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Finite-dimensional pullback attractors for parabolic equations with Hardy type potentials

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Abstract. Using the asymptotic *a priori* estimate method, we prove the existence of a pullback $\mathcal{D}$-attractor for a reaction-diffusion equation with an inverse-square potential in a bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, with the nonlinearity of polynomial type and a suitable exponential growth of the external force. Then under some additional conditions, we show that the pullback $\mathcal{D}$-attractor has a finite fractal dimension and is upper semicontinuous with respect to the parameter in the potential.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin. In this paper we consider the nonautonomous reaction-diffusion equation with the Hardy type potential of the form

$$(1.1)\qquad \begin{cases} u_t - \Delta u - \dfrac{\mu}{|x|^2}u + f(u) = g(x,t), & x \in \Omega,\ t > \tau, \\ u|_{\partial\Omega} = 0, & t > \tau, \\ u(x,\tau) = u_\tau(x), & x \in \Omega, \end{cases}$$

where $u_\tau \in L^2(\Omega)$ is given, $0 < \mu \leq \mu^*$ is a parameter, $\mu^* = \left(\frac{N-2}{2}\right)^2$ is the best constant in the Hardy inequality

$$(1.2)\qquad \mu^* \int_\Omega \frac{|u|^2}{|x|^2}\,dx \leq \int_\Omega |\nabla u|^2\,dx, \qquad \forall u \in C_0^\infty(\Omega),$$

and the nonlinearity f and the external force g satisfy some conditions specified later.

The case where $g \equiv 0$ and f has some special forms was studied in [1, 2, 6, 7, 17], which focused on global existence and dependence of the behavior of the solutions of (1.1) on the parameter μ.

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In this paper we continue the study of the long-time behavior of solutions to problem (1.1) by allowing the external force g to depend on time t. Nonautonomous equations appear in many applications in natural sciences, so they are of great importance and interest. One way of studying the long-time behavior of solutions of such equations is to use the theory of pullback attractors. This theory has been developed for both nonautonomous and random dynamical systems and has shown to be very useful in the understanding of the dynamics of nonautonomous dynamical systems (see [3] and references therein).

In this paper, we assume that the nonlinearity f and the external force g satisfy the following conditions:

(***F***) $f \in C^1(\mathbb{R})$ satisfies, for some $p \geq 2$,
$$C_1|u|^p - k_1 \leq f(u)u \leq C_2|u|^p + k_2, \qquad f'(u) \geq -\ell, \qquad \forall u \in \mathbb{R};$$

(***G***) $g \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}; L^2(\Omega))$ satisfies
$$\int_{-\infty}^{0} e^{\lambda_{1,\mu} s}(|g(s)|_2^2 + |g'(s)|_2^2)\, ds < \infty,$$
where $\lambda_{1,\mu}$ is the first eigenvalue of the operator $A_\mu = -\Delta - \mu/|x|^2$ in Ω with the homogeneous Dirichlet condition.

To study problem (1.1), we will use the space $H_\mu(\Omega)$, $0 \leq \mu \leq \mu^*$, defined as the closure of $C_0^\infty(\Omega)$ in the norm
$$\|u\|_\mu = \left(\int_\Omega \left(|\nabla u|^2 - \mu\frac{|u|^2}{|x|^2}\right) dx\right)^{1/2}.$$

The aim of this paper is to prove the existence and upper semicontinuity with respect to the parameter μ of a finite-dimensional pullback $\mathcal{D}$-attractor in the space $H_\mu(\Omega) \cap L^p(\Omega)$ for the process associated to problem (1.1). Let us describe the methods used in the paper. First, we apply the compactness method [11] to prove the global existence of a weak solution and use *a priori* estimates to show the existence of a family of pullback $\mathcal{D}$-absorbing sets $\hat{B} = \{B(t) : t \in \mathbb{R}\}$ in $H_\mu(\Omega) \cap L^p(\Omega))$ for the process. By the compactness of the embedding $H_\mu(\Omega) \hookrightarrow L^2(\Omega)$, the process is pullback $\mathcal{D}$-asymptotically compact in $L^2(\Omega)$. This immediately implies the existence of a pullback $\mathcal{D}$-attractor in $L^2(\Omega)$. When proving the existence of pullback $\mathcal{D}$-attractors in $L^p(\Omega)$ and in $H_\mu(\Omega) \cap L^p(\Omega)$, to overcome the difficulty due to the lack of embedding results, we use the asymptotic *a priori* estimate method initiated in [13] for autonomous equations. Finally, using the abstract theories developed recently in [8, 4], we prove that the resulting pullback $\mathcal{D}$-attractor has a finite fractal dimension and is upper semicontinuous with respect to the parameter μ at $\mu = 0$. In particular, we show that the pullback $\mathcal{D}$-attractors

$\hat{\mathcal{A}}_\mu$ of the singular reaction-diffusion equation converge to the pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}}_0$ of the classical reaction-diffusion equation as the parameter μ tends to 0. It is also worth noticing that, when $\mu = 0$, our results recover and improve the recent results in [16, 10, 9, 12] for the nonautonomous Laplace equation in bounded domains.

The paper is organized as follows. In Section 2, for the convenience of the reader, we recall some concepts and results on function spaces and pullback attractors which we will use. In Section 3, we prove the existence of a pullback $\mathcal{D}$-attractor $\hat{\mathcal{A}}_\mu = \{A_\mu(t) : t \in \mathbb{R}\}$ in $H_\mu(\Omega) \cap L^p(\Omega)$ by using the asymptotic *a priori* estimate method. In Section 4, we give some estimates on the fractal dimension of the pullback $\mathcal{D}$-attractor. The upper semicontinuity of $\hat{\mathcal{A}}_\mu$ at $\mu = 0$ is discussed in the last section.

Notation. For brevity, we denote by $|\cdot|_2$, $(\cdot,\cdot)$ and $\|\cdot\|_\mu$, $((\cdot,\cdot))_\mu$ the norms and scalar products in $L^2(\Omega)$ and $H_\mu(\Omega)$, respectively, and by $|\cdot|_p$ the norm in $L^p(\Omega)$. We also frequently use the notation

$$\Omega_M = \Omega(u(t) \geq M) = \{x \in \Omega : u(x,t) \geq M\}.$$

2. Preliminaries

2.1. Function spaces and operators. For each $0 \leq \mu \leq \mu^*$, we define the space $H_\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_\mu^2 = \int_\Omega \left(|\nabla u|^2 - \mu \frac{|u|^2}{|x|^2} \right) dx.$$

Then $H_\mu(\Omega)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_\mu = \int_\Omega \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2} \right) dx \quad \text{for all } u, v \in H_\mu(\Omega).$$

It is known (see [17]) that if $0 \leq \mu < \mu^*$, then $H_\mu(\Omega) \equiv H_0^1(\Omega)$. In the critical case, i.e., when $\mu = \mu^*$, we recall the improved Hardy–Poincaré inequality of [17],

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2} \right) dx \geq C(q,\Omega) \|u\|_{W^{1,q}(\Omega)}^2, \qquad 1 \leq q < 2, \tag{2.1}$$

and for $0 \leq s < 1$, $1 \leq r < r_* = \frac{2N}{N-2(1-s)}$,

$$\int_\Omega \left(|\nabla u|^2 - \mu^* \frac{|u|^2}{|x|^2} \right) dx \geq C(s,r,\Omega) \|u\|_{W^{s,r}(\Omega)}^2 \tag{2.2}$$

for all $u \in C_0^\infty(\Omega)$. These imply that the following continuous embeddings hold for $1 \leq q < 2$ and $0 \leq s < 1$:

$$H_\mu(\Omega) \hookrightarrow W_0^{1,q}(\Omega), \qquad H_\mu(\Omega) \hookrightarrow H_0^s(\Omega). \tag{2.3}$$

Moreover, since $W_0^{1,q}(\Omega)$ is compactly embedded in $H_0^s(\Omega)$ for a suitable $q = q(s)$ close enough to 2, and $H_0^s(\Omega)$ is compactly embedded in $L^2(\Omega)$, we infer that the embeddings

$$H_\mu(\Omega) \hookrightarrow L^2(\Omega), \quad H_\mu(\Omega) \hookrightarrow H_0^s(\Omega), \quad 0 \le s < 1, \tag{2.4}$$

are compact.

Recall that the embedding $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for $1 \le p \le Nq/(N-q)$ and $q < N$. Thus by denoting $p^* = Nq/(N-q)$ for $1 \le q < 2$, it follows from (2.3) that the continuous embedding $H_\mu(\Omega) \hookrightarrow L^p(\Omega)$ holds for any $1 \le p \le p^*$.

We now consider the boundary value problem

$$\begin{cases} -\Delta u - \dfrac{\mu}{|x|^2} u = \lambda u & \text{for } x \in \Omega, \\ u = 0 \quad \text{for } x \in \partial\Omega. \end{cases} \tag{2.5}$$

In order to apply the Friedrichs extension of symmetric operators (see [18]) we recall the improved Hardy inequality of [17],

$$\int_\Omega |\nabla u|^2 \, dx \ge \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} \, dx + \lambda_\Omega \int_\Omega |u|^2 \, dx, \tag{2.6}$$

where λ_Ω is a positive constant depending on Ω, and set $X = L^2(\Omega)$, $D(\widetilde{A}) = C_0^\infty(\Omega)$, $\widetilde{A}u = -\Delta u - (\mu/|x|^2)u$. Then it follows that the operator $\widetilde{A}$ is a positive and self-adjoint operator and the energy space X_E equals $H_\mu(\Omega)$ since X_E is the completion of $D(\widetilde{A}) = C_0^\infty(\Omega)$ with respect to the scalar product

$$\langle u, v\rangle_\mu = \int_\Omega \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2}\right) dx.$$

Moreover,

$$\widetilde{A} \subset A \subset A_E,$$

where $A_E : H_\mu(\Omega) \to H_\mu^{-1}(\Omega)$ is the energetic extension ($H_\mu^{-1}(\Omega)$ is the dual space of $H_\mu(\Omega)$), and $A = -\Delta - \mu/|x|^2$ is the Friedrichs extension of $\widetilde{A}$ with the domain of definition

$$D(A) = \{u \in H_\mu(\Omega) : A(u) \in X\}.$$

We also have the evolution triple $H_\mu(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow\hookrightarrow H_\mu^{-1}(\Omega)$ with compact and dense embeddings. Hence, for each $0 < \mu \le \mu^*$, there exists a complete orthonormal system of eigenvectors $(e_{j,\mu}, \lambda_{j,\mu})$ depending on μ such that

$$(e_{j,\mu}, e_{k,\mu}) = \delta_{j,k} \quad \text{and} \quad -\Delta e_{j,\mu} - \frac{\mu}{|x|^2} e_{j,\mu} = \lambda_{j,\mu} e_{j,\mu}, \qquad j,k = 1,2,\ldots,$$

$$0 < \lambda_{1,\mu} \le \lambda_{2,\mu} \le \lambda_{3,\mu} \le \cdots, \qquad \lambda_{j,\mu} \to +\infty \quad \text{as } j \to +\infty.$$

Finally we observe that for all $u \in H_\mu(\Omega)$,

$$\|u\|_\mu^2 \geq \lambda_{1,\mu}|u|_2^2. \tag{2.7}$$

2.2. Pullback attractors. Let (X, d) be a metric space. For $A, B \subset X$, we define the Hausdorff semi-distance between A and B by

$$\operatorname{dist}(A, B) = \sup_{x\in A} \inf_{x\in B} d(x, y).$$

Let $\{U(t,\tau) : t, \tau \in \mathbb{R}\}$ be a *process* in X, i.e., a two-parameter family of mappings $U(t,\tau) : X \to X$ such that $U(\tau,\tau) = \mathrm{Id}$ and $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$ in $\mathbb{R}$. The process $\{U(t,\tau)\}$ is said to be *norm-to-weak continuous* on X if $U(t,\tau)x_n$ converges weakly to $U(t,\tau)x$ as x_n converges strongly to x in X, for all $t \geq \tau$ in $\mathbb{R}$. Now, we recall a useful method to verify that a process is norm-to-weak continuous.

LEMMA 2.1 ([19]). *Let X and Y be two Banach spaces, and X^*, Y^* be their respective dual spaces. Assume that X is dense in Y, the injection $i : X \to Y$ is continuous and its adjoint $i^* : Y^* \to X^*$ is dense, and $\{U(t,\tau)\}$ is a continuous or weakly continuous process on Y. Then $\{U(t,\tau)\}$ is norm-to-weak continuous on X iff for all $t \geq \tau$ in $\mathbb{R}$, $U(t,\tau)$ maps compact subsets of X to bounded subsets of X.*

Let $\mathcal{B}(X)$ be the family of all nonempty bounded subsets of X, and $\mathcal{D}$ be a nonempty class of parameterized sets $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

DEFINITION 2.2. A process $\{U(t,\tau)\}$ is said to be pullback $\mathcal{D}$-*asymptotically compact* if for all $t \in \mathbb{R}$, $\hat{\mathcal{D}} \in \mathcal{D}$ and any $\tau_n \to -\infty$ and $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

DEFINITION 2.3. A process $\{U(t,\tau)\}$ is said to be pullback ω-$\mathcal{D}$-*limit compact* if for any $\epsilon > 0$, $t \in \mathbb{R}$, and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists a $\tau_0(\mathcal{D}, \epsilon, t) \leq t$ such that

$$\alpha\Big(\bigcup_{\tau\leq\tau_0} U(t,\tau)D(\tau)\Big) \leq \epsilon,$$

where α is the *Kuratowski measure of noncompactness* of $B \in \mathcal{B}(X)$, defined by

$$\alpha(B) = \inf\{\delta > 0 : B \text{ has a finite open cover of sets of diameter} < \delta\}.$$

LEMMA 2.4 ([10]). *A process $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact iff it is pullback ω-$\mathcal{D}$-limit compact.*

DEFINITION 2.5. A family of bounded sets $\hat{\mathcal{B}} \in \mathcal{D}$ is said to be *pullback $\mathcal{D}$-absorbing* for the process $\{U(t,\tau)\}$ if for any $t \in \mathbb{R}$ and $\hat{\mathcal{D}} \in \mathcal{D}$, there exists $\tau_0 = \tau_0(\hat{\mathcal{D}}, t)$ such that

$$\bigcup_{\tau\leq\tau_0} U(t,\tau)D(\tau) \subset B(t).$$

DEFINITION 2.6. A family $\hat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a *pullback $\mathcal{D}$-attractor* for the process $U(t, \tau)$ if

(i) $A(t)$ is compact for all $t \in \mathbb{R}$.
(ii) $\hat{\mathcal{A}}$ is invariant, i.e., $U(t, \tau)A(\tau) = A(t)$ for all $t \geq \tau$.
(iii) $\hat{\mathcal{A}}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau \to -\infty} \operatorname{dist}(U(t, \tau)D(\tau), A(t)) = 0$$

for all $\hat{\mathcal{D}} \in \mathcal{D}$ and all $t \in \mathbb{R}$.
(iv) If $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

THEOREM 2.7 ([10]). *Let $\{U(t, \tau)\}$ be a norm-to-weak continuous process such that $\{U(t, \tau)\}$ is pullback $\mathcal{D}$-asymptotically compact. If there exists a family of pullback $\mathcal{D}$-absorbing sets $\hat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathcal{D}$, then $\{U(t, \tau)\}$ has a unique pullback $\mathcal{D}$-attractor $\mathcal{A} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)}.$$

2.3. Fractal dimension of pullback attractors. Consider a given separable Hilbert space H, with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. Given a compact set $K \subset H$ and $\epsilon > 0$, we denote by $N_\epsilon(K)$ the minimum number of open balls in H with radii $< \epsilon$ that are necessary to cover K.

DEFINITION 2.8. For any nonempty compact $K \subset H$, the *fractal dimension* of K is the number

$$d_F(K) = \limsup_{\epsilon \to 0} \frac{\log(N_\epsilon(K))}{\log(1/\epsilon)}. \tag{2.8}$$

Consider a separable real Hilbert space $V \subset H$ such that the injection of V in H is continuous, and V is dense in H.

We identify H with its topological dual H', identifying $v \in V$ with the element $f_v \in H'$ defined by

$$f_v(h) = (v, h), \qquad h \in H.$$

Let $F : V \times \mathbb{R} \to V'$ be a given family of nonlinear operators such that, for all $\tau \in \mathbb{R}$ and $u_0 \in H$, there exists a unique function $u(t) = u(t; \tau, u_0)$ satisfying

$$\begin{cases} u \in L^2(\tau, T; V) \cap C([\tau, T]; H),\ F(u(t), t) \in L^1(\tau, T; V') \quad \text{for all } T > \tau, \\ \dfrac{du}{dt} = F(u(t), t), \quad t > \tau, \\ u(\tau) = u_0. \end{cases} \tag{2.9}$$

Define

$$U(t,\tau)u_0 = u(t,\tau;u_0), \quad \tau \le t,\, u_0 \in H. \tag{2.10}$$

Fix $T^* \in \mathbb{R}$. We assume that there exists a family $\{K(t) : t \le T^*\}$ of nonempty compact subsets of H with the invariance property

$$U(t,\tau)K(\tau) = K(t) \qquad \text{for all } \tau \le t \le T^*, \tag{2.11}$$

and such that, for all $\tau \le t \le T^*$ and $u_0 \in K(\tau)$, there exists a continuous linear operator $L(t,\tau,u_0) \in \mathcal{L}(H)$ such that

$$|u(t,\tau)\overline{u}_0 - U(t,\tau)u_0 - L(t,\tau,u_0)(\overline{u}_0 - u_0)| \le \gamma(t-\tau, |\overline{u}_0 - u_0|)|\overline{u}_0 - u_0| \tag{2.12}$$

for all $\overline{u}_0 \in K(\tau)$, where $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\gamma(s,\cdot)$ is nondecreasing for all $s \ge 0$, and

$$\lim_{r\to 0} \gamma(s,r) = 0 \quad \text{ for any } s \ge 0. \tag{2.13}$$

We assume that, for all $t \le T^*$, the mapping $F(\cdot,t)$ is Gateaux differentiable in V, i.e., for any $u \in V$ there exists a continuous linear operator $F'(u,t) \in \mathcal{L}(V,V')$ such that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}(F(u+\epsilon v,t) - F(u,t) - \epsilon F'(u,t)v) = 0 \in V'.$$

Moreover, we suppose that the mapping $F' : (u,t) \in V \times (-\infty, T^*] \mapsto F'(u,t) \in \mathcal{L}(V;V')$ is continuous (thus, in particular, for each $t \le T^*$, the mapping $F(\cdot,t)$ is continuously Fréchet differentiable in V).

Then, for all $\tau \le T^*$ and $u_0, v_0 \in H$, there exists a unique $v(t) = v(t;\tau,u_0,v_0)$, which is a solution of

$$\begin{cases} v \in L^2(\tau,T;V) \cap C([\tau,T];H) \quad \text{for all } \tau < T \le T^*, \\ \dfrac{dv}{dt} = F'(U(t,\tau)u_0,t)v, \quad \tau < t < T^*, \\ v(\tau) = v_0. \end{cases} \tag{2.14}$$

We make the assumption that

$$v(t;\tau,u_0,v_0) = L(t,\tau,u_0)v_0 \quad \text{ for all } \tau \le t \le T^*,\, u_0, v_0 \in K(\tau). \tag{2.15}$$

Let us write, for $j = 1, 2, \ldots$,

$$\tilde{q}_j = \limsup_{T\to+\infty} \sup_{\tau\le T^*} \sup_{u_0\in K(\tau-T)} \frac{1}{T} \int_{\tau-T}^{\tau} \operatorname{Tr}_j(F'(U(s,\tau-T)u_0,s))\,ds, \tag{2.16}$$

where

$$\operatorname{Tr}_j(F'(U(s,\tau)u_0,s)) = \sup_{v_0^i\in H,\, |v_0^i|\le 1,\, i\le j} \sum_{i=1}^{j} (F'(U(s,\tau)u_0,s)e_i, e_i),$$

$e_1, \ldots, e_j$ being an orthonormal basis for the subspace of H spanned by

$$v(s;\tau,u_0,v_0^1), \ldots, v(s;\tau,u_0,v_0^j).$$

THEOREM 2.9 ([8]). *Under the assumptions above, and in particular* (2.11)–(2.13) *and* (2.15), *suppose that*

$$\bigcup_{\tau \le T^*} K(\tau) \text{ is relatively compact in } H, \tag{2.17}$$

and there exist q_j, $j = 1, 2, \ldots$, *such that*

$$\tilde{q}_j \le q_j \quad \text{for any } j \ge 1, \tag{2.18}$$

$$q_{n_0} \ge 0, \quad q_{n_0+1} < 0, \tag{2.19}$$

for some $n_0 \ge 1$, *and*

$$q_j \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j) \quad \text{for all } j = 1, 2, \ldots. \tag{2.20}$$

Then

$$d_F(K(\tau)) \le d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}} \quad \text{for all } \tau \le T^*. \tag{2.21}$$

2.4. The upper semicontinuity of the pullback $\mathcal{D}$-attractor

DEFINITION 2.10. Let $\{U_\epsilon(t,\tau) : \epsilon \in [0,1]\}$ be a family of evolution processes in a Banach space X with corresponding pullback $\mathcal{D}$-atractors $\{A_\epsilon(t) : \epsilon \in [0,1]\}$. For any bounded interval $I \subset \mathbb{R}$, we say $\{A_\epsilon(\cdot)\}$ is *upper semicontinuous* at $\epsilon = 0$ for $t \in I$ if

$$\lim_{\epsilon \to 0} \sup_{t \in I} \operatorname{dist}(A_\epsilon(t), A_0(t)) = 0.$$

THEOREM 2.11 ([4]). *Let* $\{U_\epsilon(t,\tau) : \epsilon \in [0,\epsilon_0]\}$ *be a family of processes with corresponding pullback* $\mathcal{D}$*-atractors* $\{A_\epsilon(t) : \epsilon \in [0,\epsilon_0]\}$. *Then, for any bounded* $I \subset \mathbb{R}$, $\{U_\epsilon(t,\tau) : \epsilon \in [0,\epsilon_0]\}$ *is upper semicontinuous at* 0 *for* $t \in I$ *if for each* $t \in \mathbb{R}$, *each compact subset* K *and each* $T > 0$, *the following conditions hold:*

(i) $\sup_{\tau \in [t-T,t]} \sup_{\chi \in K} d(U_\epsilon(t,\tau)\chi, U_0(t,\tau)\chi) \to 0$ *as* $\epsilon \to 0$.
(ii) $\bigcup_{\epsilon \in [0,\epsilon_0]} \bigcup_{t \le t_0} A_\epsilon(t)$ *is bounded for any given* t_0.
(iii) $\overline{\bigcup_{0<\epsilon\le\epsilon_0} A_\epsilon(t)}$ *is compact for each* $t \in \mathbb{R}$.

3. Existence of a pullback $\mathcal{D}$-attractor in $H_\mu(\Omega) \cap L^p(\Omega)$. We denote

$$X = L^2(\tau, T; H_\mu(\Omega)) \cap L^p(\tau, T; L^p(\Omega)),$$
$$X^* = L^2(\tau, T; H_\mu^{-1}(\Omega)) \cap L^{p'}(\tau, T; L^{p'}(\Omega)),$$

where p' is the conjugate of p and $\mu \in [0, \mu^*]$.

DEFINITION 3.1. A function $u(\cdot)$ is said to be a *weak solution* of problem (1.1) on (τ, T) if $u \in X$, $du/dt \in X^*$, $u|_{t=\tau} = u_\tau$ for a.e. $x \in \Omega$ and

$$\int_\tau^T \int_\Omega \left(\frac{\partial u}{\partial t}\varphi + \nabla u \nabla \varphi - \frac{\mu}{|x|^2} u\varphi + f(u)\varphi \right) dx\, dt = \int_\tau^T \int_\Omega g(t)\varphi\, dx\, dt$$

for all test functions $\varphi \in X$.

It is known (see, for example, [5, Theorem 1.8, p. 33]) that if $u \in X$ and $du/dt \in X^*$, then $u \in C([\tau, T]; L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful.

THEOREM 3.2. *Under assumptions* ($\boldsymbol{F}$) *and* ($\boldsymbol{G}$), *for any* $T > \tau$ *in* $\mathbb{R}$, *and* u_τ *given, problem* (1.1) *has a unique weak solution* u *on* (τ, T). *Moreover, the solution* u *can be extended to* $[\tau, +\infty)$ *and for all* $t > \tau$,

$$|u(t)|_2^2 \le e^{-\lambda_{1,\mu}(t-\tau)}|u_\tau|_2^2 + \frac{2k_1}{\lambda_{1,\mu}}|\Omega| + \frac{e^{-\lambda_{1,\mu}t}}{\lambda_{1,\mu}} \int_{-\infty}^{t} e^{\lambda_{1,\mu}s}|g(s)|_2^2\,ds. \tag{3.1}$$

Proof. The proof of existence and uniqueness of solution is classical, using the compactness method (see e.g. [15]), so we omit it here. We now show that inequality (3.1) holds. Multiplying (1.1) by u and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \|u\|_\mu^2 + \int_\Omega f(u)u\,dx = \int_\Omega g(t)u\,dx.$$

Using hypothesis ($\boldsymbol{F}$) and the Cauchy inequality, we deduce that

$$\frac{d}{dt}|u|_2^2 + 2\|u\|_\mu^2 + 2C_1|u|_p^p \le 2k_1|\Omega| + \frac{1}{\lambda_{1,\mu}}|g(t)|_2^2 + \lambda_{1,\mu}|u|_2^2. \tag{3.2}$$

Combining this with the fact that $\|u\|_\mu^2 \ge \lambda_{1,\mu}|u|_2^2$, we have

$$\frac{d}{dt}|u|_2^2 + \lambda_{1,\mu}|u|_2^2 \le 2k_1|\Omega| + \frac{1}{\lambda_{1,\mu}}|g(t)|_2^2.$$

Hence applying the Gronwall lemma we get (3.1). ∎

Thanks to Theorem 3.2, we can define a process $U_\mu(t,\tau) : L^2(\Omega) \to H_\mu(\Omega) \cap L^p(\Omega)$, $t \ge \tau$, where $U_\mu(t,\tau)u_\tau$ is the unique weak solution of problem (1.1) with u_τ as initial datum at time τ.

Define $\mathcal{R}$ as the set of all functions $r : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{t \to -\infty} e^{\lambda_{1,\mu}t} r^2(t) = 0,$$

and denote by $\mathcal{D}$ the class of families $\hat{\mathcal{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(L^2(\Omega))$ satisfying $D(t) \subset \overline{B}(r(t))$ for some function $r \in \mathcal{R}$, where $\overline{B}(r(t))$ is the closed ball in $L^2(\Omega)$ with radius $r(t)$.

LEMMA 3.3. *Assume that hypotheses* ($\boldsymbol{F}$) *and* ($\boldsymbol{G}$) *are satisfied, and* $u(t)$ *is a weak solution of problem* (1.1). *Then for all* $t > \tau$,

$$\|u(t)\|_\mu^2 + |u(t)|_p^p \le C\Big(e^{-\lambda_{1,\mu}(t-\tau)}|u_\tau|_2^2 + 1 + e^{-\lambda_{1,\mu}t} \int_{-\infty}^{t} e^{\lambda_{1,\mu}s}|g(s)|_2^2\,ds\Big), \tag{3.3}$$

where C *is a positive constant. Hence, there exists a family of pullback* $\mathcal{D}$-*absorbing sets in* $H_\mu(\Omega) \cap L^p(\Omega)$ *for the process* $U_\mu(t,\tau)$.

Proof. Multiplying (1.1) by u and integrating on Ω, we have

$$\frac{1}{2}\frac{d}{dt}|u|_2^2 + \|u\|_\mu^2 + \int_\Omega f(u)u\,dx = \int_\Omega g(t)u\,dx \le \frac{1}{\lambda_{1,\mu}}|g(t)|_2^2 + \frac{\lambda_{1,\mu}}{4}|u|_2^2. \tag{3.4}$$

Using hypothesis ($\boldsymbol{F}$) and $\|u\|_\mu^2 \ge \lambda_{1,\mu}|u|_2^2$, we have

$$\frac{d}{dt}|u|_2^2 + \lambda_{1,\mu}|u|_2^2 + C(\|u\|_\mu^2 + |u|_p^p) \le C(1 + |g(t)|_2^2). \tag{3.5}$$

Let $F(s) = \int_0^s f(r)\,dr$. By ($\boldsymbol{F}$) we get

$$C(|u|_p^p - 1) \le \int_\Omega F(u)\,dx \le C(|u|_p^p + 1). \tag{3.6}$$

Now multiplying (3.5) by $e^{\lambda_{1,\mu}t}$ and using (3.6) we get

$$\begin{aligned}\frac{d}{dt}(e^{\lambda_{1,\mu}t}|u(t)|_2^2) + Ce^{\lambda_{1,\mu}t}\Big(\|u(t)\|_\mu^2 + 2\int_\Omega F(u(t))\,dx\Big)& \\ \le C(e^{\lambda_{1,\mu}t} + e^{\lambda_{1,\mu}t}|g(t)|_2^2)&.\end{aligned} \tag{3.7}$$

Integrating (3.7) from τ to $s \in [\tau, t-1]$ and from s to $s+1$ respectively, we obtain

$$\begin{aligned}&e^{\lambda_{1,\mu}s}|u(s)|_2^2 \\ &\quad\le e^{\lambda_{1,\mu}\tau}|u_\tau|_2^2 + Ce^{\lambda_{1,\mu}s} + C\int_\tau^s e^{\lambda_{1,\mu}r}|g(r)|_2^2, \quad \forall s \in [\tau, t-1],\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&C\int_s^{s+1} e^{\lambda_{1,\mu}r}\Big(\|u(r)\|_\mu^2 + 2\int_\Omega F(u(r)\,dx)\Big)\,dr \\ &\quad\le e^{\lambda_{1,\mu}s}|u(s)|_2^2 + C\int_s^{s+1}(e^{\lambda_{1,\mu}r} + e^{\lambda_{1,\mu}r}|g(r)|_2^2)\,dr \\ &\quad\le e^{\lambda_{1,\mu}\tau}|u_\tau|_2^2 + Ce^{\lambda_{1,\mu}s} + C\int_\tau^s e^{\lambda_{1,\mu}r}|g(r)|_2^2\,dr \\ &\qquad + Ce^{\lambda_{1,\mu}(s+1)} + C\int_s^{s+1} e^{\lambda_{1,\mu}r}|g(r)|_2^2 dr \qquad \text{(by (3.8))} \\ &\quad\le C\Big(e^{\lambda_{1,\mu}\tau}|u_\tau|_2^2 + e^{\lambda_{1,\mu}t} + \int_\tau^t e^{\lambda_{1,\mu}r}|g(r)|_2^2\,dr\Big).\end{aligned} \tag{3.9}$$

Multiplying (1.1) by $u_t(s)$ and integrating over Ω we have

$$|u_t(s)|_2^2 + \frac{1}{2}\frac{d}{ds}\Big(\|u(s)\|_\mu^2 + 2\int_\Omega F(u(s))\,dx\Big) = \int_\Omega g(s)u_t(s) \le \frac{1}{2}|g(s)|_2^2 + \frac{1}{2}|u_t(s)|_2^2, \tag{3.10}$$

thus

$$e^{\lambda_{1,\mu}s}|u_t(s)|_2^2 + \frac{d}{ds}\Big(e^{\lambda_{1,\mu}s}\Big(\|u(s)\|_\mu^2 + 2\int_\Omega F(u(s))\,dx\Big)\Big) \le \lambda_{1,\mu}e^{\lambda_{1,\mu}s}\Big(\|u(s)\|^2 + 2\int_\Omega F(u(s))\,dx\Big) + e^{\lambda_{1,\mu}s}|g(s)|_2^2. \tag{3.11}$$

Combining (3.9) and (3.11) and using the uniform Gronwall inequality, we have

$$e^{\lambda_{1,\mu}t}\Big(\|u(t)\|_\mu^2 + 2\int_\Omega F(u(t))\,dx\Big) \le C\Big(e^{\lambda_{1,\mu}\tau}|u_\tau|_2^2 + e^{\lambda_{1,\mu}t} + \int_{-\infty}^{t} e^{\lambda_{1,\mu}s}|g(s)|_2^2\,ds\Big). \tag{3.12}$$

Using (3.6) again, we get (3.3). ∎

From Lemma 3.3 we see that the process $U_\mu(t,\tau)$ maps compact subsets of $H_\mu(\Omega)\cap L^p(\Omega)$ to bounded subsets of $H_\mu(\Omega)\cap L^p(\Omega)$ and thus by Lemma 2.1, it is norm-to-weak continuous in $H_\mu(\Omega)\cap L^p(\Omega)$. Since $U_\mu(t,\tau)$ has a family of pullback $\mathcal{D}$-absorbing sets in $H_\mu(\Omega)\cap L^p(\Omega)$, in order to prove the existence of pullback $\mathcal{D}$-attractors, it is sufficient to verify that $U_\mu(t,\tau)$ is pullback $\mathcal{D}$-asymptotically compact.

To prove the pullback $\mathcal{D}$-asymptotic compactness of $U(t,\tau)$ in $L^p(\Omega)$, we need the following lemmas.

LEMMA 3.4 ([9]). *Let $\{U(t,\tau)\}$ be a norm-to-weak continuous process in the spaces $L^2(\Omega)$ and $L^p(\Omega)$, and suppose it satisfies the following two conditions:*

(1) *$\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^2(\Omega)$.*
(2) *For any $\epsilon > 0$, $\hat{\mathcal{B}} \in \mathcal{D}$, there exist constants $M = M(\epsilon,\hat{\mathcal{B}})$ and $\tau_0 = \tau_0(\epsilon,\hat{\mathcal{B}}) \le t$ such that*

$$\Big(\int_{\Omega(|U(t,\tau)u_\tau|\ge M)} |U(t,\tau)u_\tau|^p\,dx\Big)^{1/p} < \epsilon$$

for all $u_\tau \in B(\tau)$ and $\tau \le \tau_0$.

Then $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in $L^p(\Omega)$.

LEMMA 3.5 ([12]). *Suppose that for some $\lambda > 0$ and $\tau \in \mathbb{R}$, and for all $s > \tau$,*

$$y'(s) + \lambda y(s) \le h(s), \tag{3.13}$$

where the functions y, y', h are assumed to be locally integrable and y, h are nonnegative on the interval $t < s < t + r$ for some $t \ge \tau$. Then

$$y(t+r) \le e^{-\lambda r/2} \frac{2}{r} \int_t^{t+r/2} y(s)\,ds + e^{-\lambda(t+r)} \int_t^{t+r} e^{\lambda s} h(s)\,ds. \tag{3.14}$$

LEMMA 3.6. *Under hypotheses ($\boldsymbol{F}$) and ($\boldsymbol{G}$), the process $\{U_\mu(t,\tau)\}$ associated to problem (1.1) is pullback $\mathcal{D}$-asymptotically compact in $L^p(\Omega)$.*

Proof. It is sufficient to verify condition (2) in Lemma 3.4. From hypothesis ($\boldsymbol{F}$), we can choose a constant M large enough such that $f(u) \ge \tilde{C}_1 |u|^{p-1}$ in

$$\Omega_{2M} = \Omega(u(t) \ge 2M) = \{x \in \Omega : u(x,t) \ge 2M\}.$$

Throughout this section, we denote

$$(u-M)^+ = \begin{cases} u - M & \text{if } u \ge M, \\ 0 & \text{if } u < M. \end{cases}$$

First, in Ω_{2M} we obtain

$$\begin{aligned} g(t)((u-M)^+)^{p-1} &\le \frac{\tilde{C}_1}{2}((u-M)^+)^{2p-2} + \frac{1}{2\tilde{C}_1}|g(t)|_2^2 \\ &\le \frac{\tilde{C}_1}{2}((u-M)^+)^{p-1}|u|^{p-1} + \frac{1}{2\tilde{C}_1}|g(t)|_2^2, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &f(u)((u-M)^+)^{p-1} \\ &\quad \ge \tilde{C}_1((u-M)^+)^{p-1}|u|^{p-1} \\ &\quad \ge \frac{\tilde{C}_1}{2}((u-M)^+)^{p-1}|u|^{p-1} + \frac{\tilde{C}_1 M^{p-2}}{2}((u-M)^+)^p. \end{aligned} \tag{3.16}$$

Now, we multiply the first equation in (1.1) by $|(u-M)^+|^{p-1}$ to deduce for all $0 < \mu \le \mu^*$ that

$$\begin{aligned} \frac{du}{dt}|(u-M)^+|^{p-1} - \Delta u|(u-M)^+|^{p-1} - \frac{\mu}{|x|^2}u|(u-M)^+|^{p-1} \\ + f(u)|(u-M)^+|^{p-1} = g(t)|(u-M)^+|^{p-1}. \end{aligned}$$

This yields, by integrating over Ω_{2M},

$$\frac{1}{p}\frac{d}{dt}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx+\int\limits_{\Omega_{2M}}(p-1)\nabla u\nabla(u-M)^+|(u-M)^+|^{p-2}\,dx$$
$$-\int\limits_{\Omega_{2M}}\frac{\mu}{|x|^2}u|(u-M)^+|^{p-1}\,dx+\int\limits_{\Omega_{2M}}f(u)|(u-M)^+|^{p-1}\,dx$$
$$=\int\limits_{\Omega_{2M}}g(t)|(u-M)^+|^{p-1}\,dx.$$

We remark that $-u(u-M)^+\geq-|u|^2$ on Ω_{2M}, thus it follows from the Hardy inequality that

$$\int\limits_{\Omega_{2M}}(p-1)\nabla u\nabla(u-M)^+|(u-M)^+|^{p-2}\,dx-\int\limits_{\Omega_{2M}}\frac{\mu}{|x|^2}u|(u-M)^+|^{p-1}\,dx$$
$$\geq C\int\limits_{\Omega_{2M}}\left[|\nabla u|^2-\frac{\mu}{|x|^2}|u|^2\right]|(u-M)^+|^{p-2}dx$$
$$\geq CM^{p-2}\int\limits_{\Omega_{2M}}\left[|\nabla u|^2-\frac{\mu}{|x|^2}|u|^2\right]dx\geq 0.$$

This gives

$$\frac{1}{p}\frac{d}{dt}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx+\int\limits_{\Omega_{2M}}f(u)|(u-M)^+|^{p-1}\,dx$$
$$\leq\int\limits_{\Omega_{2M}}g(t)|(u-M)^+|^{p-1}\,dx.$$

Combining this with (3.15) and (3.16) we conclude that

$$\frac{1}{p}\frac{d}{dt}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx+\frac{\tilde{C}_1M^{p-2}}{2}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx\leq\frac{1}{2\tilde{C}_1}\int\limits_{\Omega_{2M}}|g(t)|^2\,dx,$$

and thus

$$\frac{d}{dt}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx+CM^{p-2}\int\limits_{\Omega_{2M}}|(u-M)^+|^p\,dx\leq C|g(t)|_2^2\,dx.$$

Thanks to Lemma 3.5, we have for some $t_1<t$ and for all $r>0$,

$$\int\limits_{\Omega_{2M}}|(u(t_1+r)-M)^+|^p\,dx$$
$$\leq Ce^{-CM^{p-2}r/2}\int\limits_{t_1}^{t_1+r}\int\limits_{\Omega_{2M}}|(u(s)-M)^+|^p\,dx\,ds$$
$$+Ce^{-CM^{p-2}(t_1+r)}\int\limits_{t_1}^{t_1+r}e^{CM^{p-2}s}|g(s)|_2^2\,ds. \tag{3.17}$$

Now we estimate the right hand side terms of (3.17). First, we have

$$\begin{aligned}(3.18)\qquad &\int_{t_1}^{t_1+r}\int_{\Omega_{2M}} |(u(s)-M)^+|^p\,dx\,ds\\ &\le \int_{t_1}^{t_1+r} |(u(s)-M)^+|_p^p\,ds \le C\Big(\int_{t_1}^{t_1+r} |u(s)|_p^p\,ds + rM^p|\Omega|^p\Big)\\ &\le C\Big(|u(t_1)|_2^2 + 1 + \int_{t_1}^{t_1+r} |g(s)|_2^2\,ds + rM^p|\Omega|^p\Big) \quad \text{(by (3.5))}\\ &\le C\Big(1 + e^{-\lambda_{1,\mu}t_1}\int_{-\infty}^{t_1} e^{\lambda_{1,\mu}s}|g(s)|_2^2\,ds + \int_{t_1}^{t_1+r} |g(s)|_2^2\,ds\Big) < \infty\end{aligned}$$

for sufficiently small τ by (3.1). Therefore, there exists a number N_0 independent of τ, M and u_τ such that

$$(3.19)\qquad \int_{t_1}^{t_1+r}\int_{\Omega_{2M}} |(u(s)-M)^+|^p\,ds \le N_0,$$

thus for sufficiently large M, we have

$$(3.20)\qquad Ce^{-CM^{p-2}r/2}\int_{t_1}^{t_1+r}\int_{\Omega_{2M}} |(u(s)-M)^+|^p\,dx\,ds \le \frac{\epsilon}{2}.$$

It is well known that for an integrable function h on an interval $[a,b]$ and a given $\epsilon > 0$ we have

$$(3.21)\qquad e^{-Mb}\int_a^b e^{Ms}h(s)\,ds \le \frac{\epsilon}{2}$$

for M large enough. Now combining (3.17), (3.20) and (3.21), choosing $r = t - t_1 > 0$, we get

$$(3.22)\qquad \int_{\Omega_{2M}} |(U(t,\tau)u_\tau - M)^+|^p\,dx \le \epsilon$$

for $\tau \le \tau_1$ and $M \ge M_1$. Next, we set

$$(3.23)\qquad (u+M)^- = \begin{cases} u+M & \text{if } u \le -M,\\ 0 & \text{if } u > -M,\end{cases}$$

and repeating the same steps above with $(u+M)^-$ instead of $(u-M)^+$, we deduce that there exist $M_2 > 0$ and $\tau_2 < t$ such that for any $\tau < \tau_2$ and $M \ge M_2$,

$$(3.24)\qquad \int_{\Omega(u(t)\le -2M)} |(u+M)^-|^p\,dx \le \epsilon.$$

Now, let $M_0 = \max\{M_1, M_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$. It follows from (3.22) and (3.24) that

$$\int_{\Omega(|u(t)|\geq 2M)} (|u| - M)^p \, dx \leq \epsilon \tag{3.25}$$

for all $\tau \leq \tau_0$ and $M \geq M_0$. Hence,

$$\begin{aligned} \int_{\Omega(|u(t)|\geq 2M)} |u|^p \, dx &= \int_{\Omega(|u(t)|\geq 2M)} [(|u| - M) + M]^p \, dx \\ &\leq 2^{p-1}\Big(\int_{\Omega(|u(t)|\geq 2M)} (|u| - M)^p \, dx + \int_{\Omega(|u(t)|\geq 2M)} M^p \, dx\Big) \\ &\leq 2^{p-1}\Big(\int_{\Omega(|u(t)|\geq 2M)} (|u| - M)^p \, dx + \int_{\Omega(|u(t)|\geq 2M)} (|u| - M)^p \, dx\Big) \leq 2^p \epsilon, \end{aligned} \tag{3.26}$$

which completes the proof. ∎

Lemma 3.7. *Suppose hypotheses* (***F***) *and* (***G***) *hold. Then for any* $s \in \mathbb{R}$ *and any bounded subset* $B \subset L^2(\Omega)$, *there exists a constant* $\tau_0 = \tau_0(B, s) \leq s$ *such that for all* $\tau \leq \tau_0$ *and all* $u_\tau \in B$, *the unique weak solution* u *of problem* (1.1) *with initial datum* u_τ *at time* τ *satisfies*

$$|u_t(s)|_2^2 \leq C\Big(1 + e^{-\lambda_{1,\mu} s} \int_{-\infty}^{s} e^{\lambda_{1,\mu} r}(|g(r)|_2^2 + |g'(r)|_2^2) \, dr\Big),$$

where $C > 0$ *is independent of* s *and* B.

Proof. Integrating (3.11) with respect to s from r to $r+1$ for $r \in [\tau, t-1]$ we get

$$\begin{aligned} \int_r^{r+1} e^{\lambda_{1,\mu} s} |u_t(s)|_2^2 \, ds &\leq e^{\lambda_{1,\mu} r}\Big(\|u(r)\|_\mu^2 + 2\int_\Omega F(u(r)) \, dx\Big) \\ &\quad + \lambda_{1,\mu} \int_r^{r+1} e^{\lambda_{1,\mu} s}\Big(\|u(s)\|_\mu^2 + 2\int_\Omega F(u(s)) \, dx\Big) \, ds \\ &\quad + \int_r^{r+1} e^{\lambda_{1,\mu} s} |g(s)|_2^2 \, ds \\ &\leq C\Big(e^{\lambda_{1,\mu} t} + e^{\lambda_{1,\mu}\tau} |u_\tau|_2^2 + \int_{-\infty}^{t} e^{\lambda_{1,\mu} s} |g(s)|_2^2 \, ds\Big), \end{aligned} \tag{3.27}$$

where we have used (3.9) and (3.12). Differentiating (1.1) in time and multiplying the above equality by $e^{\lambda_{1,\mu} s} u_t$, we get

$$\frac{1}{2}\frac{d}{ds}(e^{\lambda_{1,\mu}s}|u_t|_2^2) + e^{\lambda_{1,\mu}s}\|u_t\|_\mu^2 + e^{\lambda_{1,\mu}s}(f'(u)u_t, u_t)$$
$$= \frac{1}{2}e^{\lambda_{1,\mu}s}(g'(s), u_t) + \frac{\lambda_{1,\mu}}{2}e^{\lambda_{1,\mu}s}|u_t|_2^2.$$

Using hypothesis ($\boldsymbol{F}$) and the Cauchy inequality, we obtain

$$\frac{d}{dr}(e^{\lambda_{1,\mu}r}|u_t(s)|_2^2) \le C(e^{\lambda_{1,\mu}s}|g'(s)|_2^2 + e^{\lambda_{1,\mu}s}|u_t(s)|_2^2). \tag{3.28}$$

From (3.27), (3.28) and the uniform Gronwall inequality, we get

$$e^{\lambda_{1,\mu}s}|u_t(s)|_2^2 \le C\Big(e^{\lambda_{1,\mu}s} + e^{\lambda_{1,\mu}\tau}|u_\tau|_2^2 + \int_{-\infty}^{s} e^{\lambda_{1,\mu}r}(|g(r)|_2^2 + |g'(r)|_2^2)\,dr\Big). \tag{3.29}$$

This implies the desired inequality. ∎

We are in a position to prove the main result of this section.

Theorem 3.8. *Assume that hypotheses* ($\boldsymbol{F}$) *and* ($\boldsymbol{G}$) *are satisfied. Then for each* $\mu \in [0, \mu^*]$, *the process* $U_\mu(t,\tau)$ *associated to problem* (1.1) *has a pullback* $\mathcal{D}$-*attractor* $\hat{\mathcal{A}}_\mu = \{A_\mu(t) : t \in \mathbb{R}\}$ *in* $H_\mu(\Omega) \cap L^p(\Omega)$.

Proof. By Lemma 3.3, the process $U_\mu(t,\tau)$ has a family of pullback $\mathcal{D}$-absorbing sets in $H_\mu(\Omega) \cap L^p(\Omega)$. It is sufficient to show that $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact, i.e., for any $t \in \mathbb{R}$, $\hat{\mathcal{B}} \in \mathcal{D}$, and any sequences $\tau_n \to -\infty$ and $u_{\tau_n} \in B(\tau_n)$, the sequence $\{U_\mu(t,\tau_n)u_{\tau_n}\}$ is precompact in $H_\mu(\Omega) \cap L^p(\Omega)$. Due to Lemma 3.6, we need only show that the sequence $\{U_\mu(t,\tau_n)u_{\tau_n}\}$ is precompact in $H_\mu(\Omega)$.

Denoting $u_n(t_n) = U_\mu(t,\tau_n)u_{\tau_n}$, we have

$$\begin{aligned} \|u_n(t) - u_m(t)\|_\mu^2 &= -\Big\langle \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t), u_n(t) - u_m(t)\Big\rangle \\ &\quad - \langle f(u_n(t)) - f(u_m(t)), u_n(t) - u_m(t)\rangle \\ &\le \Big|\frac{d}{dt}u_n(t) - \frac{d}{dt}u_m(t)\Big|_2 |u_n(t) - u_m(t)|_2 + \ell|u_n(t) - u_m(t)|_2^2. \end{aligned} \tag{3.30}$$

Hence by Lemmas 3.6 and 3.7, we have $\|u_n(t) - u_m(t)\|_\mu \to 0$ as $n, m \to \infty$, which completes the proof. ∎

4. Estimates of the fractal dimension of the pullback $\mathcal{D}$-attractor. From now on, besides ($\boldsymbol{G}$) we assume the external force g satisfies the following additional condition:

($\boldsymbol{G'}$) $g \in L^\infty(-\infty, T^*; L^\infty(\Omega))$ for some $T^* \in \mathbb{R}$.

LEMMA 4.1. *Under conditions* ($\boldsymbol{F}$), ($\boldsymbol{G}$) *and* ($\boldsymbol{G'}$), *every trajectory* $\{u(t)\}_{t\in\mathbb{R}}$ *lying on the pullback* $\mathcal{D}$*-attractor* $\hat{\mathcal{A}}_\mu = \{A_\mu(t) : t \in \mathbb{R}\}$ *is bounded in* $L^\infty(-\infty, T^*; L^\infty(\Omega))$.

Proof. Let $u(t)$ be an arbitrary trajectory lying on $\hat{\mathcal{A}}_\mu$. First, multiply the first equation in (1.1) by $|(u-M)^+|$, then integrate over Ω to get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_M} |(u-M)^+|^2\,dx + \int_{\Omega_M} \nabla u \nabla (u-M)^+\,dx$$
$$-\int_{\Omega_M} \frac{\mu}{|x|^2} u|(u-M)^+|\,dx + \int_{\Omega_M} f(u)|(u-M)^+|\,dx = \int_{\Omega_M} g(t)|(u-M)^+|\,dx.$$

We remark that on Ω_M, $u(u-M)^+ \le |u|^2$, so it follows from the Hardy inequality that

$$\int_{\Omega_M} \nabla u\nabla(u-M)^+\,dx - \int_{\Omega_M} \frac{\mu}{|x|^2}u|(u-M)^+|\,dx \ge \int_{\Omega_M}\left[|\nabla u|^2 - \frac{\mu}{|x|^2}|u|^2\right]dx$$
$$\ge \lambda_{\Omega_M}\int_{\Omega_M}|u|^2\,dx \ge \lambda_{\Omega_M}\int_{\Omega_M}|(u-M)^+|^2\,dx.$$

This gives

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_M}|(u-M)^+|^2\,dx + \lambda_{\Omega_M}\int_{\Omega_M}|(u-M)^+|^2\,dx$$
$$\le \int_{\Omega_M}(g(t)-f(u))|(u-M)^+|\,dx.$$

Since $g \in L^\infty(-\infty, T^*; L^\infty(\Omega))$, there exists $K > 0$ such that $|g(t,x)| \le K$ for a.e. $(x,t) \in \Omega \times (-\infty, T^*)$. By hypothesis ($\boldsymbol{F}$) we can choose M large enough such that $f(u) \ge K$ when $u \ge M$. Then

$$\frac{d}{dt}\int_{\Omega_M}|(u-M)^+|^2\,dx + 2\lambda_{\Omega_M}\int_{\Omega_M}|(u-M)^+|^2\,dx \le 0.$$

By the Gronwall inequality, we have, for all $t \le T^*$,

$$\int_{\Omega_M}|(u(t)-M)^+|^2\,dx \le e^{-2\lambda_{\Omega_M}(t-\tau)}\int_{\Omega_M}|(u_\tau - M)|^2\,dx \to 0 \quad \text{as } \tau \to -\infty.$$

By the invariance of $\hat{\mathcal{A}}_\mu$, we have

$$\int_{\Omega(u(t)\ge M)} |(u(t)-M)^+|^2\,dx = 0. \tag{4.1}$$

Repeating the same steps above with $(u+M)^-$ instead of $(u-M)^+$, we deduce that

$$\int_{\Omega(u(t)\le -M)} |(u(t)+M)^-|^2 dx = 0. \tag{4.2}$$

Noticing that M we have chosen here is independent of t, it follows from (4.1) and (4.2) that

$$\|u\|_{L^\infty(-\infty,T^*;L^\infty(\Omega))} \le M. \ \blacksquare$$

LEMMA 4.2. *Under conditions* ($\boldsymbol{F}$), ($\boldsymbol{G}$) *and* ($\boldsymbol{G'}$), *the pullback* $\mathcal{D}$-*attractor* $\hat{\mathcal{A}}_\mu = \{A_\mu(t) : t \in \mathbb{R}\}$ *satisfies*

$$\bigcup_{\tau \le T^*} A_\mu(\tau) \text{ is relatively compact in } L^2(\Omega). \tag{4.3}$$

Proof. Since $g \in L^\infty(-\infty, T^*; L^\infty(\Omega))$, there exists a constant C such that $|g(t)|_2^2 \le C$ for a.e. $t \le T^*$. Therefore

$$\begin{aligned} r_0(t) &= 2c\Big(1 + e^{-\lambda_{1,\mu}t} \int_{-\infty}^{t} e^{\lambda_{1,\mu}s}|g(s)|_2^2\, ds + e^{-\lambda_{1,\mu}t} \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\lambda_{1,\mu}r}|g(r)|_2^2\, dr\, ds\Big) \\ &\le 2c\left(1 + \frac{C}{\lambda_{1,\mu}} + \frac{C}{\lambda_{1,\mu}^2}\right) =: r_0. \end{aligned}$$

We denote

$$B(t) = \{v \in L^2(\Omega) : |v|_2^2 \le r_0\}.$$

Then

$$B^* := \bigcup_{\tau \le T^*} B(\tau) \text{ is bounded in } L^2(\Omega).$$

Let us denote by M the set of all $y \in L^2(\Omega)$ for which there exists a sequence $\{(t_n, \tau_n)\}_{n\ge1} \subset \mathbb{R}^2$ satisfying $\tau_n \le t_n \le T^*$, $\lim_{n\to\infty}(t_n - \tau_n) = +\infty$ and a sequence $\{u_{0n}\} \subset B^*$ such that $\lim_{n\to\infty} |U_\mu(t_n, \tau_n)u_{0n} - y|_2 = 0$.

Observe that

$$A_\mu(t) \subset M \quad \text{ for all } t \le T^*. \tag{4.4}$$

In fact, by the definition of $\hat{\mathcal{A}}_\mu$, if $t \le T^*$ and $y \in A_\mu(t)$, there exist a sequence $\tau_n \le t$ and a sequence $u_{0n} \in B(\tau_n) \subset B^*$ such that $\lim_{n\to\infty} |U_\mu(t, \tau_n)u_{0n} - y|_2 = 0$. Consequently, taking $t_n = t$ for all $n \ge 1$ we conclude that $y \in M$.

On the other hand, M is a relatively compact subset in $L^2(\Omega)$. In fact, if $\{y_k\}_{k\ge1} \subset M$ is a given sequence, for each $k \ge 1$ we take a pair $(t_k, \tau_k) \in \mathbb{R}^2$ and an element $u_{0k} \in B^*$ such that $t_k \le T^*$, $t_k - \tau_k \ge k$ and $|U_\mu(t_k, \tau_k)u_{0k} - y_k|_2 \le 1/k$. Then we can extract from $\{y_k\}_{k\ge1}$ a subsequence that converges in $L^2(\Omega)$.

As M is relatively compact in $L^2(\Omega)$, taking into acount (4.4) we obtain (4.3). ■

Lemma 4.3. *Suppose f is a C^2 function satisfying* (***F***), *and g satisfies* (***G***) *and* (***G***$'$). *Then the process $U_\mu(t,\tau)$ associated to problem* (1.1) *has the quasidifferentiability properties* (2.12), (2.13) *and* (2.15) *with $v(t) = v(t,\tau,u_0,v_0)$ being the solution of*

$$\begin{cases} v \in L^2(\tau,T;H_\mu(\Omega)) \cap C([\tau,T];L^2(\Omega)), \\ \dfrac{dv}{dt} = \Delta v + \dfrac{\mu}{|x|^2} v - f'(u)v, \\ v(\tau) = v_0. \end{cases} \tag{4.5}$$

Proof. Fix $\tau \le T^*$, $u_0, \overline{u}_0 \in K(\tau)$ and denote $u(t) = U_\mu(t,\tau)u_0$, $\overline{u}(t) = U_\mu(t,\tau)\overline{u}_0$ and $v(t)$ the solution of (4.5) with $v_0 = \overline{u}_0 - u_0$. Let $z(t)$ be defined by $z(t) = \overline{u}(t) - u(t) - v(t)$, $t \le \tau$. Then z satisfies

$$\begin{cases} z \in L^2(\tau,T;H_\mu(\Omega)) \cap C([\tau,T];L^2(\Omega)) \\ \dfrac{dz}{dt} = \Delta z + \dfrac{\mu}{|x|^2} z - f'(u)z - h, \\ z(\tau) = 0, \end{cases} \tag{4.6}$$

with $h = f(\overline{u}) - f(u) - f'(u)(\overline{u} - u)$. Taking the inner product of (4.6) with z yields

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \|z\|_\mu^2 \le \ell|z|^2 + |h|_{L^{p'}}|z|_{L^p}, \tag{4.7}$$

where p' is the conjugate exponent to p.

On the other hand, since f is C^2, it follows from Taylor's theorem that

$$|h(x)| \le \frac{1}{2}|f''(c)|\,|u - \overline{u}|^2$$

for some c on the line segment joining $u(x)$ to $\overline{u}(x)$. Since both $u(t)$ and $\overline{u}(t)$ lie in $A(t)$, they are bounded in $L^\infty(\Omega)$ and so

$$|h(x)| \le C|u(x) - \overline{u}(x)|^2 \tag{4.8}$$

for some constant C.

It follows from (4.8), if we write $h(t) = h(u(x,t))$, that

$$\begin{aligned} \|h(t)\|_{L^{p'}}^{p'} &\le C \int_\Omega |u(t) - \overline{u}(t)|^{2p'}\,dx \\ &= C \int_\Omega |u(t) - \overline{u}(t)|^{2p'-2+\epsilon}|u(t) - \overline{u}(t)|^{2-\epsilon}\,dx \\ &\le C|u(t) - \overline{u}(t)|^{2-\epsilon}, \end{aligned}$$

where we have used the Hölder inequality and the fact that $u(t)$ and $v(t)$ are bounded in $L^\infty(\Omega)$. So we have

$$\|h(t)\|_{L^{p'}} \le C|u(t) - \overline{u}(t)|^{(2-\epsilon)/p'},$$

and if we choose $\epsilon = 2 - p'(1+\delta)$ for some $\delta \in (0, (2-p')/p')$, we obtain

$$\|h(t)\|_{L^{p'}} \le C|u(t) - \overline{u}(t)|^{1+\delta}.$$

On the other hand, it is easy to check that

$$|u(t) - \overline{u}(t)|^2 \le e^{2\ell(t-\tau)}|u_0 - \overline{u}_0|^2.$$

Therefore, $\|h(t)\|_{L^{p'}} \le Ce^{(1+\delta)\ell t}|u_0 - \overline{u}_0|^{1+\delta}$. So from (4.7) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|z|^2 + \|z\|_\mu^2 &\le \ell|z|^2 + Ce^{(1+\delta)\ell t}|u_0 - \overline{u}_0|^{1+\delta}\|z\|_\mu \\ &\le \ell|z|^2 + Ce^{2(1+\delta)\ell t}|u_0 - \overline{u}_0|^{2(1+\delta)} + \frac{1}{4}\|z\|_\mu^2.\end{aligned}$$

Hence, neglecting the $\|z\|_\mu^2$ terms, we get

$$\frac{1}{2}\frac{d}{dt}|z|^2 \le \ell|z|^2 + Ce^{2(1+\delta)\ell t}|u_0 - \overline{u}_0|^{2(1+\delta)}.$$

Using the Gronwall inequality, we obtain

$$|z|^2 \le k(t)|u_0 - \overline{u}_0|^{2(1+\delta)}.$$

Then

$$|z| \le \sqrt{k(t)}|u_0 - \overline{u}_0|^{1+\delta}.$$

Choose $\gamma(t,r) = \sqrt{k(t)}r^\delta \to 0$ as $r \to 0$. ■

THEOREM 4.4. *Suppose f is a C^2 function and satisfies ($\boldsymbol{F}$), and g satisfies ($\boldsymbol{G}$) and ($\boldsymbol{G}'$). Then there exist q_j, $j = 1, 2, \ldots$, such that*

$$\tilde{q}_j \le q_j \quad \text{for any } j \ge 1, \qquad q_{n_0} \ge 0,\ q_{n_0+1} < 0 \quad \text{for some } n_0 \ge 1,$$
$$q_j \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j) \qquad \text{for all } j = 1, 2, \ldots,$$

where $\tilde{q}_j$ is defined in (2.16) with $F(u) = \Delta u + \frac{\mu}{|x|^2}u - f(u) + g$. Thus,

$$d_F(A(\tau)) \le \max\{1, d_0\} \quad \text{for all } \tau \in \mathbb{R}, \quad \text{where } d_0 := n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}}.$$

Proof. We have

$$F'(U_\mu(s,\tau)u_\tau)e_i = \Delta e_i + \frac{\mu}{|x|^2}e_i - f'(u)e_i.$$

Then

$$\begin{aligned}\langle F'(U_\mu(s,\tau)u_\tau)e_i, e_i\rangle &= -\Big\{\int_\Omega |\nabla e_i|^2\,dx - \int_\Omega \frac{\mu}{|x|^2}e_i^2\,dx\Big\} - \int_\Omega f'(u)e_i^2\,dx \\ &\le -\Big\{\int_\Omega |\nabla e_i|^2\,dx - \int_\Omega \frac{\mu}{|x|^2}e_i^2\,dx\Big\} + \ell,\end{aligned}$$

where we have used the facts that $-f'(u) \le \ell$ and $\int_\Omega e_i^2\,dx = 1$. Therefore

$$\begin{aligned}
\mathrm{Tr}_j[F'(U_\mu(s,\tau)u_\tau)] &= \sup_{i\le j}\sum_{i=1}^{j}\langle F'(U_\mu(s,\tau)u_\tau)e_i, e_i\rangle \\
&\le -\sum_{i=1}^{j}\Big(\int_\Omega |\nabla e_i|^2\,dx - \int_\Omega \frac{\mu}{|x|^2}e_i^2\,dx\Big) + \ell j \\
&= -\sum_{i=1}^{j}(Ae_i, e_i)_{L^2(\Omega)} + \ell j \quad (Au := -\Delta u - (\mu/|x|^2)u) \\
&\le -C_\mu \sum_{i=1}^{j}\int_\Omega |\nabla e_i|^2\,dx + \ell j \quad (C_\mu = 1 - \mu/\mu^*) \\
&= -C_\mu \sum_{i=1}^{j}(-\Delta e_i, e_i)_{L^2(\Omega)} + \ell j.
\end{aligned}$$

By using the inequality

$$\sum_{i=1}^{j}(-\Delta e_i, e_i)_{L^2(\Omega)} \ge \sum_{i=1}^{j}\lambda_i(\Omega),$$

and the inequality (1.3) in [14]:

$$\sum_{i=1}^{m}\lambda_i(\Omega) \ge \frac{NC_N}{N+2}\mu_N(\Omega)^{-2/N}m^{(N+2)/N} + M_N\frac{\mu_N(\Omega)}{I(\Omega)}m,$$

where $C_N = (2\pi)^2\omega_N^{-2/N}$, ω_N is the volume of the unit ball in $\mathbb{R}^N$, $\mu_N(\Omega)$ is the N-dimensional volume of Ω, $M_N = c/(N+2)$, with $c < (2\pi)^2\omega_N^{-4/N}$, but c independent of N, and $I(\Omega) = \min_{\alpha\in\mathbb{R}^N}\int_\Omega |x-\alpha|^2\,dx$, we get

$$\begin{aligned}
&\mathrm{Tr}_j[F'(U_\mu(s,\tau)u_\tau)] \\
&\quad\le -C_\mu\frac{NC_N}{N+2}\mu_N(\Omega)^{-2/N}j^{(N+2)/N} - C_\mu M_N\mathcal{R}(\Omega)j + \ell j \\
&\qquad\qquad (\mathcal{R}(\Omega) := \mu_N(\Omega)/I(\Omega)) \\
&\quad= -C_\mu\frac{NC_N}{N+2}\mu_N(\Omega)^{-2/N}j^{(N+2)/N} + l_1 j \\
&\quad= -Kj^{(N+2)/N} + l_1 j,
\end{aligned}$$

where $l_1 = \ell - C_\mu M_N\mathcal{R}(\Omega)$ and $K = C_\mu\frac{NC_N}{N+2}\mu_N(\Omega)^{-2/N}$. Hence, we get $\tilde{q}_j \le -Kj^{(N+2)/N} + l_1 j = jK(l_1/K - j^{2/N})$.

If $0 \le l_1 < K$, then taking $q_j = jK(1 - j^{2/N})$ and $n_0 = 1$, we can apply Theorem 2.9 to obtain

$$d_F(A(\tau)) \le 1 \quad \text{for all } \tau \le T^*.$$

If $l_1 \ge K$, then taking $q_j = jK(l_1/K - j^{2/N})$ and $n_0 = [(l_1/K)^{N/2}]$, where $[m]$ denotes the integer part of a real number m, we have

$$q_{n_0} = K[(l_1/K)^{N/2}](l_1/K - [(l_1/K)^{N/2}]^{2/N}) \ge 0,$$
$$q_{n_0+1} = K[(l_1/K)^{N/2} + 1](l_1/K - ([(l_1/K)^{N/2}] + 1)^{2/N}) < 0,$$

and

$$\begin{aligned} &q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j) \\ &\quad = n_0 l_1 - K n_0^{(N+2)/N} + (K(n_0+1)^{(N+2)/N} - K n_0^{(N+2)/N} - l_1)(n_0 - j). \end{aligned}$$

In order to show that $q_j \le q_{n_0} + (q_{n_0} - q_{n_0+1})(n_0 - j)$, we will prove that

$$Kj^{(N+2)/N} - K n_0^{(N+2)/N} \ge (K(n_0+1)^{(N+2)/N} - K n_0^{(N+2)/N})(j - n_0),$$

or equivalently,

$$((n_0+1)^{(N+2)/N} - n_0^{(N+2)/N})(j - n_0) \le j^{(N+2)/N} - n_0^{(N+2)/N}.$$

The last inequality follows from the fact that for all $n \in \mathbb{N}^*$,

$$(n+2)^m - (n+1)^m \ge (n+1)^m - n^m, \quad \text{where} \quad 1 < m := \frac{N+2}{N} < 2.$$

We now apply Theorem 2.9 to get $d_F(A(\tau)) \le n_0 + \frac{q_{n_0}}{q_{n_0} - q_{n_0+1}}$ for all $\tau \le T^*$.

If $l_1 < 0$, then taking $q_j = -l_1(1-j)$ and $n_0 = 1$, we get $q_{n_0} = 0, q_{n_0+1} = l_1 < 0$; applying Theorem 2.9 we obtain

$$d_F(A(\tau)) \le 1 \quad \text{for all } \tau \le T^*.$$

Finally, since $U_\mu(t,\tau)$ is Lipschitz in $A(\tau)$, it follows from [15, Proposition 13.9] that $d_F(A(t))$ is bounded for every $t \ge \tau$ by the same bound. ∎

5. The upper semicontinuity of pullback $\mathcal{D}$-attractors at $\mu = 0$. The aim of this section is to prove the upper semicontinuity of pullback $\mathcal{D}$-attractors $\hat{A}_\mu$ at $\mu = 0$ in $L^2(\Omega)$. Notice that in this section, we let $\mu \to 0$, thus we can assume $\mu < \mu^*$.

LEMMA 5.1. *Let hypotheses* $(\mathbf{F}), (\mathbf{G})$ *and* $(\mathbf{G}')$ *hold. Then for all* $t \le T^*$, *for each compact subset* $K \subset L^2(\Omega)$ *and each* $T > 0$, *we have*

$$|U_\mu(t,\tau)u_\tau - U_0(t,\tau)u_\tau|_2^2 \le \mu C \quad \textit{for all } \tau \in [t-T, t], u_\tau \in K,$$

where the constant C *is independent of* τ *and* u_τ *(but depends on* T, K*).*

Proof. Denote $U_\mu(t,\tau)u_\tau = u(t)$ and $U_0(t,\tau)u_\tau = v(t)$. Letting $w(t) = u(t) - v(t)$, we have

$$w_t - \Delta w - \frac{\mu}{|x|^2}u + f(u) - f(v) = 0.$$

Multiplying this equation by w, then integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \int_\Omega \left(|\nabla w|^2 - \frac{\mu}{|x|^2} uw \right) dx + \int_\Omega (f(u) - f(v)) w \, dx = 0.$$

Since $f(u) - f(v)w = (f(u) - f(v))(u - v) \geq -\ell |u - v|^2 = -\ell |w|^2$, we have

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \int_\Omega \left(|\nabla u|^2 - \frac{\mu}{|x|^2} |w|^2 \right) dx - \int_\Omega \frac{\mu}{|x|^2} vw \, dx - \ell |w|_2^2 \leq 0.$$

Hence

$$\frac{d}{dt}|w|_2^2 \leq \ell |w|_2^2 + \mu \int_\Omega \frac{1}{|x|^2} vw \, dx. \tag{5.1}$$

Notice that when $\mu < \mu^*$, $H_\mu(\Omega) \equiv H_0^1(\Omega)$, so we can estimate

$$\begin{aligned} &\int_\Omega \frac{1}{|x|^2} v(s) w(s) \, dx \\ &\leq \left(\int_\Omega \frac{|v(s)|^2}{|x|^2} \right)^{1/2} \left(\int_\Omega \frac{|w(s)|^2}{|x|^2} \right)^{1/2} \\ &\leq C \left(\int_\Omega |\nabla v(s)|^2 \, dx \right)^{1/2} \left(\int_\Omega (|\nabla u(s)|^2 + |\nabla v(s)|^2) \, dx \right)^{1/2} \quad \text{(by (2.6))} \\ &\leq C \|v(s)\|_\mu (\|u(s)\|_\mu + \|v(s)\|_\mu) \\ &\leq C e^{-\lambda_{1,\mu} s} \left(e^{\lambda_{1,\mu} \tau} |u_\tau|_2^2 + e^{\lambda_{1,\mu} s} + \int_{-\infty}^{s} e^{\lambda_{1,\mu} r} |g(r)|_2^2 \, dr \right) \quad \text{(use (3.3))} \\ &\leq C^{-\lambda_{1,\mu} s} \left(e^{\lambda_{1,\mu} t} |u_\tau|_2^2 + e^{\lambda_{1,\mu} t} + \int_{-\infty}^{t} e^{\lambda_{1,\mu} r} |g(r)|_2^2 \, dr \right). \end{aligned} \tag{5.2}$$

From (5.1) and (5.2) we get

$$\begin{aligned} &\frac{d}{ds}|w(s)|_2^2 \\ &\leq \ell |w(s)|_2^2 + C\mu \left(e^{\lambda_{1,\mu} t} |u_\tau|_2^2 + e^{\lambda_{1,\mu} t} + \int_{-\infty}^{t} e^{\lambda_{1,\mu} r} |g(r)|_2^2 \, dr \right) e^{-\lambda_{1,\mu} s} \\ &\leq \ell |w(s)|_2^2 + C(K, t, g) \mu e^{-\lambda_{1,\mu} s} \end{aligned} \tag{5.3}$$

Integrating from τ to r with respect to s, where $s \leq r \leq t$, and keeping in mind that $w(\tau) = 0$, we get

$$
\begin{aligned}
(5.4) \qquad |w(r)|_2^2 &\le \ell \int_\tau^r |w(s)|_2^2\, ds + C(K,t,g)\mu \frac{e^{-\lambda_{1,\mu}(t-T)}}{\lambda_{1,\mu}} \\
&\le \ell \int_\tau^r |w(s)|_2^2 ds + C(K,t,g,T,\lambda_{1,\mu})\mu.
\end{aligned}
$$

Now applying the Gronwall inequality, we get

$$
|w(t)|_2^2 \le C\mu, \tag{5.5}
$$

where C is independent of τ and u_τ. This completes the proof. ∎

THEOREM 5.2. *Let hypotheses* (**F**), (**G**) *and* (**G′**) *hold. For any bounded interval $I \subset \mathbb{R}$, the family of pullback $\mathcal{D}$-attractors $\{\hat{\mathcal{A}}_\mu : \mu \in [0,\mu^*]\}$ is upper semicontinuous in $L^2(\Omega)$ at 0 for any $t \in I$; that is,*

$$
\lim_{\mu\to 0} \sup_{t\in I} \operatorname{dist}_{L^2(\Omega)}(A_\mu(t), A_0(t)) = 0.
$$

Proof. We will verify conditions (i)–(iii) in Theorem 2.11. First, condition (i) follows directly from Lemma 5.1.

By Lemma 3.3, there exists a family of pullback $\mathcal{D}$-absorbing sets $B(\cdot) = \overline{B}(r_0(\cdot))$ of the process $\{U_\mu(t,\tau)\}$, which is uniform with respect to the parameter $\mu \in [0,\mu^*)$. By the definition of pullback $\mathcal{D}$-absorbing sets, for any $t \in \mathbb{R}$, there exists $\tau_0 = \tau_0(t) \le t$ such that

$$
\bigcup_{\tau \le \tau_0} U_\mu(t,\tau)B(\tau) \subset B(t) = \overline{B}(r_0(t)). \tag{5.6}
$$

By Theorem 2.7, we see that

$$
A_\mu(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U_\mu(t,\tau)B(\tau)}. \tag{5.7}
$$

From (5.6), (5.7), we get

$$
A_\mu(t) \subset \overline{B}(r_0(t)). \tag{5.8}
$$

Now, for given $t_0 \in \mathbb{R}$ we can write

$$
\bigcup_{\mu\in[0,\mu^*]} \bigcup_{t\le t_0} A_\mu(t) \subset \bigcup_{t\le t_0} \overline{B}(r_0(t)). \tag{5.9}
$$

We have

(5.10)

$$
\begin{aligned}
r_0(t) &= 2c\Big(1 + e^{-\lambda_{1,\mu}t} \int_{-\infty}^{t} e^{\lambda_{1,\mu}s}|g(s)|_2^2\, ds + e^{-\lambda_{1,\mu}t} \int_{-\infty}^{t}\int_{-\infty}^{s} e^{\lambda_{1,\mu}r}|g(r)|_2^2\, dr\, ds\Big) \\
&\le 2c\Big(1 + \frac{C}{\lambda_{1,\mu}} + \frac{C}{\lambda_{1,\mu}^2}\Big).
\end{aligned}
$$

Hence, from (5.9),

$$\bigcup_{\mu\in[0,\mu^*]}\bigcup_{t\le t_0} A_\mu(t) \text{ is bounded in } L^2(\Omega) \text{ for given } t_0,$$

i.e., condition (ii) of Theorem 2.11 is satisfied.

From (5.8) we see that, for each $t\in\mathbb{R}$,

$$\bigcup_{0<\mu\le\mu^*} A_\mu(t) \subset \overline{B}(r_0(t)), \tag{5.11}$$

thus $\bigcup_{0<\mu\le\mu^*} A_\mu(t)$ is bounded in $H_\mu(\Omega)$ and hence

$$\overline{\bigcup_{0<\mu\le\mu^*} A_\mu(t)} \text{ is compact in } L^2(\Omega),$$

since $H_\mu(\Omega) \subset L^2(\Omega)$ compactly. Thus condition (iii) of Theorem 2.11 holds. ■

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