On growth and zeros of differences of some meromorphic functions

by YONG LIU (Shaoxing and Joensuu) and HONGXUN YI (Jinan)

Abstract. Let \( f \) be a transcendental meromorphic function and
\[
g_k(z) = f(z + c_1) + \cdots + f(z + c_k) - kf(z),
\]
\[
G(z) = \frac{f(z + c_1) + f(z + c_2) + \cdots + f(z + c_k) - kf(z)}{f(z)}.
\]

A number of results are obtained concerning zeros and fixed points of the difference \( g_k(z) \) and the divided difference \( G(z) \).

1. Introduction and main results. Recently, there has been an increasing interest in studying difference equations in the complex plane. Halburd and Korhonen [HK1, HK2] established a version of Nevanlinna theory based on difference operators. Bergweiler and Langley [BL] investigated the existence of zeros of \( \Delta f \) and \( \frac{\Delta f(z)}{f(z)} \), and obtained several profound and significant results, which may be viewed as discrete analogues of the following theorem on the zeros of \( f' \).

**Theorem A** ([BE, ELR, H]). Let \( f \) be transcendental and meromorphic in the plane with
\[
\liminf_{r \to \infty} T(r, f)/r = 0.
\]

Then \( f' \) has infinitely many zeros.

If \( f \) satisfies the hypotheses of Theorem A, by Hurwitz’s theorem we know that if \( z_0 \) is a zero of \( f'(z) \) then \( \Delta_c f(z) = f(z + c) - f(z) \) has a zero near \( z_0 \) for all sufficiently small \( c \in \mathbb{C} \setminus \{0\} \). Hence it is natural to ask whether \( \Delta_c f(z) \) must have infinitely many zeros or not. Bergweiler and Langley [BL] answered this problem, and obtained the following theorems.

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Theorem B ([BL]). There exists $\delta_0 \in (0, 1/2)$ with the following property. Let $f$ be a transcendental entire function with
\[ \rho(f) \leq \rho < 1/2 + \delta_0 < 1. \]
Then
\[ G(z) = \frac{f(z + 1) - f(z)}{f(z)} \]
has infinitely many zeros.

Here $\rho(f)$ denotes the order of growth of the meromorphic function $f(z)$. In what follows $\lambda(f)$ and $\lambda(1/f)$ denote the exponents of convergence of the zeros and poles of $f(z)$, respectively. In this paper, we shall assume that the reader is familiar with the basic concepts of Nevanlinna theory (see [HI, YY]).

Theorem C ([BL]). Let $f$ be a function transcendental and meromorphic of lower order $\mu(f) < 1$ in the plane. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles $z_j, z_k$ of $f$ satisfy $z_j - z_k = c$. Then $\triangle_c f(z) = f(z + c) - f(z)$ has infinitely many zeros.

Chen and Shon [CS1] considered zeros and fixed points of differences and divided differences of entire functions with $\rho(f) = 1$ and obtained the following theorem.

Theorem D ([CS1]). Let $c \in \mathbb{C} \setminus \{0\}$ and let $f$ be a transcendental entire function with $\rho(f) = \rho = 1$ that has infinitely many zeros with $\lambda(f) = \lambda < 1$. Then $\triangle_c f(z) = f(z + c) - f(z)$ has infinitely many zeros and infinitely many fixed points.

Recently, Chen and Shon [CS2] considered the following three problems:

(i) What conditions will guarantee that the difference $f(z + c) - f(z)$ has infinitely many zeros for a meromorphic function $f$?
(ii) What is the exponent of convergence of zeros of the difference $f(z + c) - f(z)$ if it has infinitely many zeros?
(iii) What can we say about the zeros of
\[ f(z + c) - f(z) - l(z) \quad \text{and} \quad \frac{f(z + c) - f(z)}{f(z)} - l(z), \]
where $l(z)$ is a polynomial?

For question (i), the following theorem shows that the conditions that $f$ satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles will guarantee that the difference $f(z + c) - f(z)$ has infinitely many zeros, without any hypothesis on $c$.

Theorem E ([CS2]). Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and $f$ a meromorphic function of order of growth $\rho(f) = \rho \leq 1$. Suppose that $f$ satisfies $\lambda(1/f) <$
λ(f) < 1 or has infinitely many zeros (with λ(f) = 0) and finitely many poles. Then

$$\triangle_c f(z) = f(z + c) - f(z)$$

has infinitely many zeros and satisfies $\lambda(\triangle_c f) = \lambda(f)$.

Concerning question (ii), Theorem E also shows that if $f(z + c) - f(z)$ has infinitely many zeros, then $\lambda(f(z + c) - f(z)) = \lambda(f)$.

As for question (iii), the following two theorems show that

$$f(z + c) - f(z) - l(z) \text{ and } \frac{f(z + c) - f(z)}{f(z)} - l(z)$$

have infinitely many zeros, respectively.

**Theorem F (CS2).** Let $c$ and $f(z)$ satisfy the conditions of Theorem E. Suppose that $l(z)$ is a polynomial. Then $\triangle_c f(z) - l(z)$ has infinitely many zeros and satisfies $\lambda(\triangle_c f - l) = \rho(f)$.

**Theorem G (CS2).** Let $c \in \mathbb{C} \setminus \{0\}$ be a constant and $f$ a transcendental meromorphic function of order of growth $\rho(f) = \rho < 1$ or of the form $f(z) = h(z)e^{az}$ where $a \neq 0$ is a constant and $h(z)$ is a transcendental meromorphic function with $\rho(h) < 1$. Suppose that $l(z)$ is a nonconstant polynomial. Then

$$G_1(z) = \frac{f(z + c) - f(z)}{f(z)} - l(z)$$

has infinitely many zeros.

The aim of the paper is to generalize Theorems E–G. In [CS2], Chen and Shon consider the zeros of the differences $\triangle_c f(z)$ under the assumption $\rho(f) \leq 1$. We study the zeros of the sum $g_k(z) = \triangle_{c_1} f(z) + \cdots + \triangle_{c_k} f(z)$ under the assumption $\rho(f) < \infty$. In particular, we study the densities of the zeros of $g_k(z) - l(z)$ and of $G_k(z) = \frac{f(z+c_1)+\cdots+f(z+c_k)-kf(z)}{f(z)} - l(z)$. We prove the following three theorems.

**Theorem 1.1.** Let $f(z)$ be a finite order meromorphic function with $\lambda(1/f) < \lambda(f) < 1$. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$, let $g_k(z) = f(z + c_1) + \cdots + f(z + c_k) - kf(z)$, and suppose $g_k(z) \neq 0$. Then:

(i) If $\rho(f) = \rho < 1$, we have $\lambda(g_k) = \lambda(f)$.
(ii) If $1 \leq \rho(f) = \rho < \infty$, we have $\lambda(g_k) \geq \lambda(f)$.

**Theorem 1.2.** Let $f, c_j$ ($j = 1, \ldots, k), g_k(z)$ satisfy the conditions of Theorem 1.1. Suppose that $l(z)$ is a nonconstant polynomial, and let $g_k(z, L) := g_k(z) - l(z)$. Then:

(i) If $\rho(f) < 1$, we have $\lambda(g_k(z, L)) = \rho(f)$.
(ii) If $1 \leq \rho(f) < \infty$, we have $\lambda(g_k(z, L)) \geq 1$. 

Theorem 1.3. Let \( f \) be a transcendental meromorphic function of order of growth \( \rho(f) = \rho < 1 \) or of the form \( f(z) = h(z)e^{az} \) where \( a \neq 0 \) is a constant and \( h(z) \) is a transcendental meromorphic function with \( \rho(h) < 1 \). Let \( c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\} \) be such that \( c_1 + \cdots + c_k \neq 0 \). Suppose that \( l(z) \) is a nonconstant polynomial. Then
\[
G_k(z) = \frac{f(z + c_1) + \cdots + f(z + c_k) - kf(z)}{f(z)} - l(z)
\]
has infinitely many zeros.

Remark. In the special case when \( l(z) = z \), one obtains results on fixed points.

2. Some lemmas. In order to prove our theorems, we need the following lemmas and notions.

Following Hayman \[H2\] pp. 75–76, we define an \( \varepsilon \)-set to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. If \( E \) is an \( \varepsilon \)-set then the set of \( r \geq 1 \) for which the circle \( S(0, r) = \{z \in \mathbb{C} : |z| = r\} \) meets \( E \) has finite logarithmic measure, and for almost all real \( \theta \) the intersection of \( E \) with the ray \( \arg z = \theta \) is bounded.

Bergweiler and Langley \[BL\] have shown that differences of meromorphic functions of order less than one behave asymptotically like their derivatives in the complex plane.

Lemma 2.1 \([BL]\). Let \( f \) be transcendental and meromorphic of order less than 1 in the plane. Let \( h > 0 \). Then there exists an \( \varepsilon \)-set \( E \) such that
\[
f(z + c) - f(z) = cf'(z)(1 + o(1)) \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E,
\]
uniformly in \( c \) for \( |c| \leq h \).

The following lemma due to Bergweiler and Langley \[BL\] gives an asymptotic identity involving a meromorphic function of order less than one, its derivative and its shift.

Lemma 2.2 \([BL]\). Let \( f \) be a function transcendental and meromorphic in the plane of order less than 1. Let \( h > 0 \). Then there exists an \( \varepsilon \)-set \( E \) such that
\[
\frac{f'(z + c)}{f(z + c)} \to 0, \quad \frac{f(z + c)}{f(z)} \to 1 \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E,
\]
uniformly in \( c \) for \( |c| \leq h \). Further, \( E \) may be chosen so that for large \( z \) not in \( E \) the function \( f \) has no zeros or poles in \( |\varsigma - z| \leq h \).

In Lemma 2.1 of \[BL\], Bergweiler and Langley prove that \( \Delta f(z) = f(z + c) - f(z) \) and \( \frac{\Delta f(z)}{f} \) are both transcendental. The following lemma
is a generalization of Lemma 2.1 of [BL] and states that $g_k(z) = \Delta c_1 f(z) + \cdots + \Delta c_k f(z)$ and $G(z) = g_k(z)/f(z)$ are also transcendental.

**Lemma 2.3.** Let $f$ be a transcendental meromorphic function with $\rho(f) = \rho < 1$. Let $c_1, \ldots, c_k \in \mathbb{C} \setminus \{0\}$ be such that $c_1 + \cdots + c_k \neq 0$. Then $g_k(z)$ and $G(z) = g_k(z)/f(z)$ are both transcendental.

**Proof.** Without loss of generality, it may be assumed that $c_1 = 1$ and $\text{Re } c_2 = \min \{\text{Re } c_i : i = 2, \ldots, k\}$. Assume that $g_k(z)$ is a rational function. Then

$$f(z + 1) + f(z + c_2) + \cdots + f(z + c_k) = R(z) + kf(z),$$

where $R(z)$ is a rational function. Suppose that $A = \{x_j + iy_j : j = 1, \ldots, s\}$ consists of all poles of $R(z)$.

Set

$$M = 2 \max \{|x_j| + |y_j| + 1 + \cdots + |c_k| : j = 1, \ldots, s\}$$

and

$$D_1 = \{z : \text{Re } z > M\}, \quad D_2 = \{z : \text{Re } z < -M\},$$

$$D_3 = \{z : \text{Im } z > M\}, \quad D_4 = \{z : \text{Im } z < -M\}.$$ 

Now we prove that $f(z)$ has at most finitely many poles. Suppose, contrary to the assertion, that $f(z)$ has infinitely many poles. Then there is at least one $D_j$ such that $f(z)$ has infinitely many poles in $D_j$.

If $f(z)$ has infinitely many poles in $D_1$, let $z_0$ be one. If $\text{Re } c_2 \geq 0$, then for each $m_i \in \mathbb{N}$, $i = 1, \ldots, k$, $z_{m_1, \ldots, m_k} = z_0 + m_1 + m_2 c_2 + \cdots + m_k c_k \in D_1$ and $R(z_{m_1, \ldots, m_k}) \neq \infty$. By (2.1), we find that $f(z)$ has an infinite sequence of poles of the form

$$\{z_{m_1, \ldots, m_k} = z_0 + m_1 + m_2 c_2 + \cdots + m_k c_k : m_i \in \mathbb{N} \ (1 \leq i \leq k)\}.$$ 

Moreover, it can be seen from (2.1) that for each pole in this sequence there is another pole within a distance of $1 + \cdots + |c_k|$, and so $\lambda(1/f) \geq 1$, a contradiction.

If $\text{Re } c_2 < 0$, and there exist some $c_j$ $(2 \leq j \leq k)$ such that $c_j = c_2$; without loss of generality, we may suppose that $c_2 = \cdots = c_t$ $(2 \leq t \leq k)$. Then we can rewrite (2.1) as

$$f(z + 1 - c_2) + f(z + c_{t+1} - c_2) + \cdots + f(z + c_k - c_2) - kf(z - c_2) = R(z - c_2) - (t - 1)f(z).$$

For each $m_i \in \mathbb{N}$, $i = 1, 2, t + 1, \ldots, k$,

$$z_{m_1, m_2, m_{t+1}, \ldots, m_k}^* = z_0 + m_1(1 - c_2) - m_2 c_2 + m_{t+1}(c_{t+1} - c_2) + \cdots + m_k(c_k - c_2) \in D_1$$
and \( R(z_{m_1, m_2, m_{t+1}, \ldots, m_k}) \neq \infty \). From (2.2), we find that \( f(z) \) has an infinite sequence of poles of the form
\[
\{ z_{m_1, m_2, m_{t+1}, \ldots, m_k} = z_0 + m_1(1 - c_2) - m_2c_2 + m_{t+1}(c_{t+1} - c_2) \\
+ \cdots + m_k(c_k - c_2), \ m_i \in \mathbb{N} \ (i = 1, 2, t + 1, \ldots, k) \}.
\]
So \( \lambda(1/f) \geq 1 \), a contradiction.

If \( f \) has infinitely many poles in \( D_2 \) (or \( D_3 \), or \( D_4 \)), using a similar method, we obtain \( \lambda(1/f) \geq 1 \), a contradiction. Hence \( f \) has at most finitely many poles.

Thus, there exists a rational function \( R_1 \) such that \( h(z) = f(z) - R_1(z) \) is a transcendental entire function. By (2.1), we have
(2.3) \( h(z + c_1) + \cdots + h(z + c_k) = kh(z) + P(z) \),
where \( P(z) = R(z) + kR_1(z) - R_1(z + c_1) - \cdots - R_k(z + c_k) \). Since \( h(z + c_j) \) \((j = 1, \ldots, k)\) and \( h(z) \) are entire functions, we infer that \( P(z) \) is a polynomial. By Lemma 2.1, there exists an \( \varepsilon \)-set \( E \) such that
(2.4) \( h(z + c_j) - h(z) = c_jh'(z)(1 + o(1))) \ (j = 1, \ldots, k) \) as \( z \to \infty \) in \( \mathbb{C} \setminus E \).
If \( P(z) \equiv 0 \), by (2.3) and (2.4), as \( z \to \infty \) in \( \mathbb{C} \setminus E \), we have
\[
(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = 0,
\]
and since \( c_1 + \cdots + c_k \neq 0 \), we obtain \( h'(z) = 0 \) (as \( z \not\in E \)). This is impossible. Hence \( P(z) \neq 0 \). Set \( \deg P = l \geq 0 \); then \( P(z) = cz^l(1 + o(1)) \), where \( c \neq 0 \) is a constant. By (2.3) and (2.4), as \( z \to \infty \) in \( \mathbb{C} \setminus E \), we get
\[
(c_1 + \cdots + c_k)h'(z)(1 + o(1)) = cz^l(1 + o(1)),
\]
which contradicts the fact that \( h'(z) \) is transcendental.

Next, we assume that \( G(z) \) is a rational function. Then
\[
\frac{f(z + c_1) + \cdots + f(z + c_k) - kf(z)}{f(z)} = \theta(z),
\]
where \( \theta(z) \) is a rational function, By Lemma 2.1, there exists an \( \varepsilon \)-set \( E \) such that
(2.5) \( \frac{(c_1 + \cdots + c_k)f'(z)(1 + o(1))}{f(z)} = \theta(z) \) as \( z \to \infty \) in \( \mathbb{C} \setminus E \);
however, since \( f(z) \) is transcendental and has either infinitely many poles or infinitely many zeros, we conclude that \( f'(z)/f(z) \) must be transcendental, so (2.5) is impossible.

Remark. Lemma 2.3 is also proved in [LY], but the methods are partly different.

The following lemma is the classical logarithmic derivative estimate due to Gundersen [G].
Lemmas 2.4 ([G]). Let \( f \) be a transcendental meromorphic function with \( \rho(f) = \rho < \infty \). Let \( \varepsilon > 0 \) be a given constant. Then there exists a set \( E \subset (1, \infty) \) with finite logarithmic measure such that for all \( |z| \notin E \cup [0, 1] \) and for any integers \( k \) and \( j \) such that \( k > j \geq 0 \), we have
\[
\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.
\]

The following lemma is a generalization of Borel’s Theorem on combinations of entire functions.

Lemmas 2.5 ([YY, pp. 79–80]). Let \( f_j(z) \) \((j = 1, \ldots, n) \) \((n \geq 2)\) be meromorphic functions, and suppose that there are entire functions \( g_j(z) \) \((j = 1, \ldots, n)\) that satisfy:

(i) \( f_1(z)e^{g_1(z)} + \cdots + f_k(z)e^{g_k(z)} \equiv 0. \)

(ii) When \( 1 \leq j < k \leq n \), then \( g_j(z) - g_k(z) \) is not a constant.

(iii) When \( 1 \leq j \leq n \), \( 1 \leq h < k \leq n \), then
\[
T(r, f_j) = o\{T(r, e^{g_h-g_k})\} \quad (r \to \infty, r \notin E),
\]
where \( E \subset (1, \infty) \) is of finite linear measure or finite logarithmic measure.

Then \( f_j \equiv 0 \) \((j = 1, \ldots, n)\).

3. Proof of Theorem 1.1

Proof of Claim (i). Suppose that \( \lambda(1/f) < \lambda(f) < 1 \) and \( \rho(f) < 1 \). By Lemma 2.3, \( g_k(z) \) is transcendental. Let \( f(z) = u(z)/v(z) \), where \( u(z) \) and \( v(z) \) are the canonical products \((v(z) \text{ may be a polynomial})\) formed by the zeros and the poles of \( f(z) \), respectively, and
\[
\lambda(1/f) = \lambda(v) = \rho(v) < \lambda(f) = \lambda(u) = \rho(u).
\]

By Lemma 2.1, there exists an \( \varepsilon \)-set \( E \) such that
\[
g_k(z) = (c_1 + \cdots + c_k)f'(z)(1 + o(1)) \quad \text{as} \quad z \to \infty \quad \text{in} \quad \mathbb{C} \setminus E.
\]

Set
\[
H = \{|z| : z \in E, g_k(z) = 0 \text{ or } f'(z) = 0\}.
\]

Then \( H \) is of finite linear measure. By (3.1), for \( |z| = r \notin H \), we obtain
\[
|g_k(z) - (c_1 + \cdots + c_k)f'(z)| = |o(1)(c_1 + \cdots + c_k)f'(z)|
\]
\[
< |g_k(z)| + |(c_1 + \cdots + c_k)f'(z)|.
\]

Thus \( g_k(z) \) and \(-(c_1 + \cdots + c_k)f'(z)\) satisfy the assumptions of Rouché’s theorem. Applying Rouché’s theorem and (3.2), for \( |z| = r \notin H \) we obtain
\[
n(r, 1/g_k) - n(r, g_k) = n(r, 1/f') - n(r, f').
\]
Since \( f' = (u'(z)v(z) - u(z)v'(z))/v^2(z), \) \( \lambda(1/f) < \lambda(f) = \rho(f) < 1, \) and \( \rho(f') = \rho(f), \) we have

\[
\lambda(1/f') = \lambda(1/f) < \lambda(f) = \rho(f) = \rho(f').
\]

From this, we get

\[
\lambda(1/g_k) \leq \lambda(1/f) < \lambda(f) = \lambda(f').
\]

Hence, with (3.3) and (3.4), we obtain

(3.4)

\[
\lambda(g_k) = \lambda(f') = \lambda(f).
\]

Thus (i) holds.

Proof of Claim (ii). Since \( 1 \leq \rho(f) < \infty \) and \( \lambda(1/f) < \lambda(f) < 1, \) it follows from the Hadamard factorization theorem that

\[
f(z) = h(z)e^{P(z)} = \frac{u(z)}{v(z)}e^{P(z)},
\]

where \( P(z) \) is a nonconstant polynomial, \( h(z) \) is a meromorphic function such that \( h(z) = u(z)/v(z), \) \( u(z) \) and \( v(z) \) are the canonical products \((v(z) \) may be a polynomial\) formed by the zeros and the poles of \( f(z), \) respectively, and

\[
\lambda(1/f) = \lambda(v) = \rho(v) = \lambda(1/h) < \lambda(f) = \lambda(u) = \rho(u) = \lambda(h) = \rho(h) < 1.
\]

Hence

\[
g_k(z) = f(z + c_1) + \cdots + f(z + c_k) = h(z + c_1)e^{P(z)+R_1(z)} + \cdots + h(z + c_k)e^{P(z)+R_k(z)} - kh(z)e^{P(z)}
\]

\[
= (h(z + c_1)e^{R_1(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z))e^{P(z)} = w(z)e^{P(z)},
\]

where \( R_j(z) = P(z + c_j) - P(z) \) \((j = 1, \ldots, k),\) and

\[
w(z) = h(z + c_1)e^{R_1(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z).
\]

From this, we get \( \lambda(1/w) \leq \lambda(1/h) = \lambda(1/f) < \lambda(f) < 1. \) Since \( g_k(z) \neq 0, \) we have \( w(z) \neq 0. \)

Next, suppose, contrary to the assertion, that \( \lambda(g_k) < \lambda(f) < 1. \)

If \( 1 \leq \rho(w) < \infty, \) then there exist a nonconstant polynomial \( R_0(z) \) and a nonzero meromorphic function \( Q(z) \) such that

(3.5)

\[
w(z) = Q(z)e^{R_0(z)} = \frac{u_1(z)}{v_1(z)}e^{R_0(z)},
\]

where \( Q(z) = u_1(z)/v_1(z) \) with \( u_1(z) \) and \( v_1(z) \) being the canonical products formed by the zeros and the poles of \( w(z), \) respectively, and

\[
\lambda(1/Q) = \lambda(v_1) = \rho(v_1) = \lambda(1/w) \leq \lambda(1/f) < 1,
\]

\[
\lambda(u_1) = \rho(u_1) = \lambda(Q) = \lambda(w) = \lambda(g_k) < 1.
\]
So, we obtain $\rho(Q) = \max\{\lambda(Q), \lambda(1/Q)\} < 1$. Let $c_k+1 = 0$, $h(z) = h(z)e^{R_k+1(z)}$, where $R_k+1(z) = 0$. We next consider two cases.

**Case (1.1):** There exist $i, j \in \{0, 1, \ldots, k + 1\}$ such that $R_j(z) - R_i(z) = A$ is a constant. We need to consider two subcases.

**Subcase (1.1.1):** $R_j(z) - R_0(z)$ is not a constant for any $j \in \{1, \ldots, k + 1\}$. Then there exist $1 \leq i, j \leq k + 1$ such that $R_j(z) - R_i(z) = A$ is a constant. Hence $P(z + c_j) - P(z + c_i) = A$. Since $P(z)$ is a polynomial, it must have the form $P(z) = az + d$ and $a \neq 0$. Hence $R_j = ac_j$ is a constant for $j = 1, \ldots, k + 1$. From

$$w(z) = h(z + c_1)e^{R_1(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z),$$

we get $\rho(w) < 1$, a contradiction.

**Subcase (1.1.2):** There exists a $j \in \{1, \ldots, k+1\}$ such that $R_j(z)-R_0(z) = A$ is a constant. If there also exists $i \in \{1, \ldots, j-1, j+1, \ldots, k+1\}$ such that $R_i(z)-R_0(z) = B$ is a constant, then $R_j(z)-R_i(z) = A-B$. By Subcase (1.1.1), $R_j$ is a constant for $j = 1, \ldots, k+1$. Therefore, $R_0$ is then a constant, a contradiction. If now for arbitrary $i, \alpha \in \{0, 1, \ldots, j-1, j+1, \ldots, k+1\}$, $R_i(z) - R_{\alpha}(z)$ is not a constant, then

$$h(z + c_1)e^{R_1(z)} + h(z + c_2)e^{R_2(z)} + \cdots + (e^Ah(z + c_j) - Q(z))e^{R_0(z)} + h(z + c_k)e^{R_k(z)} - kh(z) = 0.$$ 

Since $\deg(R_i(z) - R_{\alpha}(z)) \geq 1$, $e^{R_i(z) - R_{\alpha}(z)}$ is of regular growth (see, e.g., [III p. 7]), and $\rho(h(z + c_i)) < 1$ and $\rho(e^Ah(z + c_j) - Q(z)) < 1$, we conclude that

$$T(r, h(z + c_i)) = o\{T(r, e^{R_i(z) - R_{\alpha}(z)})\},$$

$$T(r, e^Ah(z + c_j) - Q(z)) = o\{T(r, e^{R_i(z) - R_{\alpha}(z)})\}.$$

Thus, from Lemma 2.5 and (3.6), we have $h(z) \equiv 0$, a contradiction.

**Case (1.2):** $R_j(z)-R_i(z)$ is not a constant for any $i, j \in \{0, 1, \ldots, k+1\}$, $i \neq j$. By Lemma 2.5, $h(z+c_j) \equiv 0$ ($j = 1, \ldots, k$), $h(z) \equiv 0$, a contradiction.

Therefore, $\rho(w) < 1$. Then there exists a nonzero meromorphic function $Q(z)$ such that

$$w(z) = h(z + c_1)e^{R_1(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z) = Q(z),$$

where $\rho(Q) = \max\{\lambda(Q), \lambda(1/Q)\} < 1$. We break the rest of the proof into three cases.

**Case (2.1):** There exists exactly one $j \in \{1, \ldots, k\}$ such that $R_j(z)$ is a nonconstant polynomial. From (3.7), we get $\rho(w) \geq 1$, a contradiction.

**Case (2.2):** There exist at least two $i, j \in \{1, \ldots, k\}$ such that $R_i(z)$, $R_j(z)$ are nonconstant polynomials. Without loss of generality, we suppose
$R_1(z), \ldots, R_m(z)$ ($m \geq 2$) are nonconstant polynomials, while $R_{m+1}, \ldots, R_k$ are constants. We now rewrite $w(z)$ as follows:

$$w(z) = h(z + c_1)e^{R_1(z)} + \cdots + h(z + c_m)e^{R_m(z)} + h(z + c_{m+1})e^{R_{m+1}(z)} + \cdots + h(z + c_k)e^{R_k(z)} - kh(z) = Q(z).$$

If there exist $1 \leq i, j \leq m$ such that $R_i - R_j$ is a constant, we may apply Subcase (1.1.1) to deduce that $R_i(z)$ is a constant for $i = 1, \ldots, m$, a contradiction. Thus for arbitrary $i, j \in \{1, \ldots, m\}$ with $i \neq j$, $R_i - R_j$ is not a constant. By Lemma 2.5, we have $h(z + c_j) \equiv 0$, a contradiction.

**Case (2.3): $R_j$ is constant for all $j \in \{1, \ldots, k\}$.** Using the method of Subcase (1.1.1), we see that $P(z) = az + b$, $a \neq 0$. Substituting this into $w(z)$, we have

$$w(z) = h(z + c_1)e^{ac_1} + \cdots + h(z + c_k)e^{ac_k} - kh(z).$$

By Lemma 2.2, there exists an $\varepsilon$-set $E$ such that

$$(3.8) \quad h(z + c) = h(z)(1 + o(1))$$

as $z \to \infty$ in $\mathbb{C} \setminus E$. By (3.8), we obtain

$$(3.9) \quad w(z) = (e^{ac_1} + \cdots + e^{ac_k})h(z)(1 + o(1)) - kh(z) = (e^{ac_1} + \cdots + e^{ac_k} - k)h(z)(1 + o(1)).$$

By (3.9) and $w(z) \neq 0$, we have $e^{ac_1} + \cdots + e^{ac_k} \neq k$. Since $h(z)$ is transcendental, we know that $w(z)$ is transcendental. Set

$$H = \{|z| : z \in E, w(z) = 0 \text{ or } h(z) = 0\},$$

Then $H$ is of finite linear measure. By (3.9), for $|z| = r \notin H \cup [0, 1]$, we obtain

$$(3.10) \quad |w(z) - (e^{ac_1} + \cdots + e^{ac_k} - k)h(z)| = |(e^{ac_1} + \cdots + e^{ac_k} - k)o(1)| < |w(z)| + |(e^{ac_1} + \cdots + e^{ac_k} - k)h(z)|.$$

Applying Rouché’s theorem and (3.10), and using a similar method to the proof of (i), we obtain

$$\lambda(w) = \lambda(h) = \lambda(u) = \lambda(f),$$

a contradiction. Hence $\lambda(g_k) = \lambda(w) \geq \lambda(f)$. Theorem 1.1 is thus proved.

**4. Proof of Theorem 1.2**

**Proof of Claim (i).** Since $f$ satisfies $\lambda(f) > \lambda(1/f)$ and $\rho(f) < 1$, from Theorem 1.1 and the proof of (i) there, we obtain

$$\rho(f) = \lambda(f) = \lambda(g_k) = \rho(g_k), \quad \rho(g_k) > \lambda(1/f) \geq \lambda(1/g_k).$$

Since $g_k(z, L) = g_k(z) - l(z)$, where $l(z)$ is a nonzero polynomial, we have

$$\lambda(1/g_k(z, L)) = \lambda(1/g_k) < \lambda(g_k) = \rho(g_k) = \rho(g_k(z, L)) < 1.$$
As $\lambda(1/g_k(z,L)) < \rho(g_k(z,L)) < 1$, we obtain $\lambda(g_k(z,L)) = \rho(g_k(z,L))$. Hence, $\lambda(g_k(z,L)) = \rho(g_k(z,L)) = \rho(g_k) = \lambda(f) = \rho(f)$.

Proof of Claim (ii). Suppose that $\lambda(g_k(z,L)) < 1$. Then $1 \leq \rho(g_k(z,L)) = \rho(g_k - l) = \rho(g_k) < \infty$. We rewrite $g_k(z,L)$ as follows:

$$g_k(z,L) = g_k(z) - l(z) = h_*(z)e^{L(z)},$$

where $L(z)$ is a nonconstant polynomial and $h_*(z)$ is a meromorphic function such that

$$\lambda(h_*) = \lambda(g_k(z,L)) < 1, \quad \lambda(1/h_*) = \lambda(1/g_k(z,L)) \leq \lambda(1/f) < 1.$$  

With (4.2), we have

$$\rho(h_*) = \max\{\lambda(h_*), \lambda(1/h_*)\} < 1.$$  

Since $g_k(z) - l(z) \neq 0$, we obtain $h_*(z) \neq 0$.

From (4.1) and $f(z) = h(z)e^{P(z)}$, we have

$$h(z + c_1)e^{P(z+c_1)} + \cdots + h(z + c_k)e^{P(z+c_k)} - kh(z)e^{P(z)} - l(z) - h_*(z)e^{L(z)} \equiv 0.$$  

Let $h(z)e^{P(z)} = h(z + c_0)e^{P(z+c_0)}$, where $c_0 = 0$. We consider three cases.

Case (1): There exist $i, j \in \{0, 1, \ldots, k\}$ such that $P(z+c_i) - P(z+c_j) = A$ is a constant. Since $P(z)$ is a polynomial, it must have the form $P(z) = az + d$ and $a \neq 0$. Hence

$$[h(z + c_1)e^{ac_1} + \cdots + h(z + c_k)e^{ac_k} - kh(z)]e^{az+d} - h_*(z)e^{L(z)} - l(z) = 0.$$  

If $L(z) - az - d \equiv C$, then

$$[h(z + c_1)e^{ac_1} + \cdots + h(z + c_k)e^{ac_k} - kh(z) - h_*(z)e^{C}]e^{az+d} - l(z) = 0,$$

which is impossible. If $L(z) - az - d \not\equiv C$, from Lemma 2.5 we get $l(z) \equiv h_*(z) \equiv 0$, a contradiction.

Case (2): There exists $i \in \{0, 1, \ldots, k\}$ such that $P(z+c_i) - L(z) = A$. If there also exists $j \in \{0, 1, \ldots, i - 1, i + 1, \ldots, k\}$ such that $P(z+c_j) - L(z) = B$, then $P(z+c_j) - P(z+c_i) = A - B$. Using the method of Case (1), we reach a contradiction. If for arbitrary $j \neq i$, we have $P(z_j) - L(z) \not\equiv B$, then

$$h(z + c_1)e^{P(z+c_1)} + h(z + c_2)e^{P(z+c_2)} + \cdots + (e^{A}h(z + c_j) - h_*(z))e^{L(z)} + \cdots + h(z + c_k)e^{P(z+c_k)} - kh(z)e^{P(z)} - l(z) = 0.$$  

From Lemma 2.5, we have $l(z) \equiv h(z) \equiv 0$, a contradiction.

Case (3): For arbitrary $i, t, j \in \{0, 1, \ldots, k\}$, $i \neq t$, such that $P(z+c_i) - P(z+c_t)$ is not a constant, $P(z+c_j) - L(z)$ is also not a constant.
Lemma 2.5, we obtain \( h(z + c_j) \equiv 0 \) and \( l(z) \equiv 0 \), a contradiction. Therefore, 
\( \lambda(g_k(z, L)) \geq 1 \). This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3. Let \( \rho(f) = \rho < 1 \). By Lemma 2.3, we see 
that \( \frac{\sum f(z+c_n) - kf(z)}{f(z)} \) is transcendental, and hence so is \( G_k(z) \). By Lemma 2.1, there exists an \( \varepsilon \)-set \( E \) such that 
(5.1) \[ h(z + c) - h(z) = cf'(z)(1 + o(1)) \]
as \( z \to \infty \) in \( \mathbb{C} \setminus E \). By Lemma 2.4, for a given \( \varepsilon > 0 \), there exists a set \( H_1 \subset (1, \infty) \) with finite logarithmic measure such that for all \( z \) satisfying \( |z| \not\in H_1 \cup [0, 1] \), we have

(5.2) \[ \left| \frac{f'(z)}{f(z)} \right| \leq |z|^{\rho^{-1} + \varepsilon} \].

Set 
\( H_2 = \{|z| : z \in E, G_k(z) = 0 \text{ or } l(z) = 0\} \).

Then \( H_2 \) has finite linear measure. For large \( |z| = r \not\in [0, 1] \cup H_1 \cup H_2 \), from (5.1) and (5.2), we see

(5.3) \[ |G_k(z) + l(z)| = \left| (c_1 + \cdots + c_k) \frac{f'(z)}{f(z)} (1 + o(1)) \right| \]
\[ \leq |(c_1 + \cdots + c_k)(1 + o(1))| |z|^{\rho^{-1} + \varepsilon} \]
\[ < |G_k(z)| + |l(z)|, \]

since \( \rho < 1 \). Thus \( G_k(z) \) and \( l(z) \) satisfy the conditions of Rouché's theorem. Applying Rouché's theorem and (5.3), for \( |z| = r \not\in [0, 1] \cup H_1 \cup H_2 \) we have

(5.4) \[ n(r, 1/G_k) - n(r, G_k) = n(r, 1/l) - n(r, l) = \deg l. \]

Since \( G_k \) is transcendental and \( \rho(G_k) < 1 \), we know that at least one of \( n(r, G_k) \to \infty \) and \( n(r, 1/G_k) \to \infty \) is true as \( r \to \infty \). Hence, by (5.4), both are true. Hence \( G_k(z) \) must have infinitely many zeros.

Suppose now that \( f(z) = h(z)e^{az} \), where \( a \neq 0 \) is a constant, and \( h(z) \) is a transcendental meromorphic function such that \( \rho(h) < 1 \). Substituting this into \( G_k(z) \), we obtain

(5.5) \[ G_k(z) = \frac{h(z + c_1)e^{ac_1} + \cdots + h(z + c_k)e^{ac_k} - kh(z)}{h(z)} - l(z). \]

If \( e^{ac_1} + \cdots + e^{ac_k} - k = 0 \), then using the same method as in the first part of the proof, and (5.5), we deduce that \( G_k(z) \) has infinitely many zeros.

If \( e^{ac_1} + \cdots + e^{ac_k} - k \neq 0 \), then by Lemma 2.1 and (5.2), for a given \( \varepsilon > 0 \), there exist an \( \varepsilon \)-set \( E \) and a set \( H_1 \subset (1, \infty) \) with finite logarithmic measure such that for all \( z \) satisfying \( |z| \not\in E \cup [0, 1] \cup H_1 \), we have
\begin{align}
(5.6) \quad & \left| \frac{h(z + c_1)e^{ac_1} + \cdots + h(z + c_k)e^{ack} - kh(z)}{h(z)} \right| \\
& = \left| (c_1e^{ac_1} + \cdots + c_ke^{ack}) \frac{h'}{h} + e^{ac_1} + \cdots + e^{ack} - k \right| \\
& \leq |c_1e^{ac_1} + \cdots + c_ke^{ack}| |z|^\rho + e^{ac_1} + \cdots + e^{ack} - k|.
\end{align}

Set

\[ H_2 = \{ |z| : z \in E, G_k(z) = 0 \text{ or } l(z) = 0 \}. \]

Then \( H_2 \) have finite linear measure. From (5.5) and (5.6), we see that

\begin{align}
(5.7) \quad & |G_k(z) + l(z)| \\
& = |c_1e^{ac_1} + \cdots + c_ke^{ack}| |z|^\rho + e^{ac_1} + \cdots + e^{ack} - k| \\
& < |G_k(z)| + |l(z)|.
\end{align}

Thus \( G_k(z) \) and \( l(z) \) satisfy the assumptions of Rouché’s theorem. Applying Rouché’s theorem and (5.7), for \( |z| = r \notin [0, 1] \cup H_1 \cup H_2 \) we obtain (5.4).

Using the same argument as in the proof of Lemma 2.3, we show that \( G_k(z) \) is transcendental. Applying the same method as in the first part of the proof, we obtain \( n(r, 1/G_k) \to \infty \). Theorem 1.3 is proved.

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References


Yong Liu
Department of Mathematics
Shaoxing College of Arts and Sciences
312000 Shaoxing, Zhejiang, P.R. China
and
Department of Physics and Mathematics
Joensuu Campus
University of Eastern Finland
P.O. Box 111
Joensuu FI-80101, Finland
E-mail: liuyongsdu@yahoo.cn

Hongxun Yi
School of Mathematics
Shandong University
250100 Jinan, Shandong, P.R. China
E-mail: hxyi@sdu.edu.cn

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