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The analysis of blow-up solutions to a semilinear parabolic system with weighted localized terms

by Haihua Lu (Nanjing and Nantong), Feng Wang (Changzhou) and Qiaoyun Jiang (Nantong)

Abstract. This paper deals with blow-up properties of solutions to a semilinear parabolic system with weighted localized terms, subject to the homogeneous Dirichlet boundary conditions. We investigate the influence of the three factors: localized sources $u^p(x_0,t)$, $v^n(x_0,t)$, local sources $u^m(x,t)$, $v^q(x,t)$, and weight functions $a(x), b(x)$, on the asymptotic behavior of solutions. We obtain the uniform blow-up profiles not only for the cases $m,q \le 1$ or $m,q > 1$, but also for $m > 1$ & $q < 1$ or $m < 1$ & $q > 1$.

1. Introduction and main results. In this paper, we consider the following semilinear parabolic problem with localized sources:

$$(1.1)\qquad \begin{cases} u_t = \Delta u + a(x)u^m(x,t)v^n(x_0,t), & x \in B,\ t>0,\\ v_t = \Delta v + b(x)u^p(x_0,t)v^q(x,t), & x \in B,\ t>0,\\ u(x,t)=v(x,t)=0, & x\in \partial B,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in B,\end{cases}$$

where $B = B(0,R)$ is the open ball of $\mathbb{R}^N$ centered at the origin with radius R, $m,q \ge 0$ and $n,p > 0$ are constants, and $x_0 \in B$ is a fixed point. The initial data $u_0, v_0 \in C_0(B)$ are nonnegative and nontrivial. Many localized problems arise in applications and have been widely studied. Problem (1.1) models a variety of phenomena, such as chemical reactions due to catalysis (see [14]), heat transfer with inter localized sources, or population dynamics. Using the methods of [3] and [18] we know that (1.1) has a nonnegative local solution, and that the Comparison Principle is true. Moreover, if $m,n,p,q \ge 1$ then the uniqueness result holds. Using the methods of [8], we can also prove that under some assumptions the solution to (1.1) blows up in finite time. Throughout this paper we always assume

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(H1) $a, b, u_0, v_0 \in C(\bar{B}) \cap C^2(B)$; $a(x), b(x), u_0(x), v_0(x) \geq 0, \not\equiv 0$ in B and $u_0(x) = v_0(x) = 0$ on ∂B.
(H2) $a(x), b(x), u_0(x), v_0(x)$ are radially symmetric, $a'(r), b'(r) \leq 0$ and $u_0'(x), v_0'(x) < 0$ for $r \in (0, R]$ with $r = |x|$.
(H3) $\Delta u_0(x) + u_0^m(x)v_0^n(x_0) \geq 0$ and $\Delta v_0(x) + u_0^p(x_0)v_0^q(x) \geq 0$ in $B(0, R)$.

Denote $B_a = \{x \in B : a(x) > 0\}$. Under assumptions (H1) and (H2) we can see that there exists an $R_a \in (0, R]$ such that $B_a = B_{R_a}(0)$. We may also assume $0 \neq x_0 \in B_a$. From (H1)–(H3) we can easily deduce that the solution $(u(x,t), v(x,t)) = (u(r,t), v(r,t))$ is radially symmetric and satisfies $u_r \leq 0, v_r \leq 0, u_t \geq 0, v_t \geq 0$ by the maximum principle.

In order to motivate the main results for problem (1.1), we recall some classical results. In the last few years, a lot of effort has been devoted to study the properties of solutions to localized problems. The blow-up of solutions to the scalar problem

$$
\begin{cases} u_t - \Delta u = u^m(x,t)u^p(x_0(t),t) - \mu u^q(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}
$$

has been studied by many authors (see [1, 18, 19, 20]). Souplet [18] obtained a sharp blow-up exponent for system (1.2), and later introduced a new method to investigate the profile of the blow-up solution to (1.2) with $m = \mu = 0$ in [19], where it was proved that

$$
\lim_{t\to T} (T-t)^{1/(p-1)} u(x,t) = \lim_{t\to T} (T-t)^{1/(p-1)} \|u(t)\|_\infty = (p-1)^{-1/(p-1)}.
$$

Wang [21, 22] discussed the finite time blow-up of the positive solution to the problem

$$
\begin{cases} u_t = \Delta u + u^m v^n, \quad v_t = \Delta v + u^p v^q, & x \in \Omega,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}
$$

and evaluated the blow-up rate of the solution to (1.3) with $\Omega = B_R(0)$; he found that under suitable conditions,

$$
c(T-t)^{-\alpha} \leq \max_{\bar{\Omega}} u(\cdot,t) \leq C(T-t)^{-\alpha},
$$
$$
c(T-t)^{-\beta} \leq \max_{\bar{\Omega}} v(\cdot,t) \leq C(T-t)^{-\beta}
$$

for some positive constants c and C, where T is the blow-up time and

$$
\alpha = \frac{1+n-q}{np-(1-m)(1-q)}, \qquad \beta = \frac{1+p-m}{np-(1-m)(1-q)} \tag{1.4}
$$

is the unique positive solution of the linear system

$$\begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.5}$$

Recently, Liu et al. [10] have studied the problem

$$\begin{cases} u_t - \Delta u = a(x)g(t), & x \in B,\ t > 0, \\ u(x,t) = 0, & x \in \partial B,\ t > 0, \\ u(x,0) = u_0(x), & x \in B, \end{cases} \tag{1.6}$$

and found, with $G(t) = \int_0^t g(s)\,ds$, that

$$\lim_{t\to T} \frac{u(x,t)}{G(t)} = a(x)$$

uniformly on any compact subset of B. Li and Wang [8] also studied problem (1.1) with $a(x) = b(x) = 1$, and proved: (i) when $m, q \le 1$, (1.1) has uniform blow-up profiles; (ii) when $m, q > 1$, (1.1) presents single point blow-up patterns. Kong et al. [5] also studied the single equation

$$\begin{cases} u_t = \Delta u + a(x)u^p(x,t)u^q(0,t), & x \in B,\ t > 0, \\ u(x,t) = 0, & x \in \partial B,\ t > 0, \\ u(x,0) = u_0(x), & x \in B. \end{cases} \tag{1.7}$$

They obtained the blow-up sets and blow-up rates. Han and Gao [4] extended the results of [5] with $u^q(0,t)$ replaced by $u^q(x_0,t)$ in (1.7), and they obtained the blow-up rate

$$c(T-t)^{-1/(p+q-1)} \le \max_{\bar{\Omega}} u(\cdot,t) \le C(T-t)^{-1/(p+q-1)} \tag{1.8}$$

for some positive constants c and C, where T is the blow-up time, $0 \le p < 1$ and $p + q > 1$.

There are many other results for parabolic equations with nonlocal or localized nonlinearities. We refer to [2, 7, 9, 12, 13, 15] and the references therein. Many of them considered the blow-up rates or uniform blow-up profiles for the cases $m \le 1, q \le 1$ or $m > 1, q > 1$. However, very few papers considered the case $m > 1$ and $q < 1$, or the case $m < 1$ and $q > 1$. In this paper, we give the uniform blow-up profiles for such cases in Theorem 1.4.

Our main results read as follows.

Theorem 1.1. *If $0 \le m, q \le 1$ and $np - (1-m)(1-q) > 0$, then the solution of* (1.1) *blows up everywhere in B.*

Theorem 1.2. *If $m, q > 1$ and $np - (1-m)(1-q) > 0$, then the blow-up set of the solution only consists of one point $x = 0$.*

Theorem 1.3. *Assume that* $np-(1-m)(1-q)>0$, $0\le m,q\le 1$, *and* (u,v) *is a classical solution of* (1.1) *which blows up in finite time* T.

(i) *If* $m,q<1$, *then*

$$\lim_{t\to T}\max_{\bar B} u(\cdot,t)(T-t)^{\alpha}=\left(\frac{\alpha}{a(0)}\right)^{\alpha}\left(\frac{\beta a(0)}{\alpha b(0)}\right)^{\frac{n}{np-(m-1)(q-1)}},$$

$$\lim_{t\to T}\max_{\bar B} v(\cdot,t)(T-t)^{\beta}=\left(\frac{\beta}{b(0)}\right)^{\beta}\left(\frac{\alpha b(0)}{\beta a(0)}\right)^{\frac{p}{np-(m-1)(q-1)}}.$$

(ii) *If* $m=1$ *or* $q=1$, *then*

$$\lim_{t\to T}\max_{\bar B}|\ln(T-t)|^{-1}\ln u(\cdot,t)=\frac{1+n-q}{np},$$

$$\lim_{t\to T}\max_{\bar B}|\ln(T-t)|^{-1}\ln v(\cdot,t)=\frac{1+p-m}{np}.$$

Theorem 1.4. *Assume that* $np-(1-m)(1-q)>0$, $n>q-1\neq 0$, $p>m-1\neq 0$, *and* (u,v) *is a classical solution of* (1.1) *which blows up in finite time* T. *Then*

$$\lim_{t\to T}\max_{\bar B} u(\cdot,t)(T-t)^{\alpha}=\left(\frac{\alpha}{a(0)}\right)^{\alpha}\left(\frac{\beta a(0)}{\alpha b(0)}\right)^{\frac{n}{np-(m-1)(q-1)}},$$

$$\lim_{t\to T}\max_{\bar B} v(\cdot,t)(T-t)^{\beta}=\left(\frac{\beta}{b(0)}\right)^{\beta}\left(\frac{\alpha b(0)}{\beta a(0)}\right)^{\frac{p}{np-(m-1)(q-1)}}.$$

Remark. If $q<1$ (resp. $m<1$), then the hypothesis $n>q-1\neq 0$ (resp. $p>m-1\neq 0$) in Theorem 1.4 may be omitted. Moreover, if $0\le m,q<1$, then the result of Theorem 1.4 is consistent with Theorem 1.3(i). Furthermore, the assumptions of Theorem 1.4 allow $m>1$ and $q<1$, or $m<1$ and $q>1$.

This paper is organized as follows. In Section 2, we will prove Theorems 1.1 and 1.2. Theorems 1.3 and 1.4 will be proved in Section 3.

2. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. The hypotheses (H1) and (H2) imply that

$$\max_{\bar B} u(\cdot,t)=u(0,t),\qquad \max_{\bar B} v(\cdot,t)=v(0,t).$$

Suppose $a(x)\ge\rho>0$ and $b(x)\ge\rho>0$ on some $\bar B_1\subset B$, where ρ is a positive constant. Then the solution (u,v) of (1.1) satisfies

$$\begin{cases} u_t\ge \Delta u+\rho u^m(x,t)v^n(x_0,t), & x\in B_1,\ t>0,\\ v_t\ge \Delta v+\rho u^p(x_0,t)v^q(x,t), & x\in B_1,\ t>0,\\ u(x,t)=v(x,t)=0, & x\in\partial B_1,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in B_1,\end{cases}$$

and (u,v) blows up in finite time for large initial data with $0 \le m,q \le 1$ and $np-(1-m)(1-q)>0$. Denote

$$\tag{2.1} \begin{cases} g(t)=v^n(x_0,t), & G(t)=\int_0^t g(s)\,ds,\\ h(t)=u^p(x_0,t), & H(t)=\int_0^t h(s)\,ds. \end{cases}$$

Noting that $\Delta u(0,t)\le 0$ and $\Delta v(0,t)\le 0$, we have

$$\tag{2.2} u_t(0,t)\le a(0)u^m(0,t)v^n(x_0,t), \quad v_t(0,t)\le b(0)u^p(x_0,t)v^q(0,t).$$

Integrating (2.2) from 0 to t we have

$$\tag{2.3} \begin{cases} \frac{1}{1-m}u^{1-m}(0,t)\le a(0)G(t)+\frac{1}{1-m}u_0^{1-m} & \text{if } 0\le m<1,\\ \frac{1}{1-q}v^{1-q}(0,t)\le b(0)H(t)+\frac{1}{1-q}v_0^{1-q} & \text{if } 0\le q<1,\\ \ln u(0,t)\le a(0)G(t)+\ln u_0(0) & \text{if } m=1,\\ \ln v(0,t)\le b(0)H(t)+\ln v_0(0) & \text{if } q=1. \end{cases}$$

Thus $\lim_{t\to T} G(t)=\lim_{t\to T} H(t)=\infty$, since $\lim_{t\to T} u(0,t)=\lim_{t\to T} v(0,t) =\infty$. Denote by $G_1(t,\tau;x,\xi)$ the Green's function associated with the operator $\partial/\partial t-\Delta$ along with the homogeneous Dirichlet boundary condition in $B_1\times(0,T)$. For any given $x^*\in B_1$, we know

$$\tag{2.4} \begin{cases} u(x^*,t)=\int_{B_1} G_1(t,0;x^*,\xi)u_0(\xi)\,d\xi\\ \qquad\qquad +\int_0^t\int_{B_1} G_1(t,\tau;x^*,\xi)a(\xi)u^m(\xi,\tau)g(\tau)\,d\xi\,d\tau,\\ v(x^*,t)=\int_{B_1} G_1(t,0;x^*,\xi)v_0(\xi)\,d\xi\\ \qquad\qquad +\int_0^t\int_{B_1} G_1(t,\tau;x^*,\xi)b(\xi)h(\tau)v^q(\xi,\tau)\,d\xi\,d\tau. \end{cases}$$

Then for any $t\in(0,T)$ and $\tau>0$, we have

$$\tag{2.5} \begin{cases} u(x^*,t)\ge \rho\int_0^t\int_{B_1} G_1(t,\tau;x^*,\xi)u^m(\xi,\tau)g(\tau)\,d\xi\,d\tau,\\ v(x^*,t)\ge \rho\int_0^t\int_{B_1} G_1(t,\tau;x^*,\xi)h(\tau)v^q(\xi,\tau)\,d\xi\,d\tau. \end{cases}$$

For any compact subset $B_2 \subset B_1$, by the strong maximum principle, there exists an $\varepsilon_0 = \varepsilon_0(B_2) > 0$ such that

$$\begin{cases} \int_{B_2} G_1(t,\tau;x^*,\xi)u^m(\xi,\tau)\,d\xi \ge \varepsilon_0, \\ \int_{B_2} G_1(t,\tau;x^*,\xi)v^q(\xi,\tau)\,d\xi \ge \varepsilon_0, \end{cases} \tag{2.6}$$

uniformly for all $t \in (\tau, T)$ and $\tau > 0$. It follows from (2.4) to (2.6) that

$$u(x^*,t) \ge \rho\varepsilon_0 G(t), \quad v(x^*,t) \ge \rho\varepsilon_0 H(t), \quad t \in (\tau,T),\ \tau > 0.$$

Thus, $\lim_{t\to T} u(x^*,t) = \lim_{t\to T} v(x^*,t) = \infty$ because $\lim_{t\to T} G(t) = \lim_{t\to T} H(t) = \infty$. The arbitrariness of x^* implies that (u,v) blows up everywhere in B. ■

Proof of Theorem 1.2. Suppose on the contrary that (u,v) blows up at another point $x^* \ne 0$. We may assume that $\lim_{t\to T} u(x^*,t) = \infty$. Set $r^* = |x^*|$; then $r^* > 0$. Since $u(x,t) = u(r,t)$ is nonincreasing in r, $\lim_{t\to T} \sup u(r,t) = \infty$ for any $r \in [0,r^*]$ with $r = |x|$.

Fix $0 < \delta_1 < \eta_1 < \min\{R_a, r^* N^{-1/2}\}$ and set $K_0 = \{x \in B : \delta_1 < x_i < \eta_1,\ i = 1,\dots,N\}$. Clearly, $a(x) \ge \delta_0$ on $\bar{K}_0$ for some δ_0. Define

$$J(x,t) = u_{x_1} + c(x)u^{m_1}(x,t), \quad (x,t) \in K_0 \times [0,T),$$

where $1 < m_1 < m$,

$$c(x) = \varepsilon \prod_{k=1}^{N} \sin(\mu_0(x_k - \delta_1)) \quad \text{with } \mu_0 = \frac{\pi}{\eta_1 - \delta_1}$$

and $\varepsilon > 0$ to be determined later. A direct computation yields

$$\begin{aligned} J_t - \Delta J &= u_{tx_1} + m_1 c(x)u^{m_1-1}u_t - \big(\Delta u_{x_1} + u^{m_1}\Delta c + m_1 c(x)u^{m_1-1}\Delta u \\ &\quad + 2m_1 u^{m_1-1}\nabla u\nabla c + m_1(m_1-1)c(x)u^{m_1-2}|\nabla u|^2\big) \\ &\le (a(x)u^m v^n(x_0,t))_{x_1} + m_1 a(x)c(x)u^{m+m_1-1}v^n(x_0,t) \\ &\quad - 2m_1 u^{m_1-1}\nabla u\nabla c - u^{m_1}\Delta c \\ &= a'(r)\frac{x_1}{r}u^m v^n(x_0,t) + m a(x)u^{m-1}v^n(x_0,t)u_{x_1} \\ &\quad + m_1 a(x)c(x)u^{m+m_1-1}v^n(x_0,t) - 2m_1 u^{m_1-1}\nabla u\nabla c - u^{m_1}\Delta c. \end{aligned}$$

Put $b_0 = m a(x)u^{m-1}v^n(x_0,t) - Am_1 u^{m_1-1}$, where $A = (2\nabla u\nabla c)/u_{x_1}$ is bounded by $2\varepsilon r^*\mu_0 N^{1/2}/\delta_1$. We have

$$\begin{aligned}(2.7)\qquad & J_t - \Delta J - b_0 J \\ & \le -c(x)\Big((m-m_1)a(x)u^{m+m_1-1}v^n(x_0,t) - Am_1u^{2m_1-1} + \frac{\Delta c}{c}u^{m_1}\Big) \\ & \le -c(x)u^{m_1}\Big((m-m_1)\delta_0 u^{m-1}v^n(x_0,t) - Am_1u^{m_1-1} + \frac{\Delta c}{c}\Big)\end{aligned}$$

for $(x,t) \in K_0 \times [t_1, T)$ with t_1 close to T since $m_1 < m$. Remember that $v(r,t) > \delta_2$ on B_1 for some constants $\delta_2 > 0$. As $m_1 < m$, there exists $\varepsilon_1 > 0$ so small that for all $0 < \varepsilon < \varepsilon_1$,

$$(2.8)\qquad (m-m_1)\delta_0\delta_2^n u^{m-1} - Am_1u^{m_1-1} - N\mu_0^2 \ge 0$$

for $(x,t) \in K_0 \times [t_1,T)$ with t_1 close to T. Consequently, from (2.7) and (2.8),

$$(2.9)\qquad J_t - \Delta J - b_0 J \le 0, \quad (x,t) \in K_0 \times [t_1,T).$$

Moreover, the assumption $u_0'(r) < 0$ gives that $u_r(r,t) < 0$ for $(r,t) \in B_1$, and so $u_{x_1} = u_r x_1/r < 0$ for $(x,t) \in \bar{K}_0 \times [t_1,T)$. We have

$$(2.10)\qquad J(x,t) = u_{x_1}(x,t) < 0 \quad \text{on } \partial K_0 \times (t_1,T).$$

We can choose $\varepsilon_2 > 0$ so small that for all $0 < \varepsilon < \varepsilon_2$,

$$\begin{aligned}(2.11)\qquad J(x,t_1) &= u_{x_1}(x,t_1) + c(x)u^{m_1}(x,t_1) \\ &\le \max_{x\in\bar{K}_0} u_{x_1}(x,t_1) + \varepsilon \max_{x\in\bar{K}_0} u^{m_1}(x,t_1) < 0\end{aligned}$$

for all $x \in K_0$. Fix $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. Application of the Comparison Principle to (2.9)–(2.11) shows that $J(x,t) \le 0$ for $(x,t) \in \bar{K}_0 \times [t_1,T)$, i.e.,

$$(2.12)\qquad -u_{x_1}u^{-m_1} \ge c(x), \quad (x,t) \in \bar{K}_0 \times [t_1,T).$$

Fix $(a_2,\dots,a_N) \in \mathbb{R}^{N-1}$ and take $\underline{a} = (\delta_1, a_2, \dots, a_N)$, $\overline{a} = (\eta_1, a_2, \dots, a_N)$. Integrating (2.12) from $\underline{a}$ to $\overline{a}$ yields

$$0 < \int_{\delta_1}^{\eta_1} c(x)\,dx_1 \le \frac{1}{m_1-1}u^{1-m_1}(\overline{a},t).$$

The fact that $\limsup_{t\to T} u(\bar{a},t) = \infty$ and $m_1 > 1$ leads to a contradiction. Therefore, u blows up only at a single point $x = 0$, and so does the solution (u,v) of problem (1.1). ∎

3. Proof of Theorems 1.3 and 1.4. In this section we study the uniform blow-up profiles for problem (1.1) to prove Theorem 1.3–1.4. Sometimes we write $f(t) \sim g(t)$ as $t \to T$ for $\lim_{t\to T} f(t)/g(t) = 1$. The following two lemmas hold.

Lemma 3.1. *Under the assumptions of Theorem* 1.3, *the following statements hold uniformly on any compact subset of* B_a.

(i) *If* $m<1$ *and* $q<1$, *then*

$$u^{1-m}(x,t)\sim(1-m)a(0)G(t),\qquad v^{1-q}(x,t)\sim(1-q)b(0)H(t).$$

(ii) *If* $m=1$ *and* $q<1$, *then*

$$\ln u(x,t)\sim a(0)G(t),\qquad v^{1-q}(x,t)\sim(1-q)b(0)H(t).$$

(iii) *If* $m=1$ *and* $q=1$, *then*

$$\ln u(x,t)\sim a(0)G(t),\qquad \ln v(x,t)\sim b(0)H(t).$$

(iv) *If* $m<1$ *and* $q=1$, *then*

$$u^{1-m}(x,t)\sim(1-m)a(0)G(t),\qquad \ln v(x,t)\sim b(0)H(t).$$

Proof. (i) Assume $m<1$ and $q<1$. At the maximum point $x=0$ of the solution (u,v), we have $\Delta u(0,t)\le 0$ and $\Delta v(0,t)\le 0$, and thus

$$u_t(0,t)u^{-m}(0,t)\le a(0)g(t),\qquad v_t(0,t)v^{-q}(0,t)\le b(0)h(t). \tag{3.1}$$

Integrating (3.1) from 0 to t, we have

$$\lim_{t\to T}\frac{u^{1-m}(0,t)}{(1-m)G(t)}\le a(0),\qquad \lim_{t\to T}\frac{v^{1-q}(0,t)}{(1-q)H(t)}\le b(0),$$

i.e.,

$$\lim_{t\to T}\sup_{x\in B_a}\frac{u^{1-m}(x,t)}{(1-m)G(t)}\le a(0),\qquad \lim_{t\to T}\sup_{x\in B_a}\frac{v^{1-q}(x,t)}{(1-q)H(t)}\le b(0). \tag{3.2}$$

On the other hand, direct computations demonstrate

$$\begin{aligned}\frac{1}{1-m}\frac{\partial u^{1-m}}{\partial t}&=\frac{1}{1-m}\Delta u^{1-m}+mu^{-m-1}|\nabla u|^2+a(x)g(t)\\ &\ge\frac{1}{1-m}\Delta u^{1-m}+a(x)g(t).\end{aligned}$$

Similarly,

$$\frac{1}{1-q}\frac{\partial v^{1-q}}{\partial t}\ge\frac{1}{1-q}\Delta v^{1-q}+b(x)h(t).$$

Consequently, $\big(\frac{1}{1-m}u^{1-m},\frac{1}{1-q}v^{1-q}\big)$ is a supersolution of the problem

$$\begin{cases} w_t=\Delta w+a(x)g(t),\quad z_t=\Delta z+b(x)h(t), & x\in B,\ 0<t<T,\\ w(x,t)=z(x,t)=0, & x\in\partial B,\ 0<t<T,\\ w(x,0)=\dfrac{u_0^{1-m}(x)}{1-m},\quad z(x,0)=\dfrac{v_0^{1-q}(x)}{1-q}, & x\in B.\end{cases} \tag{3.3}$$

Analogously to the proof of Theorem 4.1 in [19], we can prove

$$\lim_{t\to T}\inf_{x\in B_a}\frac{w(x,t)}{G(t)}\ge a(x),\qquad \lim_{t\to T}\inf_{x\in B_a}\frac{z(x,t)}{H(t)}\ge b(x) \tag{3.4}$$

uniformly on compact subsets of B_a, and hence

$$\lim_{t\to T}\inf_{x\in B_a}\frac{u^{1-m}(x,t)}{(1-m)G(t)}\ge a(0),\qquad \lim_{t\to T}\inf_{x\in B_a}\frac{v^{1-q}(x,t)}{(1-q)H(t)}\ge b(0) \tag{3.5}$$

uniformly on compact subsets of B_a. The inequalities (3.2) and (3.5) yield

$$\lim_{t\to T}\frac{u^{1-m}(x,t)}{(1-m)G(t)}= a(0),\qquad \lim_{t\to T}\frac{v^{1-q}(x,t)}{(1-q)H(t)}= b(0) \tag{3.6}$$

uniformly on compact subsets of B_a.

(ii) Assume $m=1$ and $q<1$. By the argument in case (i), we have, similarly to (3.2),

$$\lim_{t\to T}\sup_{x\in B_a}\frac{\ln u(x,t)}{G(t)}\le a(0),\qquad \lim_{t\to T}\sup_{x\in B_a}\frac{v^{1-q}(x,t)}{(1-q)H(t)}\le b(0). \tag{3.7}$$

We can find that $\big(\ln u, \frac{1}{1-q}v^{1-q}\big)$ is a supersolution of (3.3) with $w(x,0)=\ln u_0(x)$, $z(x,0)=\frac{1}{1-q}v_0^{1-q}(x)$. Proceeding as in case (i), we arrive at the corresponding conclusion.

Cases (iii) and (iv) can be treated similarly. ∎

Lemma 3.2. *Under the assumptions of Theorem* 1.3, *for any given constants δ,ε and ρ satisfying $0<\delta,\varepsilon<1$ and $\rho>1$, there exists $\widetilde T$ such that for all $t\in[\widetilde T,T]$, the following statements hold.*

(i) *If $m,q<1$, then*

$$\begin{aligned}\varepsilon(b(0)\delta)^{\frac{n}{1-q}}(1+p-m)((1-q)H(t))^{\frac{1+n-q}{1-q}}&\\ \le (a(0)\rho)^{\frac{p}{1-m}}(1+n-q)((1-m)G(t))^{\frac{1+p-m}{1-m}}&,\\ \varepsilon(a(0)\delta)^{\frac{p}{1-m}}(1+n-q)((1-m)G(t))^{\frac{1+p-m}{1-m}}&\\ \le (b(0)\rho)^{\frac{n}{1-q}}(1+p-m)((1-q)H(t))^{\frac{1+n-q}{1-q}}&.\end{aligned}$$

(ii) *If $m=1$ and $q<1$, then*

$$\begin{aligned}\ln(\varepsilon\rho\delta^{\frac{n}{1-q}})+\ln\frac{pa(0)b^{-1}(0)}{1+n-q}+\frac{1+n-q}{1-q}\ln((1-q)b(0)H(t))&\\ \le p\rho a(0)G(t)&,\\ p\delta a(0)G(t)\le \ln(\delta\varepsilon^{-1}\rho^{\frac{n}{1-q}})+\ln\frac{pa(0)b^{-1}(0)}{1+n-q}&\\ +\frac{1+n-q}{1-q}\ln((1-q)b(0)H(t))&.\end{aligned}$$

(iii) *If* $m = q = 1$, *then*

$$\ln \frac{\varepsilon \rho p a(0)}{\delta n b(0)} + n\delta b(0) H(t) \le p\rho a(0) G(t),$$

$$p\delta a(0) G(t) \le \ln \frac{\delta p a(0)}{\varepsilon \rho n b(0)} + n\delta b(0) H(t).$$

(iv) *If* $m < 1$ *and* $q = 1$, *then*

$$n\delta b(0) H(t) \le \ln(\delta \varepsilon^{-1} \rho^{\frac{p}{1-m}}) + \ln \frac{n a^{-1}(0) b(0)}{1+p-m} + \frac{1+p-m}{1-m} \ln((1-m)a(0)G(t)),$$

$$\ln(\varepsilon \rho \delta^{\frac{p}{1-m}}) + \ln \frac{n a^{-1}(0) b(0)}{1+p-m} + \frac{1+p-m}{1-m} \ln((1-m)a(0)G(t)) \le n\rho b(0) H(t).$$

Proof. Assume $m, q < 1$. Observe that

$$G'(t) = g(t) = v^n(x_0, t), \qquad H'(t) = h(t) = u^p(x_0, t).$$

By Lemma 3.1(i), for any $\delta < 1 < \rho$, there exists $t_0 < T$ such that

$$(\delta(1-m)a(0)G(t))^{\frac{p}{1-m}} \le H'(t) \le (\rho(1-m)a(0)G(t))^{\frac{p}{1-m}}, \quad t \in [t_0, T),$$

$$(\delta(1-q)b(0)H(t))^{\frac{n}{1-q}} \le G'(t) \le (\rho(1-q)b(0)H(t))^{\frac{n}{1-q}}, \quad t \in [t_0, T).$$

Thus, for any $t \in [t_0, T)$,

$$\frac{(\delta(1-m)a(0)G(t))^{\frac{p}{1-m}}}{(\rho(1-q)b(0)H(t))^{\frac{n}{1-q}}} \le \frac{dH}{dG} \le \frac{(\rho(1-m)a(0)G(t))^{\frac{p}{1-m}}}{(\delta(1-q)b(0)H(t))^{\frac{n}{1-q}}}. \tag{3.8}$$

In view of the right inequality of (3.8), for any $t \in [t_0, T)$,

$$(\delta(1-q)b(0)H(t))^{\frac{n}{1-q}} dH \le (\rho(1-m)a(0)G(t))^{\frac{p}{1-m}} dG. \tag{3.9}$$

Integrating (3.9) yields, for $t_0 \le t < T$,

$$\begin{aligned}\frac{(1-q)(\delta(1-q)b(0))^{\frac{n}{1-q}}}{1+n-q} H^{\frac{1+n-q}{1-q}}(s)\Big|_{t_0}^{t} &\le \frac{(1-m)(\rho(1-m)a(0))^{\frac{p}{1-m}}}{1+p-m} G^{\frac{1+p-m}{1-m}}(s)\Big|_{t_0}^{t} \\ &\le \frac{(1-m)(\rho(1-m)a(0))^{\frac{p}{1-m}}}{1+p-m} G^{\frac{1+p-m}{1-m}}(t).\end{aligned} \tag{3.10}$$

Since $\lim_{t\to T} H(t) = \infty$ and $q < 1$, for any $0 < \varepsilon < 1$ there exists $\widetilde{t_0}$ with $t_0 < \widetilde{t_0} < T$ such that

$$H^{\frac{1+n-q}{1-q}}(t_0) \le (1-\varepsilon) H^{\frac{1+n-q}{1-q}}(t), \quad t \in [\widetilde{t_0}, T).$$

Hence, from (3.10) it can be deduced that for $\widetilde{t}_0 < t < T$,

$$\varepsilon(b(0))\delta)^{\frac{n}{1-q}}(1+p-m)((1-q)H(t))^{\frac{1+n-q}{1-q}} \le (a(0)\rho)^{\frac{p}{1-m}}(1+n-q)((1-m)G(t))^{\frac{1+p-m}{1-m}}. \tag{3.11}$$

Application of a similar analysis to the left inequality of (3.8) shows that there exists $t_0^* < T$ such that for $t_0^* < t < T$,

$$\varepsilon(a(0)\delta)^{\frac{p}{1-m}}(1+n-q)((1-m)G(t))^{\frac{1+p-m}{1-m}} \le (b(0)\rho)^{\frac{n}{1-q}}(1+p-m)((1-q)H(t))^{\frac{1+n-q}{1-q}}. \tag{3.12}$$

Set $\widetilde{T} = \max\{\widetilde{t}_0, t_0^*\}$; then (3.11) and (3.12) yield (i) of Lemma 3.11.

Analogously, we can draw the other conclusions of the lemma. ∎

Proof of Theorem 1.3. Choose $\{\delta_i\}_{i=1}^\infty, \{\varepsilon_i\}_{i=1}^\infty, \{\rho_i\}_{i=1}^\infty$ satisfying $0 < \delta_i, \varepsilon_i < 1$ and $\rho_i > 1$ with $\delta_i, \varepsilon_i, \rho_i \to 1$. Putting $(\delta, \varepsilon, \rho) = (\delta_i, \varepsilon_i, \rho_i)$ in Lemma 3.2, we have $\widetilde{T}_i < T$.

(i) Assume $m, q < 1$. From Lemma 3.1(i) it follows that there exist $\{t_i\}_{i=1}^\infty$ with $t_i < T$ and $t_i \to T$ as $i \to \infty$, such that for any t with $t_i < t < T$,

$$(\delta_i(1-m)a(0)G(t))^{\frac{1}{1-m}} \le u(x_0,t) \le (\rho_i(1-m)a(0)G(t))^{\frac{1}{1-m}}. \tag{3.13}$$

Denote $T_i^* = \max\{t_i, \widetilde{T}_i\}$. Then (3.13) and Lemma 3.2(i) assert that for $T_i^* \le t < T$,

$$\begin{aligned} H'(t) &\ge \delta_i^{\frac{p}{1-m}}((1-m)a(0)G(t))^{\frac{p}{1-m}} \\ &\ge \delta_i^{\frac{p\alpha}{\beta(1-q)}}(\delta_i/\rho_i)^{\frac{p^2}{(1-m)(1+p-m)}}(\varepsilon_i\beta/\alpha)^{\frac{p}{1+p-m}} \\ &\quad\cdot (a(0))^{\frac{p}{1+p-m}}(b(0))^{\frac{np}{(1-q)(1+p-m)}}((1-q)H(t))^{\frac{p\alpha}{\beta(1-q)}}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} H'(t) &\le \rho_i^{\frac{p\alpha}{\beta(1-q)}}(\rho_i/\delta_i)^{\frac{p^2}{(1-m)(1+p-m)}}(\beta/(\varepsilon_i\alpha))^{\frac{p}{1+p-m}} \\ &\quad\cdot (a(0))^{\frac{p}{1+p-m}}(b(0))^{\frac{np}{(1-q)(1+p-m)}}((1-q)H(t))^{\frac{p\alpha}{\beta(1-q)}}, \end{aligned} \tag{3.15}$$

where α, β are given by (1.4). Notice that

$$1 - \frac{p\alpha}{\beta(1-q)} = -\frac{np-(1-m)(1-q)}{(1-q)(1+p-m)} = -\frac{1}{\beta(1-q)} < 0.$$

Integrating (3.14) and (3.15) from t to T and using $\lim_{t\to T} H(t) = \infty$, we find that, for $T_i^* \le t < T$,

$$\begin{aligned} &C_i^{-1}\beta(\beta/\alpha)^{-\frac{p}{1+p-m}} \\ &\quad\le (a(0))^{\frac{p}{1+p-m}}(b(0))^{\frac{np}{(1-q)(1+p-m)}}(T-t)((1-q)H(t))^{\frac{1}{\beta(1-q)}} \\ &\quad\le c_i^{-1}\beta(\beta/\alpha)^{-\frac{p}{1+p-m}}, \end{aligned} \tag{3.16}$$

where

$$c_i = \delta_i^{\frac{p\alpha}{\beta(1-q)}} \left(\frac{\delta_i}{\rho_i}\right)^{\frac{p^2}{(1-m)(1+p-m)}} \varepsilon_i^{\frac{p}{1+p-m}},$$

$$C_i = \rho_i^{\frac{p\alpha}{\beta(1-q)}} \left(\frac{\rho_i}{\delta_i}\right)^{\frac{p^2}{(1-m)(1+p-m)}} \varepsilon_i^{\frac{p}{1+p-m}}.$$

Since $c_i \to 1$ and $C_i \to 1$ on account of $\delta_i, \varepsilon_i, \rho_i \to 1$, and $T_i^* \to T$ as $i \to \infty$, by letting $i \to \infty$ in (3.16) we find

$$((1-q)H(t))^{\frac{1}{1-q}} \sim (a(0))^{\frac{-p\beta}{1+p-m}} (b(0))^{\frac{-np\beta}{(1-q)(1+p-m)}} \beta^\beta \left(\frac{\alpha}{\beta}\right)^{\frac{p}{np-(1-m)(1-q)}} (T-t)^{-\beta},$$

i.e.

$$\text{(3.17)} \quad ((1-q)b(0)H(t))^{\frac{1}{1-q}} \sim \left(\frac{\beta}{b(0)}\right)^{\beta} \left(\frac{\alpha b(0)}{\beta a(0)}\right)^{\frac{p}{np-(m-1)(q-1)}} (T-t)^{-\beta}.$$

Similarly, it can be inferred that

$$\text{(3.18)} \quad ((1-m)a(0)G(t))^{\frac{1}{1-m}} \sim \left(\frac{\alpha}{a(0)}\right)^{\alpha} \left(\frac{\beta a(0)}{\alpha b(0)}\right)^{\frac{n}{np-(m-1)(q-1)}} (T-t)^{-\alpha}.$$

From Lemma 3.1(i), (3.17) and (3.18), we know that

$$u(x,t)(T-t)^{\alpha} \sim \left(\frac{\alpha}{a(0)}\right)^{\alpha} \left(\frac{\beta a(0)}{\alpha b(0)}\right)^{\frac{n}{np-(m-1)(q-1)}},$$

$$v(x,t)(T-t)^{\beta} \sim \left(\frac{\beta}{b(0)}\right)^{\beta} \left(\frac{\alpha b(0)}{\beta a(0)}\right)^{\frac{p}{np-(m-1)(q-1)}}$$

uniformly on any compact subset of B_a. That is, the conclusion (i) holds uniformly on any compact subset of B_a.

(ii) Assume $m = 1$ or $q = 1$. We divide this case into three subcases: (1) $m = 1, q < 1$; (2) $m = q = 1$; and (3) $m < 1, q = 1$. We first discuss subcase (1). As in the proof of case (i), it follows from Lemmas 3.1(ii) and 3.2(ii) that for $T_i^* \le t \le T$,

$$\begin{aligned} G'(t) &\ge \delta_i^{\frac{n}{1-q}} ((1-q)b(0)H(t))^{\frac{n}{1-q}} \\ &\ge \delta_i^{\frac{n}{1-q}} (\varepsilon_i(1+n-q)(p\delta_i)^{-1} \rho_i^{-\frac{n}{1-q}} a^{-1}(0)b(0))^{\frac{n}{1+n-q}} \\ &\quad \cdot \exp\left\{\frac{np\delta_i}{1+n-q} a(0)G(t)\right\} \\ &= (\delta_i/\rho_i)^{\frac{n^2}{(1-q)(1+n-q)}} (\varepsilon_i p^{-1}(1+n-q)a^{-1}(0)b(0))^{\frac{n}{1+n-q}} \\ &\quad \cdot \exp\left\{\frac{np\delta_i}{1+n-q} a(0)G(t)\right\}, \end{aligned}$$

$$G'(t) \le (\rho_i/\delta_i)^{\frac{n^2}{(1-q)(1+n-q)}} \big((p\varepsilon_i)^{-1}(1+n-q)a^{-1}(0)b(0)\big)^{\frac{n}{1+n-q}} \cdot \exp\Big\{\frac{np\rho_i}{1+n-q}a(0)G(t)\Big\}.$$

Hence, for $T_i^* \le t < T$,

$$\tag{3.19} \begin{cases} \exp\Big\{-\dfrac{np\rho_i a(0)G(t)}{1+n-p}\Big\}G'(t) \\ \quad \ge (\delta_i/\rho_i)^{\frac{n^2}{(1-q)(1+n-q)}} \big(\varepsilon_i p^{-1}(1+n-p)a^{-1}(0)b(0)\big)^{\frac{n}{1+n-q}}, \\ \exp\Big\{-\dfrac{np\delta_i a(0)G(t)}{1+n-p}\Big\}G'(t) \\ \quad \le (\rho_i/\delta_i)^{\frac{n^2}{(1-q)(1+n-q)}} \big((p\varepsilon_i)^{-1}(1+n-p)a^{-1}(0)b(0)\big)^{\frac{n}{1+n-q}}. \end{cases}$$

Define $A = -\ln(np) + \frac{(1-q)\ln(1+n-q)}{1+n-q}$. Integrating (3.19) from t to T and using $\lim_{t\to T} G(t) = \infty$, we deduce that for $t \in [T_i^*, T)$,

$$\tag{3.20} \frac{1}{\rho_i}(\widehat{c}_i + |\ln(T-t)|) \le \frac{npa(0)}{1+n-q}G(t) \le \frac{1}{\delta_i}(\widehat{C}_i + |\ln(T-t)|),$$

where

$$\begin{aligned} \widehat{c} &= A - \frac{n^2 + (1-q)(1+n-q)}{(1-q)(1+n-q)}\ln\rho_i \\ &\quad + \frac{n\ln(p\varepsilon_i\delta_i^{\frac{n}{1-q}})}{1+n-q} - \frac{(1-q)\ln a(0)}{1+n-q} - \frac{n\ln b(0)}{1+n-q}, \\ \widehat{C} &= A - \frac{n^2 + (1-q)(1+n-q)}{(1-q)(1+n-q)}\ln\delta_i \\ &\quad + \frac{n\ln(p\varepsilon_i^{-1}\rho_i^{\frac{n}{1-q}})}{1+n-q} - \frac{(1-q)\ln a(0)}{1+n-q} - \frac{n\ln b(0)}{1+n-q}. \end{aligned}$$

By combining (3.20) and Lemma 3.2(ii), it follows that for $T_i^* \le t < T$,

$$\tag{3.21} \begin{aligned} \frac{\delta_i}{\rho_i}(c_i + |\ln(T-t)|) &\le \frac{n}{1-q}\ln((1-q)b(0)H(t)) \\ &\le \frac{\rho_i}{\delta_i}(C_i + |\ln(T-t)|), \end{aligned}$$

where

$$\begin{aligned} c_i &= \hat{c}_i - \frac{n\rho_i}{\delta_i(1+n-q)}\ln\big(p(1+n-q)^{-1}\varepsilon_i^{-1}\delta_i\rho_i^{\frac{n}{1-q}}a(0)b^{-1}(0)\big), \\ C_i &= \hat{C}_i - \frac{n\delta_i}{\rho_i(1+n-q)}\ln\big(p(1+n-q)^{-1}\varepsilon_i\rho_i\delta_i^{\frac{n}{1-q}}a(0)b^{-1}(0)\big). \end{aligned}$$

Consequently, (3.20) and (3.21) guarantee that for $T_i^* < t < T$,

$$
\left\{
\begin{aligned}
\frac{\widehat{c}_i + |\ln(T-t)|}{\rho_i |\ln(T-t)|} &\le \frac{npa(0)G(t)}{(1+n-p)|\ln(T-t)|} \\
&\le \frac{\widehat{C}_i + |\ln(T-t)|}{\delta_i |\ln(T-t)|}, \\
\frac{\delta_i (c_i + |\ln(T-t)|)}{\rho_i |\ln(T-t)|} &\le \frac{n \ln((1-q)b(0)H(t))}{(1-q)|\ln(T-t)|} \\
&\le \frac{\rho_i (C_i + |\ln(T-t)|)}{\delta_i |\ln(T-t)|}.
\end{aligned}
\right. \tag{3.22}
$$

Notice that $\widehat{c}_i \to \widehat{C}_i$ and $c_i \to C_i$ and are bounded because $\delta_i, \varepsilon_i, \rho_i \to 1$ as $i \to \infty$. By letting $i \to \infty$ in (3.22), we get

$$
\begin{aligned}
&\lim_{t\to T} a(0)G(t)|\ln(T-t)|^{-1} = \frac{1+n-q}{np}, \\
&\lim_{t\to T} \ln((1-q)b(0)H(t))|\ln(T-t)|^{-1} = \frac{1-q}{n}.
\end{aligned}
$$

As $v^{1-q}(x,t) \sim (1-q)b(0)H(t)$ uniformly on compact subsets of B_a, we find that, uniformly on compact subsets of B_a,

$$
\ln v(x,t) \sim \frac{1}{1-q} \ln((1-q)b(0)H(t)). \tag{3.23}
$$

Therefore, it can be deduced from Lemma 3.1(ii), (3.22) and (3.23) that uniformly on compact subsets of B_a,

$$
\left\{
\begin{aligned}
&\ln u(x,t) \sim a(0)G(t) \sim \frac{1+n-q}{np} |\ln(T-t)|, \\
&\ln v(x,t) \sim \frac{1}{n} |\ln(T-t)|.
\end{aligned}
\right. \tag{3.24}
$$

Thereby, uniformly on any compact subset of B_a,

$$
\left\{
\begin{aligned}
&\lim\nolimits_{t\to T} |\ln(T-t)|^{-1} \ln u(x,t) = \frac{1+n-q}{np}, \\
&\lim\nolimits_{t\to T} |\ln(T-t)|^{-1} \ln v(x,t) = \frac{1}{n}.
\end{aligned}
\right. \tag{3.25}
$$

Subcases (2) and (3) can be verified similarly. ∎

Proof of Theorem 1.4. To consider the uniform blow-up profiles of $\max_{\bar{B}} u(\cdot,t)$ and $\max_{\bar{B}} v(\cdot,t)$, we only need to consider the problem

$$
\left\{
\begin{aligned}
&u_t = \Delta u + a(0)u^m(x,t)v^n(x_0,t), && x \in B,\ t > 0, \\
&v_t = \Delta v + b(0)u^p(x_0,t)v^q(x,t), && x \in B,\ t > 0.
\end{aligned}
\right. \tag{3.26}
$$

Since for $N = 1$ problem (1.1) has the blow-up rate given by (1.8), we can denote $u = C_1(T-t)^{-\beta_1}$, $v = C_2(T-t)^{-\beta_2}$, where $C_1, C_2, \beta_1, \beta_2$ are constants to be determined later. By (3.26), we have

$$(3.27)\qquad \begin{cases} C_1\beta_1(T-t)^{-\beta_1-1} = a(0)C_1^m(T-t)^{-\beta_1 m}C_2^n(T-t)^{-\beta_2 n}, \\ C_2\beta_2(T-t)^{-\beta_2-1} = b(0)C_1^p(T-t)^{-\beta_1 p}C_2^q(T-t)^{-\beta_2 q}. \end{cases}$$

Then

$$\begin{cases} -\beta_1 - 1 = -\beta_1 m - \beta_2 n, \\ -\beta_2 - 1 = -\beta_1 p - \beta_2 q, \end{cases} \qquad \begin{cases} C_1\beta_1 = a(0)C_1^m C_2^n, \\ C_2\beta_2 = b(0)C_1^p C_2^q. \end{cases}$$

That is,

$$(3.28)\qquad \begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{cases} C_1^{m-1}C_2^n = \beta_1/a(0), \\ C_1^p C_2^{q-1} = \beta_2/b(0). \end{cases}$$

It is obvious that $(\alpha,\beta)^T$ solves the first equations of (3.28), where $(\alpha,\beta)^T$ is given by (1.4). To obtain the solution $(C_1,C_2)^T$, we first consider the problem

$$\begin{pmatrix} m-1 & n \\ p & q-1 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \ln\frac{\alpha}{a(0)} \\ \ln\frac{\beta}{b(0)} \end{pmatrix},$$

which has a unique solution

$$(l_1,l_2)^T = \left(\frac{n\ln\frac{\beta}{b(0)} - (q-1)\ln\frac{\alpha}{a(0)}}{np-(m-1)(q-1)}, \frac{p\ln\frac{\alpha}{a(0)} - (m-1)\ln\frac{\beta}{b(0)}}{np-(m-1)(q-1)}\right)^T.$$

Let $l_1 = \ln C_1, l_2 = \ln C_2$. Then $(C_1,C_2)^T = (e^{l_1}, e^{l_2})^T$ solves the second equations of (3.28). Notice that

$$\begin{aligned} n\ln\frac{\beta}{b(0)} - (q-1)\ln\frac{\alpha}{a(0)} &= n\ln\frac{\beta}{b(0)} - n\ln\frac{\alpha}{a(0)} + n\ln\frac{\alpha}{a(0)} - (q-1)\ln\frac{\alpha}{a(0)} \\ &= n\ln\frac{\beta a(0)}{\alpha b(0)} + (n+1-q)\ln\frac{\alpha}{a(0)}, \end{aligned}$$

and

$$p\ln\frac{\alpha}{a(0)} - (m-1)\ln\frac{\beta}{b(0)} = p\ln\frac{\alpha b(0)}{\beta a(0)} + (p+1-m)\ln\frac{\beta}{b(0)}.$$

Then

$$\begin{aligned} C_1 &= \left(\frac{\alpha}{a(0)}\right)^{\alpha}\left(\frac{\beta a(0)}{\alpha b(0)}\right)^{\frac{n}{np-(m-1)(q-1)}}, \\ C_2 &= \left(\frac{\beta}{b(0)}\right)^{\beta}\left(\frac{\alpha b(0)}{\beta a(0)}\right)^{\frac{p}{np-(m-1)(q-1)}}. \end{aligned}$$

Thus, the proof is complete. ■

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Haihua Lu
Department of Mathematics
Southeast University
Nanjing 211189, P.R. China
and
Department of Mathematics
Nantong University
Nantong 226019, P.R. China
E-mail: hhlu188@163.com

Feng Wang
School of Mathematics and Physics
Changzhou University
Changzhou 213164, P.R. China
E-mail: fengwang188@163.com

Qiaoyun Jiang
Department of Mathematics
Nantong University
Nantong 226019, P.R. China
E-mail: jiangqy@ntu.edu.cn

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