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On continuous composition operators

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Abstract. Let $I \subset \mathbb{R}$ be an interval, Y be a normed linear space and Z be a Banach space. We investigate the Banach space $\mathrm{Lip}_2(I, Z)$ of all functions $\psi : I \to Z$ such that

$$M_\psi := \sup\{\|[r, s, t; \psi]\| : r < s < t,\ r, s, t \in I\} < \infty,$$

where

$$[r, s, t; \psi] := \frac{(s-r)\psi(t) + (t-s)\psi(r) - (t-r)\psi(s)}{(t-r)(t-s)(s-r)}.$$

We show that $\psi \in \mathrm{Lip}_2(I, Z)$ if and only if ψ is differentiable and its derivative ψ' is Lipschitzian.

Suppose the composition operator N generated by $h : I \times Y \to Z$,

$$(N\varphi)(t) := h(t, \varphi(t)),$$

maps the set $\mathcal{A}(I, Y)$ of all affine functions $\varphi : I \to Y$ into $\mathrm{Lip}_2(I, Z)$. We prove that if N is continuous and $M_\psi \le M$ for some constant $M > 0$, where $\psi(t) = N(t, \varphi(t))$, then

$$h(t, y) = a(y) + b(t), \qquad t \in I,\ y \in Y,$$

for some continuous linear $a : Y \to Z$ and $b \in \mathrm{Lip}_2(I, Z)$. Lipschitzian and Hölder composition operators are also investigated.

1. Introduction. Let I be an interval in $\mathbb{R}$, let Z be a Banach space and $\psi : I \to Z$ be a function. For distinct $r, s, t \in I$ we put

$$[r, s; \psi] := \frac{\psi(s) - \psi(r)}{s - r}$$

and

$$\begin{aligned}[r, s, t; \psi] &:= \frac{1}{t-r}\left(\frac{\psi(t) - \psi(s)}{t-s} - \frac{\psi(s) - \psi(r)}{s-r}\right) \\ &= \frac{(s-r)\psi(t) + (t-s)\psi(r) - (t-r)\psi(s)}{(t-s)(s-r)(t-r)}.\end{aligned}$$

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These expressions are called the *first divided difference* of ψ at r, s and the *second divided difference* of ψ at r, s, t, respectively (see e.g. [1, p. 371] and [6, p. 237]). Moreover, if we define the function $\Delta_s(\psi)$ by

$$(\Delta_s\psi)(r) = \frac{\psi(s) - \psi(r)}{s - r}, \qquad r \in I \setminus \{s\},$$

then

$$(\Delta_t\Delta_s)(\psi)(r) = [r, s, t; \psi], \qquad r \in I \setminus \{s, t\}.$$

We consider the space $\mathrm{Lip}_2(I, Z)$ of all functions $\psi : I \to Z$ such that

$$M_\psi := \sup\{\|[r, s, t; \psi]\| : r < s < t,\ r, s, t \in I\} \tag{1}$$

is finite. In Section 2 we show that $\psi \in \mathrm{Lip}_2(I, Z)$ if and only if ψ is differentiable and ψ' is Lipschitzian.

Let $h : I \times \mathbb{R} \to \mathbb{R}$. The mapping $N : \mathbb{R}^I \to \mathbb{R}^I$ defined by

$$(N\varphi)(t) = h(t, \varphi(t))$$

is called the *composition* (*Nemytskiĭ*) *operator* determined by the *generator* h. In 1982 J. Matkowski proved that if a composition operator mapping the Banach space $\mathrm{Lip}(I, \mathbb{R})$ of Lipschitz functions $\varphi : I \to \mathbb{R}$ into itself is globally Lipschitzian, then there exist functions $a, b \in \mathrm{Lip}(I, \mathbb{R})$ such that $h(t, y) = a(t)y + b(t)$, $t \in I$, $y \in \mathbb{R}$ (cf. [3]). This result has then been extended to some other Banach function spaces by J. Matkowski and his collaborators (cf., e.g., [2], [4], [5]).

Let $(Y, \|\cdot\|)$ be a normed linear space and let $\mathcal{A}(I, Y)$ denote the space of all functions $\varphi : I \to Y$ of the form $t \mapsto ct + d$, where $c, d \in Y$. By $\mathcal{L}_M$ we denote the set of all functions $\psi \in \mathrm{Lip}_2(I, Z)$ such that $M_\psi \le M$, where M is a positive number and M_ψ is given by (1). In the next section we will prove that every continuous composition operator N mapping $\mathcal{A}(I, Y)$ into $\mathcal{L}_M$ is generated by a function h of the form $h(t, y) = a(y) + b(t)$, $t \in I$, $y \in Y$, where $a : Y \to Z$ is a continuous linear map and $b \in \mathrm{Lip}_2(I, Z)$.

In the last two sections we will examine composition operators mapping $\mathcal{A}(I, Y)$ into $\mathrm{Lip}_2(I, Z)$ and satisfying the Lipschitz and Hölder conditions.

2. On properties of functions in $\mathrm{Lip}_2(I, Z)$. Let $\psi : I \to Z$ be Lipschitz and let $L(\psi)$ denote the smallest number L such that

$$\|\psi(t) - \psi(s)\| \le L|t - s| \qquad \text{for all } s, t \in I.$$

If, e.g., $I = [\alpha, \beta)$, then $\psi : I \to Z$ is *differentiable at a* if the right-hand derivative exists at this point.

LEMMA 1. *If Z is a Banach space and $\psi \in \mathrm{Lip}_2(I, Z)$, then ψ is differentiable, ψ' is Lipschitz and $L(\psi') \le 2M_\psi$.*

Proof. Fix $t_0 \in I$ with $t_0 < \sup I$ and take $s, t \in I$ such that $t_0 < s < t$. By the definition of M_ψ we have

$$\left\| \frac{\psi(t) - \psi(s)}{t - s} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right\| \le M_\psi (t - t_0). \tag{2}$$

Since

$$\frac{\psi(t) - \psi(s)}{t - s} - \frac{\psi(s) - \psi(t_0)}{s - t_0} = \frac{t - t_0}{t - s} \left[\frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right],$$

inequality (2) yields

$$\left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right\| \le M_\psi (t - s). \tag{3}$$

Thus the limit

$$\lim_{t \to t_0+} \frac{\psi(t) - \psi(t_0)}{t - t_0} = \psi'_+(t_0)$$

exists and by (3),

$$\left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \psi'_+(t_0) \right\| \le M_\psi (t - t_0) \tag{4}$$

for $t_0 < t$, $t \in I$.

Next fix $t_0 \in I$ with $\inf I < t_0$ and take $r, s \in I$ such that $r < s < t_0$. In a similar manner to (3), it can be established that

$$\left\| \frac{\psi(t_0) - \psi(s)}{t_0 - s} - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \le M_\psi (s - r).$$

Hence the left-hand derivative $\psi'_-(t_0)$ exists and it satisfies the inequality

$$\left\| \psi'_-(t_0) - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \le M_\psi (t_0 - r) \tag{5}$$

for all $r \in I$ such that $r < t_0$. To show that

$$\psi'_+(t_0) = \psi'_-(t_0) \tag{6}$$

in the case $\inf I < t_0 < \sup I$, we choose $r, t \in I$ such that $r < t_0 < t$. As in (2) we have

$$\left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \le M_\psi (t - r). \tag{7}$$

Combining (4), (5) and (7) we conclude that

$$\|\psi'_+(t_0) - \psi'_-(t_0)\| \le 2 M_\psi (t - r),$$

whence equality (6) follows.

Assume that $r \in I$, $r \ne \sup I$. For $s, t, u \in I$ such that $r < s < t < u$ we have

$$\left\| \frac{\psi(u) - \psi(t)}{u - t} - \frac{\psi(t) - \psi(s)}{t - s} \right\| \le M_\psi (u - s)$$

and

$$\left\| \frac{\psi(t)-\psi(s)}{t-s} - \frac{\psi(s)-\psi(r)}{s-r} \right\| \le M_\psi(t-r).$$

These inequalities yield

$$\left\| \frac{\psi(u)-\psi(t)}{u-t} - \frac{\psi(s)-\psi(r)}{s-r} \right\| \le M_\psi(u-s+t-r).$$

Therefore letting $u \to t+$ and $s \to r+$, we obtain

$$\|\psi'(t)-\psi'(r)\| \le 2M_\psi(t-r).$$

A similar argument may be used when $r \in I$, $r \ne \inf I$. ∎

LEMMA 2. *If ψ is differentiable in I and ψ' satisfies the Lipschitz condition, then $\psi \in \mathrm{Lip}_2(I,Z)$ and $M_\psi \le L(\psi')$.*

Proof. Take $u,v,w \in I$ such that $u<v<w$. It is sufficient to show that $\|z\| \le L(\psi')(w-u)$, where

$$z := \frac{\psi(w)-\psi(v)}{w-v} - \frac{\psi(v)-\psi(u)}{v-u}.$$

We may assume that $z \ne 0$. Take a continuous linear functional $p : Z \to \mathbb{R}$ with $\|p\| = 1$ such that $p(z) = \|z\|$. The function $p\circ\psi : I \to \mathbb{R}$ is differentiable and $(p\circ\psi)'(t) = p(\psi'(t))$, $t \in I$. By the Lagrange mean-value theorem, for some $\sigma \in (u,v)$ and $\tau \in (v,w)$, we have

$$p\left(\frac{\psi(v)-\psi(u)}{v-u}\right) = \frac{p\circ\psi(v) - p\circ\psi(u)}{v-u} = p(\psi'(\sigma))$$

and

$$p\left(\frac{\psi(w)-\psi(v)}{w-v}\right) = p(\psi'(\tau)).$$

Therefore

$$\begin{aligned}
\left\| \frac{\psi(w)-\psi(v)}{w-v} - \frac{\psi(v)-\psi(u)}{v-u} \right\| &= \|z\| = p(z) \\
&= p\left(\frac{\psi(w)-\psi(v)}{w-v}\right) - p\left(\frac{\psi(v)-\psi(u)}{v-u}\right) = p(\psi'(\tau)) - p(\psi'(\sigma)) \\
&= p(\psi'(\tau)-\psi'(\sigma)) \le \|p\|\,\|\psi'(\tau)-\psi'(\sigma)\| \\
&\le L(\psi')(\tau-\sigma) \le L(\psi')(w-u). \ \blacksquare
\end{aligned}$$

THEOREM 1. *Assume that $I \subset \mathbb{R}$ is an interval and Z is a Banach space. Then $\psi \in \mathrm{Lip}_2(I,Z)$ if and only if ψ is differentiable and its derivative ψ' satisfies the Lipschitz condition in I.*

Theorem 1 is a consequence of Lemmas 1 and 2.

3. Continuous composition operator. We shall assume that I is an interval containing 0. We introduce the norm $\|\cdot\|_2$ in $\mathrm{Lip}_2(I,Z)$ putting

$$\|\psi\|_2 = \|\psi(0)\| + \|\psi'(0)\| + M_\psi,$$

where M_ψ is given by (1). By Lemma 1, $\|\cdot\|_2$ is a norm. Moreover, $\mathrm{Lip}_2(I,Z)$ is a Banach space.

The inequality $L(\psi') \le 2M_\psi$ and the Lagrange mean-value theorem lead to the next lemma.

LEMMA 3. *If $\psi_n \to \psi$ in $\mathrm{Lip}_2(I,Z)$, then $\psi_n(t) \to \psi(t)$ in Z for every $t \in I$.*

It is easily seen that $\|\varphi\|_2 = \|c\| + \|d\|$ if $\varphi \in \mathcal{A}(I,Y)$ $(\subset \mathrm{Lip}_2(I,Y))$ is of the form $\varphi(t) = ct + d$.

Every function $h : I \times Y \to Z$ generates the Nemytskiĭ operator N defined by

$$(N\varphi)(t) = h(t,\varphi(t)), \qquad t \in I,\ \varphi \in \mathcal{A}(I,Y). \tag{8}$$

LEMMA 4. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$. Assume that $(Y,\|\cdot\|)$ is a normed linear space and $(Z,\|\cdot\|)$ is a Banach space. If the composition operator $N : \mathcal{A}(I,Y) \to \mathrm{Lip}_2(I,Z)$ is continuous, then its generator h is continuous with respect to each variable.*

Proof. Take an arbitrary $y \in Y$ and define an affine function φ assuming $\varphi(t) = y$, $t \in I$. Since $h(\cdot,y) = N\varphi \in \mathrm{Lip}_2(I,Z)$, h is continuous with respect to the first variable. The continuity of h with respect to the second variable follows from Lemma 3.

Recall that $\mathcal{L}_M$ denotes the set of all functions $\psi \in \mathrm{Lip}_2(I,Z)$ such that $M_\psi \le M$, where M_ψ is given by (1) and M is a fixed constant.

THEOREM 2. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$. Assume that $(Y,\|\cdot\|)$ is a normed linear space and $(Z,\|\cdot\|)$ is a Banach space. If the composition operator N generated by $h : I \times Y \to Z$ maps $\mathcal{A}(I,Y)$ into $\mathcal{L}_M$, then there exists an additive mapping $a : Y \to Z$ and a mapping $b \in \mathcal{L}_M$ such that*

$$h(t,y) = a(y) + b(t), \qquad t \in I,\ y \in Y.$$

Moreover, if the operator N is continuous, then a is a continuous linear mapping.

Proof. Take $r,t \in I$ with $r < t$ and $y,\overline{y} \in Y$. Define an affine function by setting

$$\varphi(u) = y + \frac{\overline{y} - y}{t - r}(u - r), \qquad u \in I.$$

Since $N\varphi \in \mathcal{L}_M$, we obtain

$$\|(t-s)(N\varphi)(r) + (s-r)(N\varphi)(t) - (t-r)(N\varphi)(s)\| \le M(t-r)(t-s)(s-r)$$

for all $s \in (r,t)$. Choosing $s = (1/2)(r+t)$ and taking into account the relations

$$\varphi(r) = y, \qquad \varphi(t) = \overline{y}, \qquad \varphi\left(\frac{r+t}{2}\right) = \frac{y+\overline{y}}{2}$$

we get

$$\left\| h(r,y) + h(t,\overline{y}) - 2h\left(\frac{r+t}{2}, \frac{y+\overline{y}}{2}\right)\right\| \le \frac{1}{2}M(t-r)^2. \tag{9}$$

Letting $r \to t-$ and making use of the continuity of $h(\cdot,y)$ we deduce that

$$h(t,y) + h(t,\overline{y}) - 2h\left(t, \frac{y+\overline{y}}{2}\right) = 0, \qquad t \in I,\ y,\overline{y} \in Y,$$

so $h(t,\cdot)$ satisfies the Jensen functional equation in a normed linear space Y. Hence there exist functions $a : I \times Y \to Z$ and $b : I \to Z$ such that

$$h(t,y) = a(t,y) + b(t) \tag{10}$$

and $a(t,\cdot) : Y \to Z$ is an additive mapping (cf., e.g., [1, Theorem 1, p. 315]). We conclude from (10) that $b = h(\cdot,0)$, hence $b \in \mathcal{L}_M$ and finally $a(\cdot,y) \in \mathcal{L}_M$ for each $y \in Y$.

Combining (10) and (9) we get

$$\left\| a(r,y) + b(r) + a(t,\overline{y}) + b(t) - 2a\left(\frac{r+t}{2}, \frac{y+\overline{y}}{2}\right) - 2b\left(\frac{r+t}{2}\right)\right\| \le \frac{1}{2}M(t-r)^2$$

for every $r,t \in I$ and $y,\overline{y} \in Y$. Take ny and $n\overline{y}$, $n \in \mathbb{N}$, instead of y and $\overline{y}$, respectively. Next, since $a(t,ny) = na(t,y)$, dividing both sides of the resulting inequality by n and letting $n \to \infty$, we conclude that

$$a(r,y) + a(t,\overline{y}) = 2a\left(\frac{r+t}{2}, \frac{y+\overline{y}}{2}\right)$$

for all $r,t \in I$ and $y,\overline{y} \in Y$, which means that the function $(r,y) \mapsto a(r,y)$ is Jensen. Since $a(0,0) = 0$, the function a is additive with respect to the pair of variables $(t,y) \in I \times Y$. We observe that

$$a(t,y) = a((t,0) + (0,y)) = a(t,0) + a(0,y) = a(0,y),$$

since $a(t,0) = 0$ for all $t \in I$. Thus $a(y) := a(0,y)$ does not depend on the first variable t and

$$h(t,y) = a(y) + b(t), \qquad t \in I,\ y \in Y. \tag{11}$$

This finishes the proof of the first part of Theorem 2.

It remains to prove that a is continuous if so is N. But this follows from Lemma 4. ∎

An easy verification shows that the inverse result is valid.

THEOREM 3. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$, Y be a normed linear space and Z be a Banach space. If $a : Y \to Z$ is a continuous linear map and $b \in \mathrm{Lip}_2(I, Z)$, then the composition operator N generated by $h(t, y) = a(y) + b(t)$, $t \in I$, $y \in Y$, is continuous and maps the space $\mathcal{A}(I, Y)$ into $\mathcal{L}_M$, where $M = M_b$.*

4. Lipschitzian composition operators. The generator of a Lipschitzian composition operator has a form slightly different from that in Theorem 2.

THEOREM 4. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$, Y be a normed linear space and Z be a Banach space. If the composition operator N generated by $h : I \times Y \to Z$ maps $\mathcal{A}(I, Y)$ into $\mathrm{Lip}_2(I, Z)$ and satisfies the Lipschitz condition, i.e., there exists a positive constant L such that*

$$(12) \qquad \|N\varphi - N\psi\|_2 \le L\|\varphi - \psi\|_2, \qquad \varphi, \psi \in \mathcal{A}(I, Y),$$

then there exist functions $a : I \times Y \to Z$ and $b : I \to Z$ such that for each $y \in Y$ and $t \in I$, $a(\cdot, y), b \in \mathrm{Lip}_2(I, Z)$ and $a(t, \cdot)$ is a continuous linear map of Y into Z and

$$h(t, y) = a(t, y) + b(t), \qquad t \in I,\ y \in Y.$$

In particular, N is affine.

Proof. We mimic the first part of the proof of Theorem 2. By Lemma 4 the generator h of N is continuous with respect to each variable. Making use of (12) and the definition of the norm $\|\cdot\|_2$ we infer that

$$(13) \qquad \|[r, s, t; h(\cdot, \varphi(\cdot)) - h(\cdot, \psi(\cdot))]\| \le L\|\varphi - \psi\|_2$$

for all $r, s, t \in I, r < s < t$. Take arbitrary $r, t \in I$ with $r < t$ and define the functions

$$\varphi(u) = y + \frac{\overline{y} - y}{t - r}(u - r), \qquad \psi(u) = 0, \qquad u \in I.$$

Of course,

$$\varphi(r) = y, \qquad \varphi(t) = \overline{y}, \qquad \varphi\left(\frac{r + t}{2}\right) = \frac{y + \overline{y}}{2}$$

and

$$\varphi(0) = \frac{ty - r\overline{y}}{t - r}, \qquad \varphi'(0) = \frac{y - \overline{y}}{t - r}.$$

Setting $s = (r + t)/2$ in (13) we obtain

$$\left\| h(t, \overline{y}) - h(t, 0) - 2h\left(\frac{r + t}{2}, \frac{y + \overline{y}}{2}\right) + 2h\left(\frac{r + t}{2}, 0\right) + h(r, y) - h(r, 0) \right\| \\ \le \frac{1}{2}L(t - r)(\|ty - r\overline{y}\| + \|y - \overline{y}\|).$$

Letting t tend to r and making use of the continuity of h with respect to the first variable we hence get

$$h(r,\overline{y}) + h(r,y) = 2h\left(r, \frac{y+\overline{y}}{2}\right),$$

which shows that, for every fixed $r \in I$, the function $h(r,\cdot)$ satisfies the Jensen functional equation in the normed linear space Y. As in the proof of Theorem 2, there exist $a : I \times Y \to Z$ and $b : I \to Z$ such that

$$h(r,y) = a(r,y) + b(r), \qquad r \in I,\ y \in Y, \tag{14}$$

where $a(r,\cdot)$ is an additive map for every $r \in I$. Now the remainder is clear. ∎

To obtain a converse to the last theorem we will require that I is a compact interval such that $0 \in I$.

As an application of the uniform boundedness principle one obtains the following lemma.

LEMMA 5. *Let $I = [0,1]$ and let Y, Z be Banach spaces. If $a : I \times Y \to Z$ is such that $a(\cdot,y) \in \mathrm{Lip}_2(I,Z)$ for $y \in Y$ and each $a(t,\cdot)$ $(t \in I)$ is linear and continuous, then $a'_t(t,\cdot)$ is also linear and continuous.*

THEOREM 5. *Let $I = [0,1]$ and let Y, Z be Banach spaces. If $a : I \times Y \to Z$ and $b : I \to Z$ are such that $a(\cdot,y), b \in \mathrm{Lip}_2(I,Z)$ and $a(t,\cdot)$ is a continuous linear map of Y into Z and*

$$h(t,y) = a(t,y) + b(t), \qquad (t,y) \in I \times Y,$$

then the operator N, $(N\varphi)(t) = h(t,\varphi(t))$, maps $\mathcal{A}(I,Y)$ into $\mathrm{Lip}_2(I,Z)$ and

$$\|N\varphi_1 - N\varphi_2\|_2 \le L\|\varphi_1 - \varphi_2\|_2 \quad \textit{for some } L > 0.$$

Proof. Without loss of generality we may assume that $b \equiv 0$. In that case N is linear. Take $\varphi(t) = ct + d$, $t \in I = [0,1]$, where $c, d \in Y$. We have

$$(N\varphi)(t) = a(t,\varphi(t)) = a(t, ct+d) = ta(t,c) + a(t,d).$$

Of course, the function $N\varphi$ is differentiable and

$$(N\varphi)'(t) = a(t,c) + ta'_t(t,c) + a'_t(t,d), \qquad t \in I.$$

The function $a'_t(\cdot,d)$ is Lipschitz (cf. Lemma 1). Since the product of two bounded Lipschitz functions is Lipschitz as well and $a(\cdot,c)$ has a bounded derivative in I, we see that $N(\varphi) \in \mathrm{Lip}_2(I,Z)$ (cf. Lemma 2).

Further, from the compactness of I, the continuity of $a(\cdot,y)$ and $a'_t(\cdot,y)$ for each $y \in Y$ and the uniform boundedness principle we conclude that $\|a(t,\cdot)\|, \|a'_t(t,\cdot)\| \le K$ for all $t \in I$ and some constant $K > 0$. Hence

$$\frac{\|a(t,y) - a(r,y)\|}{t-r} \le \sup_{s\in I} \|a'_t(s,y)\| \le K\|y\| \tag{15}$$

for all $y \in Y$ and $0 \le r < t \le 1$. For $y \in Y$ and $0 \le r < s < t \le 1$ we have $\|[r,s,t;a(\cdot,y)]\| \le M_{a(\cdot,y)} < \infty$. Again by the uniform boundedness principle we can find a constant $L > 0$ such that

$$\|[r,s,t;a(\cdot,y)]\| \le L\|y\| \tag{16}$$

for all $y \in Y$ and $0 \le r < s < t \le 1$.

Let $\varphi \in \mathcal{A}(I,Y)$, $\varphi(t) = ct + d$, and $\psi(t) = (N\varphi)(t)$, $t \in I$. Since

$$[r,s,t;\psi] = t[r,s,t;a(\cdot,c)] + \frac{a(r,c) - a(s,c)}{s-r} + [r,s,t;a(\cdot,d)]$$

for all $0 \le r < s < t \le 1$, by (15) and (16) we have

$$M_\psi \le K\|c\| + L(\|c\| + \|d\|).$$

Consequently, $\|N\varphi\|_2 \le (2K + L)\|\varphi\|_2$. ∎

5. Hölder composition operators. The following result deals with composition operators mapping $\mathcal{A}(I,Y)$ into $\mathrm{Lip}_2(I,Z)$ satisfying the Hölder condition.

THEOREM 6. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I, Y$ be a normed linear space and Z be a Banach space. The composition operator N generated by $h : I \times Y \to Z$ mapping $\mathcal{A}(I,Y)$ into $\mathrm{Lip}_2(I,Z)$ satisfies the Hölder condition, i.e., there exist positive constants L and $\alpha < 1$ for which*

$$\|N\varphi - N\psi\|_2 \le L\|\varphi - \psi\|^\alpha, \qquad \varphi,\psi \in \mathcal{A}(I,Y), \tag{17}$$

if and only if N is a constant map, that is, there exists $b \in \mathrm{Lip}_2(I,Z)$ such that

$$h(t,y) = b(t), \qquad t \in I,\ y \in Y.$$

Proof. The "if" part is clear. We will prove the "only if" part. As in the proof of Theorem 4, inequality (17) gives

$$\begin{aligned}\Big\| h(t,\overline{y}) - h(t,0) - 2h\Big(\frac{r+t}{2}, \frac{y+\overline{y}}{2}\Big) &\\ + 2h\Big(\frac{r+t}{2}, 0\Big) + h(r,y) - h(r,0)\Big\| &\\ \le \frac{1}{2}L(t-r)^{2-\alpha}(\|ty - r\overline{y}\| + \|y - \overline{y}\|)^\alpha &\end{aligned} \tag{18}$$

for all $r,t \in I$ with $r < t$ and all $y,\overline{y} \in Y$. Analysis similar to that in the proof of Theorem 4 shows that

$$h(r,y) = a(r,y) + b(r), \qquad r \in I,\ y \in Y, \tag{19}$$

where $a(t,\cdot)$ is a continuous linear map and $a(\cdot,y), b \in \mathrm{Lip}_2(I,Z)$. Combining (19) and (18) we obtain

$$\left\|a(t,\overline{y})-2a\left(\frac{r+t}{2},\frac{y+\overline{y}}{2}\right)+a(r,y)\right\|\le\frac{1}{2}L(t-r)^{2-\alpha}(\|ty-r\overline{y}\|+\|y-\overline{y}\|)^{\alpha}.$$

Now replacing y and $\overline{y}$ by ny and $n\overline{y}$, $n\in\mathbb{N}$, respectively, then applying the additivity of $a(t,\cdot)$, and finally dividing by n we deduce that

$$\left\|a(t,\overline{y})-2a\left(\frac{r+t}{2},\frac{y+\overline{y}}{2}\right)+a(r,y)\right\| \\ \le\frac{1}{2}L(t-r)^{2-\alpha}n^{\alpha-1}(\|ty-r\overline{y}\|+\|y-\overline{y}\|)^{\alpha}.$$

Letting $n\to\infty$ we can assert that

$$a(r,y)+a(t,\overline{y})=2a\left(\frac{r+t}{2},\frac{y+\overline{y}}{2}\right),\qquad r,t\in I,\ y,\overline{y}\in Y,$$

which means that the mapping $(t,y)\mapsto a(t,y)$ satisfies the Jensen functional equation in $I\times Y$. As in Theorem 2, we have

$$h(t,y)=a(y)+b(t),\qquad t\in I,\ y\in Y$$

with $a(y)=a(0,y)$. Since a is linear and satisfies the Hölder inequality with $\alpha<1$, it follows that $a\equiv 0$. ∎

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References

[1] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Sci. Publ. and Silesian Univ., Warszawa–Kraków–Katowice, 1985.

[2] A. Matkowska, *On characterization of Lipschitzian operators of substitution in the class of Hölder's functions*, Zeszyty Nauk. Politech. Łódz. Mat. 17 (1984), 81–85.

[3] J. Matkowski, *Functional equations and Nemytskii operators*, Funkcial. Ekvac. 25 (1982), 127–132.

[4] J. Matkowski, *Lipschitzian composition operators in some function spaces*, Nonlinear Anal. 30 (1997), 719–726.

[5] J. Matkowski and J. Miś, *On a characterization of Lipschitzian operators of substitution in the space* BV$[a,b]$, Math. Nachr. 117 (1984), 155–159.

[6] A. W. Roberts and D. E. Varberg, *Convex Functions*, Academic Press, New York and London, 1973.

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