poles or infinitely many zeros, we conclude that $f'(z)/f(z)$ must be transcendental, so (2.5) is impossible.

Remark. Lemma 2.3 is also proved in [LY], but the methods are partly different.

The following lemma is the classical logarithmic derivative estimate due to Gundersen [G].

Lemma 2.4 ([G]). *Let f be a transcendental meromorphic function with $\rho(f) = \rho < \infty$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| \notin E \cup [0, 1]$ and for any integers k and j such that $k > j \geq 0$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

The following lemma is a generalization of Borel's Theorem on combinations of entire functions.

Lemma 2.5 ([YY, pp. 79–80]). *Let $f_j(z)$ $(j = 1, \dots, n)$ $(n \geq 2)$ be meromorphic functions, and suppose that there are entire functions $g_j(z)$ $(j = 1, \dots, n)$ that satisfy:*

(i) $f_1(z)e^{g_1(z)} + \dots + f_k(z)e^{g_k(z)} \equiv 0$.
(ii) *When $1 \leq j < k \leq n$, then $g_j(z) - g_k(z)$ is not a constant.*
(iii) *When $1 \leq j \leq n$, $1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty,\ r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure.

Then $f_j \equiv 0$ $(j = 1, \dots, n)$.

3. Proof of Theorem 1.1

Proof of Claim (i). Suppose that $\lambda(1/f) < \lambda(f) < 1$ and $\rho(f) < 1$. By Lemma 2.3, $g_k(z)$ is transcendental. Let $f(z) = u(z)/v(z)$, where $u(z)$ and $v(z)$ are the canonical products ($v(z)$ may be a polynomial) formed by the zeros and the poles of $f(z)$, respectively, and

$$\lambda(1/f) = \lambda(v) = \rho(v) < \lambda(f) = \lambda(u) = \rho(u).$$

By Lemma 2.1, there exists an ε-set E such that

$$g_k(z) = (c_1 + \dots + c_k)f'(z)(1 + o(1)) \quad \text{as } z \to \infty \text{ in } \mathbb{C} \setminus E. \tag{3.1}$$

Set

$$H = \{|z| : z \in E,\ g_k(z) = 0 \text{ or } f'(z) = 0\}.$$

Then H is of finite linear measure. By (3.1), for $|z| = r \notin H$, we obtain

$$\begin{aligned} |g_k(z) - (c_1 + \dots + c_k)f'(z)| &= |o(1)(c_1 + \dots + c_k)f'(z)| \\ &< |g_k(z)| + |(c_1 + \dots + c_k)f'(z)|. \end{aligned} \tag{3.2}$$

Thus $g_k(z)$ and $-(c_1 + \dots + c_k)f'(z)$ satisfy the assumptions of Rouché's theorem. Applying Rouché's theorem and (3.2), for $|z| = r \notin H$ we obtain

$$n(r, 1/g_k) - n(r, g_k) = n(r, 1/f') - n(r, f'). \tag{3.3}$$

Since $f' = (u'(z)v(z) - u(z)v'(z))/v^2(z)$, $\lambda(1/f) < \lambda(f) = \rho(f) < 1$, and $\rho(f') = \rho(f)$, we have

$$\lambda(1/f') = \lambda(1/f) < \lambda(f) = \rho(f) = \rho(f').$$

From this and $g_k(z) = f(z+c_1) + \cdots + f(z+c_k) - kf(z)$, we obtain

$$\lambda(1/g_k) \leq \lambda(1/f) < \lambda(f) = \lambda(f'). \tag{3.4}$$

Hence, with (3.3) and (3.4), we obtain

$$\lambda(g_k) = \lambda(f') = \lambda(f).$$

Thus (i) holds.

Proof of Claim (ii). Since $1 \leq \rho(f) < \infty$ and $\lambda(1/f) < \lambda(f) < 1$, it follows from the Hadamard factorization theorem that

$$f(z) = h(z)e^{P(z)} = \frac{u(z)}{v(z)}e^{P(z)},$$

where $P(z)$ is a nonconstant polynomial, $h(z)$ is a meromorphic function such that $h(z) = u(z)/v(z)$, $u(z)$ and $v(z)$ are the canonical products ($v(z)$ may be a polynomial) formed by the zeros and the poles of $f(z)$, respectively, and

$$\lambda(1/f) = \lambda(v) = \rho(v) = \lambda(1/h) < \lambda(f) = \lambda(u) = \rho(u) = \lambda(h) = \rho(h) < 1.$$

Hence

$$\begin{aligned} g_k(z) &= f(z+c_1) + \cdots + f(z+c_k) - kf(z) \\ &= h(z+c_1)e^{P(z)+R_1(z)} + \cdots + h(z+c_k)e^{P(z)+R_k(z)} - kh(z)e^{P(z)} \\ &= \big(h(z+c_1)e^{R_1(z)} + \cdots + h(z+c_k)e^{R_k(z)} - kh(z)\big)e^{P(z)} = w(z)e^{P(z)}, \end{aligned}$$

where $R_j(z) = P(z+c_j) - P(z)$ $(j = 1, \ldots, k)$, and

$$w(z) = h(z+c_1)e^{R_1(z)} + \cdots + h(z+c_k)e^{R_k(z)} - kh(z).$$

From this, we get $\lambda(1/w) \leq \lambda(1/h) = \lambda(1/f) < \lambda(f) < 1$. Since $g_k(z) \not\equiv 0$, we have $w(z) \not\equiv 0$.

Next, suppose, contrary to the assertion, that $\lambda(g_k) < \lambda(f) < 1$.

If $1 \leq \rho(w) < \infty$, then there exist a nonconstant polynomial $R_0(z)$ and a nonzero meromorphic function $Q(z)$ such that

$$w(z) = Q(z)e^{R_0(z)} = \frac{u_1(z)}{v_1(z)}e^{R_0(z)}, \tag{3.5}$$

where $Q(z) = u_1(z)/v_1(z)$ with $u_1(z)$ and $v_1(z)$ being the canonical products formed by the zeros and the poles of $w(z)$, respectively, and

$$\begin{gathered} \lambda(1/Q) = \lambda(v_1) = \rho(v_1) = \lambda(1/w) \leq \lambda(1/f) < 1, \\ \lambda(u_1) = \rho(u_1) = \lambda(Q) = \lambda(w) = \lambda(g_k) < 1. \end{gathered}$$

So, we obtain $\rho(Q) = \max\{\lambda(Q), \lambda(1/Q)\} < 1$. Let $c_{k+1} = 0$, $h(z) = h(z)e^{R_{k+1}(z)}$, where $R_{k+1}(z) = 0$. We next consider two cases.

Case (1.1): *There exist $i, j \in \{0, 1, \dots, k+1\}$ such that $R_j(z) - R_i(z) = A$ is a constant.* We need to consider two subcases.

Subcase (1.1.1): *$R_j(z) - R_0(z)$ is not a constant for any $j \in \{1, \dots, k+1\}$.* Then there exist $1 \le i, j \le k+1$ such that $R_j(z) - R_i(z) = A$ is a constant. Hence $P(z + c_j) - P(z + c_i) = A$. Since $P(z)$ is a polynomial, it must have the form $P(z) = az + d$ and $a \neq 0$. Hence $R_j = ac_j$ is a constant for $j = 1, \dots, k+1$. From

$$w(z) = h(z + c_1)e^{R_1(z)} + \dots + h(z + c_k)e^{R_k(z)} - kh(z),$$

we get $\rho(w) < 1$, a contradiction.

Subcase (1.1.2): *There exists a $j \in \{1, \dots, k+1\}$ such that $R_j(z) - R_0(z) = A$ is a constant.* If there also exists $i \in \{1, \dots, j-1, j+1, \dots, k+1\}$ such that $R_i(z) - R_0(z) = B$ is a constant, then $R_j(z) - R_i(z) = A - B$. By Subcase (1.1.1), R_j is a constant for $j = 1, \dots, k+1$. Therefore, R_0 is then a constant, a contradiction. If now for arbitrary $i, \alpha \in \{0, 1, \dots, j-1, j+1, \dots, k+1\}$, $R_i(z) - R_\alpha(z)$ is not a constant, then

$$h(z + c_1)e^{R_1(z)} + h(z + c_2)e^{R_2(z)} + \dots + (e^A h(z + c_j) - Q(z))e^{R_0(z)} + h(z + c_k)e^{R_k(z)} - kh(z) = 0. \tag{3.6}$$

Since $\deg(R_i(z) - R_\alpha(z)) \ge 1$, $e^{R_i(z) - R_\alpha(z)}$ is of regular growth (see, e.g., [H1, p. 7]), and $\rho(h(z + c_i)) < 1$ and $\rho(e^A h(z + c_j) - Q(z)) < 1$, we conclude that

$$T(r, h(z + c_i)) = o\{T(r, e^{R_i(z) - R_\alpha(z)})\},$$
$$T(r, e^A h(z + c_j) - Q(z)) = o\{T(r, e^{R_i(z) - R_\alpha(z)})\}.$$

Thus, from Lemma 2.5 and (3.6), we have $h(z) \equiv 0$, a contradiction.

Case (1.2): *$R_j(z) - R_i(z)$ is not a constant for any $i, j \in \{0, 1, \dots, k+1\}$, $i \neq j$.* By Lemma 2.5, $h(z + c_j) \equiv 0$ $(j = 1, \dots, k)$, $h(z) \equiv 0$, a contradiction.

Therefore, $\rho(w) < 1$. Then there exists a nonzero meromorphic function $Q(z)$ such that

$$w(z) = h(z + c_1)e^{R_1(z)} + \dots + h(z + c_k)e^{R_k(z)} - kh(z) = Q(z), \tag{3.7}$$

where $\rho(Q) = \max\{\lambda(Q), \lambda(1/Q)\} < 1$. We break the rest of the proof into three cases.

Case (2.1): *There exists exactly one $j \in \{1, \dots, k\}$ such that $R_j(z)$ is a nonconstant polynomial.* From (3.7), we get $\rho(w) \ge 1$, a contradiction.

Case (2.2): *There exist at least two $i, j \in \{1, \dots, k\}$ such that $R_i(z)$, $R_j(z)$ are nonconstant polynomials.* Without loss of generality, we suppose

$R_1(z), \dots, R_m(z)$ ($m \geq 2$) are nonconstant polynomials, while $R_{m+1}, \dots, R_k$ are constants. We now rewrite $w(z)$ as follows:

$$\begin{aligned} w(z) &= h(z+c_1)e^{R_1(z)} + \cdots + h(z+c_m)e^{R_m(z)} \\ &\quad + h(z+c_{m+1})e^{R_{m+1}(z)} + \cdots + h(z+c_k)e^{R_k(z)} - kh(z) = Q(z). \end{aligned}$$

If there exist $1 \leq i, j \leq m$ such that $R_i - R_j$ is a constant, we may apply Subcase (1.1.1) to deduce that $R_i(z)$ is a constant for $i = 1, \dots, m$, a contradiction. Thus for arbitrary $i, j \in \{1, \dots, m\}$ with $i \neq j$, $R_i - R_j$ is not a constant. By Lemma 2.5, we have $h(z + c_j) \equiv 0$, a contradiction.

Case (2.3): *R_j is constant for all $j \in \{1, \dots, k\}$.* Using the method of Subcase (1.1.1), we see that $P(z) = az + b$, $a \neq 0$. Substituting this into $w(z)$, we have

$$w(z) = h(z+c_1)e^{ac_1} + \cdots + h(z+c_k)e^{ac_k} - kh(z).$$

By Lemma 2.2, there exists an ε-set E such that

$$h(z+c) = h(z)(1+o(1)) \tag{3.8}$$

as $z \to \infty$ in $\mathbb{C} \setminus E$. By (3.8), we obtain

$$\begin{aligned} w(z) &= (e^{ac_1} + \cdots + e^{ac_k})h(z)(1+o(1)) - kh(z) \\ &= (e^{ac_1} + \cdots + e^{ac_k} - k)h(z)(1+o(1)). \end{aligned} \tag{3.9}$$

By (3.9) and $w(z) \not\equiv 0$, we have $e^{ac_1} + \cdots + e^{ac_k} \neq k$. Since $h(z)$ is transcendental, we know that $w(z)$ is transcendental. Set

$$H = \{|z| : z \in E,\ w(z) = 0 \text{ or } h(z) = 0\},$$

Then H is of finite linear measure. By (3.9), for $|z| = r \notin H \cup [0,1]$, we obtain

$$\begin{aligned} &|w(z) - (e^{ac_1} + \cdots + e^{ac_k} - k)h(z)| \\ &\quad = |(e^{ac_1} + \cdots + e^{ac_k} - k)o(1)| < |w(z)| + |(e^{ac_1} + \cdots + e^{ac_k} - k)h(z)|. \end{aligned} \tag{3.10}$$

Applying Rouché's theorem and (3.10), and using a similar method to the proof of (i), we obtain

$$\lambda(w) = \lambda(h) = \lambda(u) = \lambda(f),$$

a contradiction. Hence $\lambda(g_k) = \lambda(w) \geq \lambda(f)$. Theorem 1.1 is thus proved.

4. Proof of Theorem 1.2

Proof of Claim (i). Since f satisfies $\lambda(f) > \lambda(1/f)$ and $\rho(f) < 1$, from Theorem 1.1 and the proof of (i) there, we obtain

$$\rho(f) = \lambda(f) = \lambda(g_k) = \rho(g_k), \qquad \rho(g_k) > \lambda(1/f) \geq \lambda(1/g_k).$$

Since $g_k(z, L) = g_k(z) - l(z)$, where $l(z)$ is a nonzero polynomial, we have

$$\lambda(1/g_k(z,L)) = \lambda(1/g_k) < \lambda(g_k) = \rho(g_k) = \rho(g_k(z,L)) < 1.$$

As $\lambda(1/g_k(z,L)) < \rho(g_k(z,L)) < 1$, we obtain $\lambda(g_k(z,L)) = \rho(g_k(z,L))$. Hence, $\lambda(g_k(z,L)) = \rho(g_k(z,L)) = \rho(g_k) = \lambda(f) = \rho(f)$.

Proof of Claim (ii). Suppose that $\lambda(g_k(z,L)) < 1$. Then $1 \le \rho(g_k(z,L)) = \rho(g_k - l) = \rho(g_k) < \infty$. We rewrite $g_k(z,L)$ as follows:

$$g_k(z,L) = g_k(z) - l(z) = h_*(z)e^{L(z)},$$

where $L(z)$ is a nonconstant polynomial and $h_*(z)$ is a meromorphic function such that

$$\lambda(h_*) = \lambda(g_k(z,L)) < 1, \qquad \lambda(1/h_*) = \lambda(1/g_k(z,L)) \le \lambda(1/f) < 1. \tag{4.2}$$

With (4.2), we have

$$\rho(h_*) = \max\{\lambda(h_*), \lambda(1/h_*)\} < 1.$$

Since $g_k(z) - l(z) \not\equiv 0$, we obtain $h_*(z) \not\equiv 0$.

From (4.1) and $f(z) = h(z)e^{P(z)}$, we have

$$\begin{aligned} h(z+c_1)e^{P(z+c_1)} + \cdots + h(z+c_k)e^{P(z+c_k)} \\ - kh(z)e^{P(z)} - l(z) - h_*(z)e^{L(z)} \equiv 0. \end{aligned} \tag{4.3}$$

Let $h(z)e^{P(z)} = h(z+c_0)e^{P(z+c_0)}$, where $c_0 = 0$. We consider three cases.

Case (1): *There exist $i,j \in \{0,1,\dots,k\}$ such that $P(z+c_i) - P(z+c_j) = A$ is a constant.* Since $P(z)$ is a polynomial, it must have the form $P(z) = az + d$ and $a \ne 0$. Hence

$$[h(z+c_1)e^{ac_1} + \cdots + h(z+c_k)e^{ac_k} - kh(z)]e^{az+d} - h_*(z)e^{L(z)} - l(z) = 0. \tag{4.4}$$

If $L(z) - az - d \equiv C$, then

$$[h(z+c_1)e^{ac_1} + \cdots + h(z+c_k)e^{ac_k} - kh(z) - h_*(z)e^{C}]e^{az+d} - l(z) = 0,$$

which is impossible. If $L(z) - az - d \not\equiv C$, from Lemma 2.5 we get $l(z) \equiv h_*(z) \equiv 0$, a contradiction.

Case (2): *There exists $i \in \{0,1,\dots,k\}$ such that $P(z+c_i) - L(z) = A$.* If there also exists $j \in \{0,1,\dots,i-1,i+1,\dots,k\}$ such that $P(z+c_j) - L(z) = B$, then $P(z+c_j) - P(z+c_i) = A - B$. Using the method of Case (1), we reach a contradiction. If for arbitrary $j \ne i$, we have $P(z_j) - L(z) \not\equiv B$, then

$$\begin{aligned} h(z+c_1)e^{P(z+c_1)} + h(z+c_2)e^{P(z+c_2)} + \cdots + (e^{A}h(z+c_j) - h_*(z))e^{L(z)} \\ + \cdots + h(z+c_k)e^{P(z+c_k)} - kh(z)e^{P(z)} - l(z) = 0. \end{aligned}$$

From Lemma 2.5, we have $l(z) \equiv h(z) \equiv 0$, a contradiction.

Case (3): *For arbitrary $i,t,j \in \{0,1,\dots,k\}$, $i \ne t$, such that $P(z+c_i) - P(z+c_t)$ is not a constant, $P(z+c_j) - L(z)$ is also not a constant.* From

Lemma 2.5, we obtain $h(z+c_j) \equiv 0$ and $l(z) \equiv 0$, a contradiction. Therefore, $\lambda(g_k(z,L)) \geq 1$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3. Let $\rho(f) = \rho < 1$. By Lemma 2.3, we see that $\frac{f(z+c_1)+\cdots+f(z+c_k)-kf(z)}{f(z)}$ is transcendental, and hence so is $G_k(z)$. By Lemma 2.1, there exists an ε-set E such that

$$h(z+c) - h(z) = cf'(z)(1+o(1)) \tag{5.1}$$

as $z \to \infty$ in $\mathbb{C} \setminus E$. By Lemma 2.4, for a given $\varepsilon > 0$, there exists a set $H_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin H_1 \cup [0,1]$, we have

$$\left|\frac{f'(z)}{f(z)}\right| \leq |z|^{\rho-1+\varepsilon}. \tag{5.2}$$

Set

$$H_2 = \{|z| : z \in E,\ G_k(z) = 0 \text{ or } l(z) = 0\}.$$

Then H_2 has finite linear measure. For large $|z| = r \notin [0,1] \cup H_1 \cup H_2$, from (5.1) and (5.2), we see

$$\begin{aligned} |G_k(z) + l(z)| &= \left|(c_1 + \cdots + c_k)\frac{f'(z)}{f(z)}(1+o(1))\right| \\ &\leq |(c_1 + \cdots + c_k)(1+o(1))|\,|z|^{\rho-1+\varepsilon} \\ &< |G_k(z)| + |l(z)|, \end{aligned} \tag{5.3}$$

since $\rho < 1$. Thus $G_k(z)$ and $l(z)$ satisfy the conditions of Rouché's theorem. Applying Rouché's theorem and (5.3), for $|z| = r \notin [0,1] \cup H_1 \cup H_2$ we have

$$n(r, 1/G_k) - n(r, G_k) = n(r, 1/l) - n(r, l) = \deg l. \tag{5.4}$$

Since G_k is transcendental and $\rho(G_k) < 1$, we know that at least one of $n(r, G_k) \to \infty$ and $n(r, 1/G_k) \to \infty$ is true as $r \to \infty$. Hence, by (5.4), both are true. Hence $G_k(z)$ must have infinitely many zeros.

Suppose now that $f(z) = h(z)e^{az}$, where $a \neq 0$ is a constant, and $h(z)$ is a transcendental meromorphic function such that $\rho(h) < 1$. Substituting this into $G_k(z)$, we obtain

$$G_k(z) = \frac{h(z+c_1)e^{ac_1} + \cdots + h(z+c_k)e^{ac_k} - kh(z)}{h(z)} - l(z). \tag{5.5}$$

If $e^{ac_1} + \cdots + e^{ac_k} - k = 0$, then using the same method as in the first part of the proof, and (5.5), we deduce that $G_k(z)$ has infinitely many zeros.

If $e^{ac_1} + \cdots + e^{ac_k} - k \neq 0$, then by Lemma 2.1 and (5.2), for a given $\varepsilon > 0$, there exist an ε-set E and a set $H_1 \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin E \cup [0,1] \cup H_1$, we have

$$\begin{aligned}(5.6)\qquad &\left|\frac{h(z+c_1)e^{ac_1}+\cdots+h(z+c_k)e^{ac_k}-kh(z)}{h(z)}\right|\\ &\quad=\left|(c_1e^{ac_1}+\cdots+c_ke^{ac_k})\frac{h'}{h}+e^{ac_1}+\cdots+e^{ac_k}-k\right|\\ &\quad\le|c_1e^{ac_1}+\cdots+c_ke^{ac_k}|\,|z|^{\rho-1+\varepsilon}+|e^{ac_1}+\cdots+e^{ac_k}-k|.\end{aligned}$$

Set

$$H_2=\{|z|:z\in E,\,G_k(z)=0\text{ or }l(z)=0\}.$$

Then H_2 have finite linear measure. From (5.5) and (5.6), we see that

$$\begin{aligned}(5.7)\qquad &|G_k(z)+l(z)|\\ &\quad=|c_1e^{ac_1}+\cdots+c_ke^{ac_k}||z|^{\rho-1+\varepsilon}+|e^{ac_1}+\cdots+e^{ac_k}-k|\\ &\quad<|G_k(z)|+|l(z)|.\end{aligned}$$

Thus $G_k(z)$ and $l(z)$ satisfy the assumptions of Rouché's theorem. Applying Rouché's theorem and (5.7), for $|z|=r\notin[0,1]\cup H_1\cup H_2$ we obtain (5.4). Using the same argument as in the proof of Lemma 2.3, we show that $G_k(z)$ is transcendental. Applying the same method as in the first part of the proof, we obtain $n(r,1/G_k)\to\infty$. Theorem 1.3 is proved.

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