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Analytic solutions of a nonlinear two variables difference system whose eigenvalues are both 1

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Abstract. For nonlinear difference equations, it is difficult to obtain analytic solutions, especially when all the eigenvalues of the equation are of absolute value 1.

We consider a second order nonlinear difference equation which can be transformed into the following simultaneous system of nonlinear difference equations:

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases}$$

where $X(x,y) = \lambda_1 x + \mu y + \sum_{i+j\ge 2} c_{ij}x^i y^j$, $Y(x,y) = \lambda_2 y + \sum_{i+j\ge 2} d_{ij}x^i y^j$ satisfy some conditions. For these equations, we have obtained analytic solutions in the cases "$|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$" or "$\mu = 0$" in earlier studies. In the present paper, we will prove the existence of an analytic solution for the case $\lambda_1 = \lambda_2 = 1$ and $\mu = 1$.

1. Introduction. We start by considering the following second order nonlinear difference equation:

$$\begin{cases} u(t+1) = U(u(t), v(t)), \\ v(t+1) = V(u(t), v(t)), \end{cases} \tag{1.1}$$

where $U(u,v)$ and $V(u,v)$ are holomorphic functions of t. We suppose that the equation (1.1) admits an equilibrium point (u^*, v^*):

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} U(u^*, v^*) \\ V(u^*, v^*) \end{pmatrix}.$$

We can assume, without loss of generality, that $(u^*, v^*) = (0,0)$. Furthermore we suppose that U and V can be written in the form

$$\begin{pmatrix} u(t+1) \\ v(t+1) \end{pmatrix} = M \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} U_1(u(t), v(t)) \\ V_1(u(t), v(t)) \end{pmatrix},$$

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where $U_1(u,v)$ and $V_1(u,v)$ have degree greater than one with respect to u and v, and M is a constant matrix. Let λ_1, λ_2 be the characteristic values of the matrix M. For some regular matrix P determined by M, put $\binom{u}{v} = P\binom{x}{y}$. Then we can transform the system (1.1) into the following simultaneous system of first order difference equations:

$$\begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases} \tag{1.2}$$

where $X(x,y)$ and $Y(x,y)$ are supposed to be holomorphic and expanded in a neighborhood of $(0,0)$ as

$$\begin{cases} X(x,y) = \lambda_1 x + \sum_{i+j\geq 2} c_{ij}x^i y^j = \lambda_1 x + X_1(x,y), \\ Y(x,y) = \lambda_2 y + \sum_{i+j\geq 2} d_{ij}x^i y^j = \lambda_2 y + Y_1(x,y), \end{cases} \tag{1.3}$$

or

$$\begin{cases} X(x,y) = \lambda x + y + \sum_{i+j\geq 2} c'_{ij}x^i y^j = \lambda x + X'_1(x,y), \\ Y(x,y) = \lambda y + \sum_{i+j\geq 2} d'_{ij}x^i y^j = \lambda y + Y'_1(x,y), \end{cases} \tag{1.4}$$

where $\lambda = \lambda_1 = \lambda_2$.

In this paper we consider analytic solutions of difference system (1.2). In [S5] and [S6], we have obtained general analytic solutions of (1.2) in the case $|\lambda_1| \neq 1$ or $|\lambda_2| \neq 1$. However, when $|\lambda_1| = |\lambda_2| = 1$, it is even difficult to prove the existence of an analytic solution.

Kimura [K] studied the cases in which one eigenvalue is equal to 1, and Yanagihara [Y] investigated the cases in which the absolute value of one eigenvalue is 1. Here we will look for analytic solutions of nonlinear second order difference equations in which the absolute values of the eigenvalues of the matrix M are both equal to 1.

In [S7], we have proved the existence of an analytic solution and found a solution of (1.2) in which X and Y are defined by (1.3) under the condition $\lambda_1 = \lambda_2 = 1$. In this paper, we will consider the equation (1.2) in which X and Y are defined by (1.4) under the condition $\lambda = 1$, i.e., we assume that

$$\begin{cases} X(x,y) = x + y + \displaystyle\sum_{i+j\geq 2} c_{ij}x^i y^j = x + X_1(x,y), \\ Y(x,y) = y + \displaystyle\sum_{i+j\geq 2} d_{ij}x^i y^j = y + Y_1(x,y). \end{cases} \tag{1.5}$$

Here we suppose that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$, and we need some other conditions. In this case, we need Theorem C (see Section 2.1) which we have proved in [S8].

As examples of (1.2), we earlier studied some economic models and a population model (see [S1], [S4]). However we had to exclude the case $|\lambda_1| = |\lambda_2| = 1$. In Section 3, making use of Theorem 1.1 below, we will prove the existence of an analytic solution of the population model considered in [S4] in the case of $\lambda_1 = \lambda_2 = 1$. Further we will obtain an expansion of the solution in this case.

Next we consider a functional equation

$$\Psi(X(x, \Psi(x))) = Y(x, \Psi(x)), \tag{1.6}$$

where $X(x, y)$ and $Y(x, y)$ are holomorphic functions in $|x| < \delta_1$, $|y| < \delta_1$. We assume that $X(x, y)$ and $Y(x, y)$ are expanded there as in (1.5).

Now we discuss the meaning of the equation (1.6).

First we consider the simultaneous system of difference equations (1.2). Suppose (1.2) admits a solution $(x(t), y(t))$. If $dx/dt \neq 0$ for some t_0, then we can write $t = \psi(x)$ with a function ψ in a neighborhood of $x_0 = x(t_0)$, and we can write

$$y = y(t) = y(\psi(x)) = \Psi(x) \tag{1.7}$$

there. Then the function Ψ satisfies the equation (1.6).

Conversely, assume that a function Ψ is a solution of the functional equation (1.6). If the first order difference equation

$$x(t+1) = X(x(t), \Psi(x(t))) \tag{1.8}$$

has a solution $x(t)$, we put $y(t) = \Psi(x(t))$. Then $(x(t), y(t))$ is a solution of (1.2). Hence if there is a solution Ψ of (1.6), then we can reduce the system (1.2) to a single equation (1.8).

This relation is important in order to derive analytic solutions of (1.2). In the earlier studies [S2] , [S3] and [S5], we proved the existence of solutions Ψ of (1.6) whenever X and Y are defined by (1.3) or $\lambda \neq 1$ in (1.4). Further in [S8], we proved existence of solutions Ψ of (1.6) for X and Y defined by (1.5). On the other hand, in [K], Kimura considered the first order difference equation (1.8) under the condition $\lambda = 1$. We will prove the existence of an analytic solution and obtain an analytic solution of (1.2) in which X and Y are defined by (1.5).

Hereafter we consider t to be a complex variable, and concentrate on the difference system (1.2). We define

$$D_1(\kappa_0, R_0) = \{t : |t| > R_0,\ |\arg[t]| < \kappa_0\}, \tag{1.9}$$

where κ_0 is any constant such that $0 < \kappa_0 \leq \pi/4$, and R_0 is a sufficiently large number which may depend on X and Y. Further we define

$$D^*(\kappa, \delta) = \{x : |\arg[x]| < \kappa,\ 0 < |x| < \delta\}, \tag{1.10}$$

where δ is a small constant, and κ is a constant such that $\kappa = 2\kappa_0$, i.e., $0 < \kappa \leq \pi/2$.

We define $g_0^{\pm}$ as follows by the coefficients of $X(x,y)$ and $Y(x,y)$:

$$g_0^+(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20}-d_{11}) + \sqrt{(2c_{20}-d_{11})^2 + 8d_{30}}}{4}, \tag{1.11}$$

$$g_0^-(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20}-d_{11}) - \sqrt{(2c_{20}-d_{11})^2 + 8d_{30}}}{4}. \tag{1.12}$$

Our aim in this paper is to prove the following theorem.

THEOREM 1.1. *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the forms* (1.5) *such that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$. Define $A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20}$ and $A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20}$.*

(1) *Suppose*

$$d_{20} = 0, \tag{1.13}$$

and

$$(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}), \tag{1.14}$$

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}), \tag{1.15}$$

for all $n \in \mathbb{N}$ $(n \geq 4)$. Then we have formal solutions $x(t)$ of (1.2) *of the form*

$$\begin{aligned} &-\frac{1}{A_2 t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \\ &-\frac{1}{A_1 t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \end{aligned} \tag{1.16}$$

where $\hat{q}_{jk}$ are constants which are determined by X and Y.

(2) *Further suppose $R_1 = \max(R_0, 2/(|A_2|\delta))$ and*

$$A_2 < 0. \tag{1.17}$$

There are two solutions $x_1(t)$ and $x_2(t)$ of (1.2) *such that*

(i) *$x_s(t)$ are holomorphic and $x_s(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$, $s = 1, 2$,*

(ii) *$x_s(t)$ $(s = 1, 2)$ are expressible in the form*

$$x_s(t) = -\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{1.18}$$

where $b_s(t, (\log t)/t)$ has an asymptotic expansion

$$b_s\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$, and $\hat{q}_{jk(s)}$ are constants which are determined by X, Y and s.

2. Proof of Theorem 1.1

2.1. Preparation. In [K], Kimura considered the first order difference equation

$$w(t+\lambda) = F(w(t)), \tag{D1}$$

where F is represented in a neighborhood of ∞ by a Laurent series

$$F(z) = z\Big(1 + \sum_{j=m}^{\infty} b_j z^{-j}\Big), \qquad b_m = \lambda \neq 0. \tag{2.1}$$

He defined the following domains:

$$\begin{aligned} D(\epsilon, R) = \{t : \text{“}|t| > R \text{ and } |\arg[t] - \theta| < \pi/2 - \epsilon\text{”}, \\ \text{or “}\mathrm{Im}(e^{i(\theta-\epsilon)}t) > R\text{”, or “}\mathrm{Im}(e^{i(\theta+\epsilon)}t) < -R\text{”}\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \hat{D}(\epsilon, R) = \{t : \text{“}|t| > R \text{ and } |\arg[t] - \theta - \pi| < \pi/2 - \epsilon\text{”}, \\ \text{or “}\mathrm{Im}(e^{-i(\theta+\pi-\epsilon)}t) > R\text{”, or “}\mathrm{Im}(e^{-i(\theta+\pi+\epsilon)}t) < -R\text{”}\}, \end{aligned} \tag{2.3}$$

where ϵ is an arbitrarily small positive number, R is a sufficiently large number which may depend on ϵ and F, and θ is defined by $\theta = \arg \lambda$. In this present paper, we consider the case $\lambda = 1$ in (D1). Kimura proved the following theorems.

Theorem A. *Equation* (D1) *admits a formal solution of the form*

$$t\left(1 + \sum_{j+k \geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right) \tag{2.4}$$

containing an arbitrary constant $\hat{q}_{m0}$, *where* $\hat{q}_{jk}$ *are constants determined by* F.

Theorem B. *Given a formal solution of* (D1) *of the form* (2.4), *there exists a unique solution* $w(t)$ *satisfying the following conditions:*

(i) $w(t)$ *is holomorphic in* $D(\epsilon, R)$,

(ii) $w(t)$ *is expressible in the form*

$$w(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \tag{2.5}$$

where the domain $D(\epsilon, R)$ *is defined by* (2.2) *and* $b(t, \eta)$ *is holomorphic for* $t \in D(\epsilon, R)$, $|\eta| < 1/R$, *and has an expansion*

$$b(t, \eta) \sim \sum_{k=1}^{\infty} b_k(t) \eta^k.$$

Here

$$b_k(t) \sim \sum_{j+k \geq 1}^{\infty} \hat{q}_{jk} t^{-j}$$

as $t \to \infty$ *through* $D(\epsilon, R)$, *where* $\hat{q}_{jk}$ *are constants determined by* F.

Also there exists a unique solution $\hat{w}$ *which is holomorphic in* $\hat{D}(\epsilon, R)$ *and satisfies a condition analogous to* (ii), *where the domain* $\hat{D}(\epsilon, R)$ *is defined by* (2.3).

In Theorems A and B, Kimura defined the function F as in (2.1). In our method, we do not have a Laurent series for F.

In the following, A_2 and A_1 will be the constants defined in Theorem 1.1. We assume that $A_2 < 0$ and $A_1 < 0$.

Proposition 2.1. *Suppose* $\tilde{F}(t)$ *is a formal power series*

$$\tilde{F}(t) = t\Big(1 + \sum_{j=1}^{\infty} b_j t^{-j}\Big), \qquad b_1 = \lambda \neq 0. \tag{2.6}$$

Then the equation

$$\psi(\tilde{F}(t)) = \psi(t) + \lambda \tag{2.7}$$

has a formal solution

$$\psi(t) = t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right), \tag{2.8}$$

where q_1 *can be arbitrarily prescribed while the other coefficients* q_j $(j \geq 2)$ *and* q *are uniquely determined by* b_j $(j = 1, 2, \ldots)$, *independently of* q_1.

Proposition 2.2. *Suppose* $A_2 < 0$ *and* $\tilde{F}(t)$ *is holomorphic and has an asymptotic expansion*

$$\tilde{F}(t) \sim t\Big(1 + \sum_{j=1}^{\infty} b_j t^{-j}\Big), \qquad b_1 = \lambda \neq 0,$$

in $\{t : -1/(A_2 t) \in D^*(\kappa, \delta)\}$, *where* $D^*(\kappa, \delta)$ *is defined in* (1.10). *Then the equation* (2.7) *has a solution* $w = \psi(t)$, *which is holomorphic in* $\{t : -1/(A_2 t) \in D^*(\kappa/2, \delta/2)\}$ *and has an asymptotic expansion*

$$\psi(t) \sim t\left(1 + \sum_{j=1}^{\infty} q_j t^{-j} + q\frac{\log t}{t}\right)$$

there.

These propositions are proved as in [K, pp. 212–222].

Since $A_1 \leq A_2 < 0$ and $\kappa_0 = \kappa/2$, we see that $x = -1/(A_2 t) \in D^*(\kappa/2, \delta/2)$ is equivalent to $t \in D_1(\kappa/2, 2/(|A_2|\delta)) = D_1(\kappa_0, 2/(|A_2|\delta))$. Further we see that $x = -1/(A_1 t) \in D^*(\kappa/2, \delta/2)$ is equivalent to $t \in D_1(\kappa/2, 2/(|A_1|\delta)) = D_1(\kappa_0, 2/(|A_1|\delta))$, where $D_1(\kappa_0, R_0)$ is defined in (1.9). Since $A_1 \leq A_2 < 0$ and $D_1(\kappa_0, 2/(|A_2|\delta)) \subset D_1(\kappa_0, 2/(|A_1|\delta))$, we have $x = -1/(A_1 t) \in D^*(\kappa_0, \delta/2)$ for $t \in D_1(\kappa_0, 2/(|A_2|\delta))$.

We define a function ϕ to be the inverse of ψ, so that $w = \psi^{-1}(t) = \phi(t)$. Then $\phi\circ\psi(w) = w$, $\psi\circ\phi(t) = t$, and ϕ is a solution of the difference equation

$$\text{(D)} \qquad w(t+\lambda) = \tilde{F}(w(t)),$$

where $\tilde{F}$ is defined as in Proposition 2.1 (see p. 236 in [K]). Hereafter, we put $\lambda = 1$. Since $\theta = 0$, we then have the following Propositions 2.3 and 2.4, analogous to Theorems A and B.

PROPOSITION 2.3. *Suppose $\tilde{F}(t)$ is a formal power series as in* (2.6). *Then the equation* (D) *has a formal solution*

$$w = \phi(t) = t\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right), \tag{2.9}$$

where $\hat{q}_{jk}$ are constants determined by $\tilde{F}$ as in Theorem A.

PROPOSITION 2.4. *Suppose ϕ is the inverse of ψ, so $w = \psi^{-1}(t) = \phi(t)$. Given a formal solution of* (D) *of the form* (2.9)*, there exists a unique solution $w(t) = \phi(t)$ which is holomorphic and admits an asymptotic expansion for $t \in D_1(\kappa_0, 2/(|A_2|\delta))$ such that*

$$w = \phi(t) = t\left(1 + b\left(t, \frac{\log t}{t}\right)\right), \tag{2.10}$$

where

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k.$$

This function $\phi(t)$ is a solution of the difference equation (D).

In [S8], we have proved the following theorem.

THEOREM C. *Suppose $X(x,y)$ and $Y(x,y)$ are defined in* (1.5). *Assume $d_{20} = 0$ and*

$$(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^+(c_{20}, d_{11}, d_{30}), \tag{2.11}$$

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}), \tag{2.12}$$

for all $n \in \mathbb{N}$ $(n \geq 4)$, where

$$g_0^+(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4},$$

$$g_0^-(c_{20}, d_{11}, d_{30}) = \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4},$$

respectively. Then we have two formal solutions $\Psi^+(x) = \sum_{n\geq 2}^{\infty} a_n^+ x^n$, $\Psi^-(x) = \sum_{n\geq 2}^{\infty} a_n^- x^n$ of (1.6)*, where a_n^+, a_n^- are given by X and Y. For any κ with $0 < \kappa \leq \pi/2$ and small $\delta > 0$, define*

$$D^*(\kappa, \delta) = \{x : |\arg x| < \kappa,\ 0 < |x| < \delta\}. \tag{1.10}$$

Further assume $\frac{2c_{20}+d_{11}\pm\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4} \in \mathbb{R}$ *and*

$$\frac{2c_{20}+d_{11}+\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4} < 0. \tag{2.13}$$

Then there is a constant δ *and two solutions* $\Psi^+(x)$ *and* $\Psi^-(x)$ *of* (1.6), *which are holomorphic and have asymptotic expansions*

$$\Psi^+(x) \sim \sum_{n=2}^{\infty} a_n^+ x^n \quad \text{and} \quad \Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n \tag{2.14}$$

as $x \to 0$ *through* $D^*(\kappa, \delta)$.

If $d_{20} \neq 0$, *then* (1.6) *has no analytic solution.*

Note that $a_2^+ = g_0^+$, $a_2^- = g_0^-$. We have the following proposition, analogous to Theorem C.

Proposition 2.5. *Suppose* $X(x,y)$ *and* $Y(x,y)$ *are defined in* (1.5). *Assume* $d_{20} = 0$ *and*

$$(g_0^-(c_{20},d_{11},d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20},d_{11},d_{30}) \tag{2.12}$$

for all $n \in \mathbb{N}$ $(n \geq 4)$. *Then* (1.6) *has a formal solution* $\Psi^-(x) = \sum_{n\geq 2}^{\infty} a_n^- x^n$, *where* a_n^- *are given by* X *and* Y. *Further, assume* $\frac{2c_{20}+d_{11}\pm\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4}$ $\in \mathbb{R}$ *and*

$$\frac{2c_{20}+d_{11}-\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4} < 0. \tag{2.13}$$

Then for any κ *with* $0 < \kappa \leq \pi/2$, *there is a* $\delta > 0$ *and a solution* $\Psi^-(x)$ *of* (1.6) *which is holomorphic and has an asymptotic expansion*

$$\Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n \tag{2.14'}$$

as $x \to 0$ *through* $D^*(\kappa,\delta)$ *defined in* (1.10).

If $d_{20} \neq 0$, *then there is no analytic solution of* (1.6).

2.2. Proof of Theorem 1.1. We first prove (1). From Theorem C, we have formal solutions

$$\Psi(x) = \sum_{n=2}^{\infty} a_n x^j \tag{2.15}$$

of (1.6), where $a_2 = g_0^{\pm}(c_{20}, d_{11}, d_{30})$. We write the formal solutions as

$$\Psi_s(x) = \sum_{n=2}^{\infty} a_{n(s)} x^n \quad (s = 1, 2), \tag{2.16}$$

where $a_{2(1)} = g_0^+(c_{20}, d_{11}, d_{30})$, $a_{2(2)} = g_0^-(c_{20}, d_{11}, d_{30})$.

On the other hand putting $w_1(t) = -\frac{1}{A_1 x(t)}$ and $w_2(t) = -\frac{1}{A_2 x(t)}$ in (1.8), we have

$$\begin{aligned}
(2.17)\qquad w_s(t+1) &= -\frac{1}{A_s x(t+1)} \\
&= -\frac{1}{A_s X(x(t), \Psi_s(x(t)))} = -\frac{1}{A_s X\big(-\frac{1}{A_s w(t)}, \Psi_s\big(-\frac{1}{A_s w(t)}\big)\big)}
\end{aligned}$$

for $s = 1, 2$. From (1.5), we have

$$\begin{aligned}
X(x(t), \Psi_s(x(t))) &= x(t) + \Psi_s(x(t)) + \sum_{i+j\geq 2,\, i\geq 1} c_{ij} x(t)^i (\Psi_s(x(t)))^j \\
&= x(t)\Big\{1 + \sum_{\substack{(i+j\geq 2,\, i\geq 1)\\ \text{or } (i=0,\, j=1)}} c_{ij} x(t)^{i-1} (\Psi_s(x(t)))^j\Big\}.
\end{aligned}$$

where $c_{01} = 1$. Thus

$$\begin{aligned}
\frac{1}{X(x(t), \Psi_s(x(t)))} &= \frac{1}{x(t)\{1 - \sum_{\substack{(i+j\geq 2,\, i\geq 1)\\ \text{or } (i=0,\, j=1)}} -c_{ij} x(t)^{i-1} (\Psi_s(x(t)))^j\}} \\
&= \frac{1}{x(t)}\Big[1 + \sum_{k=1}^{\infty}\Big(\sum_{\substack{(i+j\geq 2,\, i\geq 1)\\ \text{or } (i=0,\, j=1)}} -c_{ij} x(t)^{i-1} (\Psi_s(x(t)))^j\Big)^k\Big].
\end{aligned}$$

Since $w_s(t) = -\frac{1}{A_s x(t)}$ $(s = 1, 2)$, we have

$$\begin{aligned}
&\frac{1}{X(x(t), \Psi_s(x(t)))} \\
&= -A_s w_s(t)\Big[1 + \sum_{k=1}^{\infty}\Big(\sum_{\substack{(i+j\geq 2,\, i\geq 1)\\ \text{or } (i=0,\, j=1)}} -c_{ij}\Big(-\frac{1}{A_s w_s(t)}\Big)^{i-1}\Big(\Psi_s\Big(-\frac{1}{A_s w_s(t)}\Big)\Big)^j\Big)^k\Big].
\end{aligned}$$

Since $\Psi_s(x)$ are formal solutions of (1.6) such that

$$\Psi_s(x) = \Psi_s\Big(-\frac{1}{A_s w_s}\Big) = \sum_{n=2}^{\infty} a_{n(s)}\Big(-\frac{1}{A_s w_s}\Big)^n \quad (s = 1, 2),$$

we have

$$(2.18)\qquad -\frac{1}{A_s X(x, \Psi_s(x))} = w_s\Big[1 + \frac{a_{2(s)} + c_{20}}{A_s} w_s^{-1} + \sum_{k\geq 2} \tilde{c}_{k(s)} (w_s)^{-k}\Big],$$

where $\tilde{c}_{k(s)}$ are determined by c_{ij} and $a_k(s)$ $(i+j \geq 2,\, i \geq 1,\, k \geq 2,\, s = 1, 2)$. From (2.18) and the definition of A_s, we have $a_{2(s)} + c_{20} = A_s$. Therefore we can write (2.17) in the form

$$w_s(t+1) = \tilde{F}_s(w_s(t)) = w_s(t)\Big\{1 + w_s(t)^{-1} + \sum_{k\geq 2} \tilde{c}_{k(s)}(w_s(t))^{-k}\Big\}. \tag{2.19}$$

On the other hand, putting $\lambda = 1$ and $m = 1$ in (2.1), i.e. $\theta = \arg[\lambda] = \arg[1] = 0$, then making use of Proposition 2.3, we have the following formal solutions of the first order difference equation (2.19):

$$w_s(t) = t\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k\right) \quad (s = 1, 2), \tag{2.20}$$

where $\hat{q}_{jk(s)}$ are determined by $\tilde{F}_s$ in (2.19). From (2.18), (2.19) and (1.6), $\tilde{F}_s$ is defined by X and Y. Hence $\hat{q}_{jk(s)}$ are determined by X and Y.

Since $x(t) = -\frac{1}{A_s w_s(t)}$, we have formal solutions $x(t)$ of (1.2) such that

$$x(t) = -\frac{1}{A_s t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1} \quad (s = 1, 2). \tag{2.21}$$

From the relation of (1.2) and (1.8) to (1.6), we have proved (1) of Theorem 1.1.

Next we will prove (2) of Theorem 1.1, that is, the existence of solutions $x^+(t)$ and $x^-(t)$ of (1.2). We suppose that $R_0 > R$ and $\kappa_0 < \pi/4 - \epsilon$. Since $\theta = 0$, we have

$$D_1(\kappa_0, R_0) \subset D(\epsilon, R). \tag{2.22}$$

For $x \in D^*(\kappa, \delta)$, making use of Theorem C, we have solutions $\Psi_{(s)}(x)$ $(s = 1, 2)$ of (1.6) which are holomorphic and can be expanded asymptotically in $D^*(\kappa, \delta)$ such that

$$\Psi_{(s)}(x) \sim \sum_{j=k}^{\infty} a_{j(s)} x^j \quad (s = 1, 2).$$

From the assumption $R_1 = \max(R_0, 2/(|A_2|\delta))$ in Theorem 1.1, making use of Proposition 2.4, we have solutions $w_s(t)$ $(s = 1, 2)$ of (2.19) which have an asymptotic expansion

$$w_s(t) = t\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right),$$

in $t \in D_1(\kappa_0, R_1)$, where

$$b_s\left(t, \frac{\log t}{t}\right) \sim t\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk(s)} t^{-j} \left(\frac{\log t}{t}\right)^k\right) \quad (s = 1, 2).$$

Thus we have solutions $x_s(t)$ of (1.2) which have asymptotic expansions

$$x_s(t) = -\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1} \qquad (s = 1, 2)$$

there. First we take a small $\delta > 0$. For sufficiently large R, since $R_1 \geq R_0 > R$, we have

$$\left|\frac{1}{A_1 t}\right|\left|1 + b_1\left(t, \frac{\log t}{t}\right)\right|^{-1} < \frac{1}{|A_1|R}(1+1) < \frac{1}{|A_2|R}(1+1) < \delta,$$

$$\left|\frac{1}{A_2 t}\right|\left|1 + b_2\left(t, \frac{\log t}{t}\right)\right|^{-1} < \frac{1}{|A_2|R}(1+1) < \delta \tag{2.23}$$

for $t \in D_1(\kappa_0, R_1)$. Since $A_1 \leq A_2 < 0$ by (1.17),

$$\arg\left[-\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}\right] = -\arg[t] - \arg\left[1 + b_1\left(t, \frac{\log t}{t}\right)\right],$$

$$\arg\left[-\frac{1}{A_2 t}\left(1 + b_2\left(t, \frac{\log t}{t}\right)\right)^{-1}\right] = -\arg[t] - \arg\left[1 + b_2\left(t, \frac{\log t}{t}\right)\right].$$

For sufficiently large R_1, we then have

$$\left|\arg\left[1 + b_1\left(t, \frac{\log t}{t}\right)\right]\right|, \left|\arg\left[1 + b_2\left(t, \frac{\log t}{t}\right)\right]\right| < \kappa_0 \qquad \text{for } t \in D_1(\kappa_0, R_1).$$

Hence

$$-\kappa_0 - \kappa_0 \leq \arg\left[-\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}\right] \leq \kappa_0 + \kappa_0 \qquad (s = 1, 2).$$

From the assumption $\kappa = 2\kappa_0$, we have

$$\left|\arg\left[-\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}\right]\right| < \kappa \leq \frac{\pi}{2} \tag{2.24}$$

$$\text{for } t \in D_1(\kappa_0, R_1) \qquad (s = 1, 2).$$

From (2.23) and (2.24), we obtain

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}, \; x_2(t) = -\frac{1}{A_2 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}$$

such that $x_s(t) \in D^*(\kappa, \delta)$ for some κ $(0 < \kappa \leq \pi/2)$. Hence we have $\Psi_{(s)}(x(t))$ $(s = 1, 2)$ which satisfies the equation (1.6).

Therefore from existence of solutions $\Psi_{(s)}$ $(s = 1, 2)$ of (1.6) and Proposition 2.4, we have holomorphic solutions $w_s(t)$ of the first order difference equation (2.19) for $t \in D_1(\kappa_0, R_1)$. Hence we obtain solutions $x_s(t)$ of (1.2) for t there, which satisfy the following conditions:

(i) $x_s(t)$ are holomorphic and $x_s(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$, $s = 1, 2$,

(ii) $x_s(t)$ $(s=1,2)$ are expressible in the form

$$x_s(t) = -\frac{1}{A_s t}\left(1 + b_s\left(t, \frac{\log t}{t}\right)\right)^{-1}. \tag{2.25}$$

Here $b_s(t, (\log t)/t)$ has an asymptotic expansion

$$b_s\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geq 1} \hat{q}_{jk(s)} t^{-j}\left(\frac{\log t}{t}\right)^k$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$. ■

Finally, we have a solution $(u(t),\ v(t))$ of (1.1) by the transformation

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = P\begin{pmatrix} x_1(t) \\ \Psi_{(s)}(x_1(t)) \end{pmatrix} \text{ and } P\begin{pmatrix} x_2(t) \\ \Psi_{(s)}(x_2(t)) \end{pmatrix}.$$

From Proposition 2.5 and Theorem 1.1, we obtain

Lemma 2.6. *Suppose $X(x,y)$ and $Y(x,y)$ are expanded in the forms* (1.5) *such that $X_1(x,y) \not\equiv 0$ or $Y_1(x,y) \not\equiv 0$. Define $A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20}$, $A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20}$ $(A_1 \leq A_2)$.*

(1) *Suppose $d_{20} = 0$ and*

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) \tag{2.26}$$

for all $n \in \mathbb{N}$ $(n \geq 4)$. Then we have a formal solution $x(t)$ of (1.2) *of the form*

$$-\frac{1}{A_1 t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j}\left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{2.27}$$

where $\hat{q}_{jk}$ are constants determined by X and Y.

(2) *Further suppose $R_1 = \max(R_0, 2/(|A_1|\delta))$, and assume*

$$A_1 < 0. \tag{2.28}$$

Then there is a solution $x_1(t)$ of (1.2) *such that*

(i) *$x_1(t)$ is holomorphic and $x_1(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$,*

(ii) *$x_1(t)$ is expressible in the form*

$$x_1(t) = -\frac{1}{A_1 t}\left(1 + b_1\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{2.29}$$

where $b_1(t, (\log t)/t)$ has an asymptotic expansion

$$b_1\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geq 1} \hat{q}_{jk(1)} t^{-j}\left(\frac{\log t}{t}\right)^k$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

3. An application. Consider the following population model:

$$\text{(P)} \qquad u(t+2) = \alpha u(t+1) + \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)},$$

where $\alpha = 1 + r$ and β are constants. This model was proposed by Prof. D. S. Dendrinos [D]. Here r is the net (births minus deaths) endogenous population (stock) growth rate. The second term on the right hand side is a function depicting net in-migration (immigration) at $t+1$, which should be considered as a "momentum" to grow from t to $t+1$. We assume that α and β are constants such that $\alpha > 0$ ($r > -1$) and $\beta > 0$ in (P).

Let

$$u(t+2) = u_1(t+2) + u_2(t+2),$$

where $u_1(t+2) = \alpha u(t+1)$, $u_2(t+2) = \beta \frac{u(t+1)-\alpha u(t)}{\alpha u(t)}$. Then $u_1(t+2)$ is a term for endogenous population growth rate from $t+1$ to $t+2$, and $u_2(t+2)$ is due to net in-migration (immigration) rate. Indeed we can write

$$\begin{aligned} u_1(t+2) &= \alpha u(t+1) = \alpha\{u_1(t+1) + u_2(t+1)\}, \\ u_2(t+2) &= \beta \frac{u(t+1) - \alpha u(t)}{\alpha u(t)} = \beta \frac{u(t+1) - u_1(t+1)}{u_1(t+1)} = \beta \frac{u_2(t+1)}{u_1(t+1)}, \end{aligned}$$

and we see that $u_1(t+1)$ is the endogenous population growth rate from t to $t+1$, and $u_2(t+1)$ is due to net in-migration (immigration) ratio at $t+1$.

We may write (P) as

$$u(t+2) - \alpha u(t+1) = \frac{c}{u(t)}\{u(t+1) - \alpha u(t)\}, \quad c = \frac{\beta}{\alpha}.$$

When $\alpha \neq 1$, (P) admits the unique equilibrium value $c = \beta/\alpha$, and we have eigenvalues λ_1 and λ_2 of this equation such that $|\lambda_1| \neq 1$ and $|\lambda_2| \neq 1$. Therefore we can have general analytic solutions such that $u(t+n) \to c$ as $n \to \infty$ ($n \in \mathbb{N}$), making use of Theorems of [S6].

If $\alpha = 1$, then any value can be an equilibrium point of (P). Suppose the equation (P) has a solution $u(t)$ such that $u(t+n) \to u_0 > 0$ as $n \to \infty$. From [S4], we have the following three cases.

1) $u(t_0 + n) \downarrow u_0 \geq c$ as $n \to \infty$,
2) $u(t_0 + n) \uparrow u_0 > c$ as $n \to \infty$,
3) there is n_0 such that $u(t_0 + n_0) \leq 0$ (extermination).

However in [S4] we have not been able to prove the existence of a solution of (P) under this condition. In this paper, we will obtain a solution of (P) by Lemma 2.6 for the case $\alpha = 1$.

Putting $u(t) = v(t) + \beta/\alpha$, we have

$$\begin{aligned}
&v(t+2) + \frac{\beta}{\alpha} \\
&\quad = \alpha v(t+1) + \beta + \frac{v(t+1) - \alpha v(t) + \frac{\beta}{\alpha} - \beta}{1 + \frac{\alpha}{\beta} v(t)} \\
&\quad = \alpha v(t+1) + \beta \\
&\qquad + \left\{ v(t+1) - \alpha v(t) + \frac{\beta}{\alpha} - \beta \right\} \\
&\qquad \times \left\{ 1 - \frac{\alpha}{\beta} v(t) + \frac{\alpha^2}{\beta^2} v(t)^2 - \frac{\alpha^3}{\beta^3} v(t)^3 + \frac{\alpha^4}{\beta^4} v(t)^4 - \cdots \right\} \\
&\quad = (1+\alpha) v(t+1) + (-\alpha - 1 + \alpha) v(t) + \beta + \frac{\beta}{\alpha} - \beta + F(v(t), v(t+1)),
\end{aligned}$$

i.e.,

$$v(t+2) = (1+\alpha) v(t+1) - v(t) + F(v(t), v(t+1)),$$

where

$$\begin{aligned}
F(v(t), v(t+1)) = {} & -\frac{\alpha}{\beta} v(t) v(t+1) + \frac{\alpha^2}{\beta} v(t)^2 \\
& + \left(v(t+1) - \alpha v(t) + \frac{\beta}{\alpha} - \beta \right) \sum_{i=2}^{\infty} \left(-\frac{\alpha}{\beta} \right)^i v(t)^i.
\end{aligned} \tag{3.1}$$

Next put $v(t+1) = \xi(t)$, $v(t) = \eta(t)$. Then

$$\begin{pmatrix} \xi(t+1) \\ \eta(t+1) \end{pmatrix} = \begin{pmatrix} \alpha+1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} + \begin{pmatrix} F(\eta(t), \xi(t)) \\ 0 \end{pmatrix}.$$

Set

$$M = \begin{pmatrix} \alpha+1 & -1 \\ 1 & 0 \end{pmatrix}.$$

When $\alpha = 1$, the eigenvalues λ_1 and λ_2 of M are $\lambda_1 = \lambda_2 = 1$. Further put $P = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) = P \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Then we obtain the difference equation

$$\begin{aligned}
&\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + P^{-1} \begin{pmatrix} F(x(t) + y(t), x(t) + 2y(t)) \\ 0 \end{pmatrix}.
\end{aligned} \tag{3.2}$$

Since

$$P^{-1} \begin{pmatrix} F(x(t) + y(t), x(t) + 2y(t)) \\ 0 \end{pmatrix} = \begin{pmatrix} -F(x(t) + y(t), x(t) + 2y(t)) \\ F(x(t) + y(t), x(t) + 2y(t)) \end{pmatrix},$$

we can write the equations (3.2) as follows:

$$\tag{1.3} \begin{cases} x(t+1) = X(x(t), y(t)), \\ y(t+1) = Y(x(t), y(t)), \end{cases}$$

where

$$\tag{1.2$'$} \begin{cases} X(x,y) = x + y - F(x+y, x+2y) \\ \qquad\qquad = x + \Big(y + \sum\limits_{i+j\geq 2} c_{ij}x^i y^j\Big) = x + X_1(x,y), \\ Y(x,y) = y + F(x+y, x+2y) \\ \qquad\qquad = y + \Big(\sum\limits_{i+j\geq 2} d_{ij}x^i y^j\Big) = y + Y_1(x,y), \end{cases}$$

with $d_{ij} = -c_{ij}$.

From the definition (3.1) of F, when $\alpha = 1$, we have

$$\tag{3.3} F(\eta,\xi) = \frac{1}{\beta}\eta^2 - \frac{1}{\beta}\eta\xi - \frac{1}{\beta^2}\eta^3 + \frac{1}{\beta^2}\eta^2\xi + \sum_{i\geq 3}\frac{1}{\beta^i}(-1)^i\eta^i(\xi - \eta).$$

Thus

$$\tag{3.4} F(x+y, x+2y) = -\frac{1}{\beta}(xy+y^2) + \frac{1}{\beta^2}(x^2y + 2xy^2 + y^3) + \sum_{i+j\geq 4, j\geq 1}\gamma_{ij}x^i y^j,$$

where $\gamma_{ij} = \gamma_{ij}(\beta)$ are constants. From (3.4), we have $c_{20} = d_{20} = 0$, $c_{n0} = d_{n0} = 0$ $(n \geq 3)$, $d_{11} = -1/\beta < 0$, $d_{02} = -1/\beta < 0$, $d_{21} = 1/\beta^2$. Thus

$$\begin{aligned} A_1 = g_0^-(c_{20}, d_{11}, d_{30}) + c_{20} &= \frac{-(2c_{20} - d_{11}) - \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} + c_{20} \\ &= \frac{-(0 + 1/\beta) - \sqrt{(0 + 1/\beta)^2 + 0}}{4} + 0 \\ &= -\frac{1}{2\beta} < 0, \\ A_2 = g_0^+(c_{20}, d_{11}, d_{30}) + c_{20} &= \frac{-(2c_{20} - d_{11}) + \sqrt{(2c_{20} - d_{11})^2 + 8d_{30}}}{4} + c_{20} \\ &= \frac{-(0 + 1/\beta) + \sqrt{(0 + 1/\beta)^2 + 0}}{4} + 0 = 0, \\ c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30}) &= \frac{3}{2\beta} > 0. \end{aligned}$$

Here we cannot have $A_1 \leq A_2 < 0$, but we have $A_1 < A_2 = 0$, which is the condition (2.28) in Lemma 2.6. Thus putting $a_2 = g_0^-(c_{20}, d_{11}, d_{30})$, we have $a_2 + c_{20} < 0$. Further

$$(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n \neq c_{20} - d_{11} - g_0^-(c_{20}, d_{11}, d_{30})$$

for all $n \in \mathbb{N}$. By Proposition 2.5, the functional equation (1.6) with X and Y defined by (1.2′) has a formal solution $\Psi^-(x) = \sum_{n\geq 2}^{\infty} a_n^- x^n$, where a_n^- are given by X and Y. Here the function F is defined by (3.3). Further, for any κ with $0 < \kappa \leq \pi/2$, there are a $\delta > 0$ and a solution $\Psi^-(x)$ of (1.6), which is holomorphic and can be expanded asymptotically as

$$\Psi^-(x) \sim \sum_{n=2}^{\infty} a_n^- x^n$$

in the domain $D^*(\kappa, \delta)$ defined in (1.10).

Making use of Lemma 2.6, we have a formal solution $x(t)$ of (3.2),

$$-\frac{1}{A_1 t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1} = \frac{2\beta}{t}\left(1 + \sum_{j+k\geq 1} \hat{q}_{jk} t^{-j} \left(\frac{\log t}{t}\right)^k\right)^{-1}, \tag{3.5}$$

where $\hat{q}_{jk}$ are constants determined by X and Y in (1.2′). Further suppose $R_1 = \max(R_0, 2/(|A_1|\delta))$. Since $A_1 = -1/(2\beta) < A_2 = 0$, there is a solution $x(t)$ of (3.2) such that

(i) $x(t)$ is holomorphic and $x(t) \in D^*(\kappa, \delta)$ for $t \in D_1(\kappa_0, R_1)$,
(ii) $x(t)$ is expressible in the form

$$x(t) = -\frac{1}{A_1 t}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1} = \frac{2\beta}{t}\left(1 + b\left(t, \frac{\log t}{t}\right)\right)^{-1}, \tag{3.6}$$

where $b(t, (\log t)/t)$ has an asymptotic expansion

$$b\left(t, \frac{\log t}{t}\right) \sim \sum_{j+k\geq 1} \hat{q}_{jk(1)} t^{-j} \left(\frac{\log t}{t}\right)^k$$

as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

By the definition (1.7), we have $y(t) = \Psi(x(t))$. Since

$$\begin{pmatrix} u(t+1) - \beta/\alpha \\ u(t) - \beta/\alpha \end{pmatrix} = \begin{pmatrix} v(t+1) \\ v(t) \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

we have a solution $u(t)$ of the population model (P) such that

$$u(t) = x(t) + y(t) + \frac{\beta}{\alpha} = x(t) + \Psi(x(t)) + \frac{\beta}{\alpha},$$

where $x(t)$ is given in the equation (3.6) as $t \to \infty$ through $D_1(\kappa_0, R_1)$.

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