On continuous composition operators

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Abstract. Let $I \subset \mathbb{R}$ be an interval, $Y$ be a normed linear space and $Z$ be a Banach space. We investigate the Banach space $\text{Lip}_2(I, Z)$ of all functions $\psi : I \to Z$ such that

$$M_\psi := \sup \{ \| [r, s, t; \psi] \| : r < s < t, r, s, t \in I \} < \infty,$$

where

$$[r, s, t; \psi] := \frac{(s - r)\psi(t) + (t - s)\psi(r) - (t - r)\psi(s)}{(t - r)(t - s)(s - r)}.$$

We show that $\psi \in \text{Lip}_2(I, Z)$ if and only if $\psi$ is differentiable and its derivative $\psi'$ is Lipschitzian.

Suppose the composition operator $N$ generated by $h : I \times Y \to Z$,

$$(N\varphi)(t) := h(t, \varphi(t)),$$

maps the set $A(I, Y)$ of all affine functions $\varphi : I \to Y$ into $\text{Lip}_2(I, Z)$. We prove that if $N$ is continuous and $M_\psi \leq M$ for some constant $M > 0$, where $\psi(t) = N(t, \varphi(t))$, then

$$h(t, y) = a(y) + b(t), \quad t \in I, y \in Y,$$

for some continuous linear $a : Y \to Z$ and $b \in \text{Lip}_2(I, Z)$. Lipschitzian and Hölder composition operators are also investigated.

1. Introduction. Let $I$ be an interval in $\mathbb{R}$, let $Z$ be a Banach space and $\psi : I \to Z$ be a function. For distinct $r, s, t \in I$ we put

$$[r, s; \psi] := \frac{\psi(s) - \psi(r)}{s - r},$$

and

$$[r, s, t; \psi] := \frac{1}{t - r} \left( \frac{\psi(t) - \psi(s)}{t - s} - \frac{\psi(s) - \psi(r)}{s - r} \right) = \frac{(s - r)\psi(t) + (t - s)\psi(r) - (t - r)\psi(s)}{(t - s)(s - r)(t - r)}.$$

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These expressions are called the first divided difference of $\psi$ at $r, s$ and the second divided difference of $\psi$ at $r, s, t$, respectively (see e.g. [1, p. 371] and [6, p. 237]). Moreover, if we define the function $\Delta_s(\psi)$ by

$$ (\Delta_s \psi)(r) = \frac{\psi(s) - \psi(r)}{s - r}, \quad r \in I \setminus \{s\}, $$

then

$$ (\Delta_t \Delta_s)(\psi)(r) = [r, s, t; \psi], \quad r \in I \setminus \{s, t\}. $$

We consider the space $\text{Lip}_2(I, Z)$ of all functions $\psi : I \to Z$ such that

$$ (1) \quad M_\psi := \sup\{\|r, s, t; \psi\| : r < s < t, r, s, t \in I\} $$

is finite. In Section 2 we show that $\psi \in \text{Lip}_2(I, Z)$ if and only if $\psi$ is differentiable and $\psi'$ is Lipschitzian.

Let $h : I \times \mathbb{R} \to \mathbb{R}$. The mapping $N : \mathbb{R}^I \to \mathbb{R}^I$ defined by

$$ (N \varphi)(t) = h(t, \varphi(t)) $$

is called the composition (Nemytskii) operator determined by the generator $h$. In 1982 J. Matkowski proved that if a composition operator mapping the Banach space $\text{Lip}(I, \mathbb{R})$ of Lipschitz functions $\varphi : I \to \mathbb{R}$ into itself is globally Lipschitzian, then there exist functions $a, b \in \text{Lip}(I, \mathbb{R})$ such that $h(t, y) = a(t)y + b(t)$, $t \in I$, $y \in \mathbb{R}$ (cf. [3]). This result has then been extended to some other Banach function spaces by J. Matkowski and his collaborators (cf., e.g., [2], [4], [5]).

Let $(Y, \| \cdot \|)$ be a normed linear space and let $\mathcal{A}(I, Y)$ denote the space of all functions $\varphi : I \to Y$ of the form $t \mapsto ct + d$, where $c, d \in Y$. By $\mathcal{L}_M$ we denote the set of all functions $\psi \in \text{Lip}_2(I, Z)$ such that $M_\psi \leq M$, where $M$ is a positive number and $M_\psi$ is given by (1). In the next section we will prove that every continuous composition operator $N$ mapping $\mathcal{A}(I, Y)$ into $\mathcal{L}_M$ is generated by a function $h$ of the form $h(t, y) = a(y) + b(t)$, $t \in I$, $y \in Y$, where $a : Y \to Z$ is a continuous linear map and $b \in \text{Lip}_2(I, Z)$.

In the last two sections we will examine composition operators mapping $\mathcal{A}(I, Y)$ into $\text{Lip}_2(I, Z)$ and satisfying the Lipschitz and Hölder conditions.

2. On properties of functions in $\text{Lip}_2(I, Z)$. Let $\psi : I \to Z$ be Lipschitz and let $L(\psi)$ denote the smallest number $L$ such that

$$ \|\psi(t) - \psi(s)\| \leq L|t - s| \quad \text{for all } s, t \in I. $$

If, e.g., $I = [\alpha, \beta)$, then $\psi : I \to Z$ is differentiable at $a$ if the right-hand derivative exists at this point.

**Lemma 1.** If $Z$ is a Banach space and $\psi \in \text{Lip}_2(I, Z)$, then $\psi$ is differentiable, $\psi'$ is Lipschitz and $L(\psi') \leq 2M_\psi$. 


**Proof.** Fix \( t_0 \in I \) with \( t_0 < \sup I \) and take \( s, t \in I \) such that \( t_0 < s < t \). By the definition of \( M_\psi \) we have

\[
(2) \quad \left\| \frac{\psi(t) - \psi(s)}{t - s} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right\| \leq M_\psi(t - t_0).
\]

Since

\[
\frac{\psi(t) - \psi(s)}{t - s} - \frac{\psi(s) - \psi(t_0)}{s - t_0} = \frac{t - t_0}{t - s} \left[ \frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right],
\]

inequality (2) yields

\[
(3) \quad \left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(s) - \psi(t_0)}{s - t_0} \right\| \leq M_\psi(t - s).
\]

Thus the limit

\[
\lim_{t \to t_0^+} \frac{\psi(t) - \psi(t_0)}{t - t_0} = \psi'_+(t_0)
\]

exists and by (3),

\[
(4) \quad \left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \psi'_+(t_0) \right\| \leq M_\psi(t - t_0)
\]

for \( t_0 < t, t \in I \).

Next fix \( t_0 \in I \) with \( \inf I < t_0 \) and take \( r, s \in I \) such that \( r < s < t_0 \). In a similar manner to (3), it can be established that

\[
\left\| \frac{\psi(t_0) - \psi(s)}{t_0 - s} - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \leq M_\psi(s - r).
\]

Hence the left-hand derivative \( \psi'_-(t_0) \) exists and it satisfies the inequality

\[
(5) \quad \left\| \psi'_-(t_0) - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \leq M_\psi(t_0 - r)
\]

for all \( r \in I \) such that \( r < t_0 \). To show that

\[
(6) \quad \psi'_+(t_0) = \psi'_-(t_0)
\]

in the case \( \inf I < t_0 < \sup I \), we choose \( r, t \in I \) such that \( r < t_0 < t \). As in (2) we have

\[
(7) \quad \left\| \frac{\psi(t) - \psi(t_0)}{t - t_0} - \frac{\psi(t_0) - \psi(r)}{t_0 - r} \right\| \leq M_\psi(t - r).
\]

Combining (4), (5) and (7) we conclude that

\[
\left\| \psi'_+(t_0) - \psi'_-(t_0) \right\| \leq 2M_\psi(t - r),
\]

whence equality (6) follows.

Assume that \( r \in I, r \neq \sup I \). For \( s, t, u \in I \) such that \( r < s < t < u \) we have

\[
\left\| \frac{\psi(u) - \psi(t)}{u - t} - \frac{\psi(t) - \psi(s)}{t - s} \right\| \leq M_\psi(u - s)
\]
and 
\[
\left\| \frac{\psi(t) - \psi(s)}{t-s} - \frac{\psi(s) - \psi(r)}{s-r} \right\| \leq M_\psi(t-r).
\]
These inequalities yield 
\[
\left\| \frac{\psi(u) - \psi(t)}{u-t} - \frac{\psi(s) - \psi(r)}{s-r} \right\| \leq M_\psi(u-s+t-r).
\]
Therefore letting \( u \to t^+ \) and \( s \to r^+ \), we obtain 
\[
\| \psi'(t) - \psi'(r) \| \leq 2M_\psi(t-r).
\]
A similar argument may be used when \( r \in I, r \neq \inf I \).

**Lemma 2.** If \( \psi \) is differentiable in \( I \) and \( \psi' \) satisfies the Lipschitz condition, then \( \psi \in \text{Lip}_2(I, Z) \) and \( M_\psi \leq L(\psi') \).

**Proof.** Take \( u, v, w \in I \) such that \( u < v < w \). It is sufficient to show that 
\[
\| z \| \leq L(\psi')(w-u),
\]
where 
\[
z := \frac{\psi(w) - \psi(v)}{w-v} - \frac{\psi(v) - \psi(u)}{v-u}.
\]
We may assume that \( z \neq 0 \). Take a continuous linear functional \( p : Z \to \mathbb{R} \) with \( \| p \| = 1 \) such that \( p(z) = \| z \| \). The function \( p \circ \psi : I \to \mathbb{R} \) is differentiable and \( (p \circ \psi)'(t) = p(\psi'(t)), \ t \in I \). By the Lagrange mean-value theorem, for some \( \sigma \in (u, v) \) and \( \tau \in (v, w) \), we have 
\[
p\left( \frac{\psi(v) - \psi(u)}{v-u} \right) = \frac{p \circ \psi(v) - p \circ \psi(u)}{v-u} = p(\psi'(\sigma))
\]
and 
\[
p\left( \frac{\psi(w) - \psi(v)}{w-v} \right) = p(\psi'(\tau)).
\]
Therefore 
\[
\left\| \frac{\psi(w) - \psi(v)}{w-v} - \frac{\psi(v) - \psi(u)}{v-u} \right\| = \| z \| = p(z)
\]
\[
= p\left( \frac{\psi(w) - \psi(v)}{w-v} \right) - p\left( \frac{\psi(v) - \psi(u)}{v-u} \right) = p(\psi'(\tau)) - p(\psi'(\sigma))
\]
\[
= p(\psi'(\tau) - \psi'(\sigma)) \leq \| p \| \| \psi'(\tau) - \psi'(\sigma) \|
\]
\[
\leq L(\psi')(\tau - \sigma) \leq L(\psi')(w-u).
\]

**Theorem 1.** Assume that \( I \subset \mathbb{R} \) is an interval and \( Z \) is a Banach space. Then \( \psi \in \text{Lip}_2(I, Z) \) if and only if \( \psi \) is differentiable and its derivative \( \psi' \) satisfies the Lipschitz condition in \( I \).

Theorem 1 is a consequence of Lemmas 1 and 2.
3. Continuous composition operator. We shall assume that $I$ is an interval containing 0. We introduce the norm $\| \cdot \|_2$ in Lip$_2(I, Z)$ putting
\[ \|\psi\|_2 = \|\psi(0)\| + \|\psi'(0)\| + M_\psi, \]
where $M_\psi$ is given by (1). By Lemma 1, $\| \cdot \|_2$ is a norm. Moreover, Lip$_2(I, Z)$ is a Banach space.

The inequality $L(\psi') \leq 2M_\psi$ and the Lagrange mean-value theorem lead to the next lemma.

**Lemma 3.** If $\psi_n \to \psi$ in Lip$_2(I, Z)$, then $\psi_n(t) \to \psi(t)$ in $Z$ for every $t \in I$.

It is easily seen that $\| \varphi \|_2 = \| c \| + \| d \|$ if $\varphi \in A(I, Y)$ ($\subset$ Lip$_2(I, Y)$) is of the form $\varphi(t) = ct + d$.

Every function $h : I \times Y \to Z$ generates the Nemytskii operator $N$ defined by
\[ (N\varphi)(t) = h(t, \varphi(t)), \quad t \in I, \varphi \in A(I, Y). \]

**Lemma 4.** Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$. Assume that $(Y, \| \cdot \|)$ is a normed linear space and $(Z, \| \cdot \|)$ is a Banach space. If the composition operator $N : A(I, Y) \to$ Lip$_2(I, Z)$ is continuous, then its generator $h$ is continuous with respect to each variable.

**Proof.** Take an arbitrary $y \in Y$ and define an affine function $\varphi$ assuming $\varphi(t) = y$, $t \in I$. Since $h(\cdot, y) = N\varphi \in$ Lip$_2(I, Z)$, $h$ is continuous with respect to the first variable. The continuity of $h$ with respect to the second variable follows from Lemma 3.

Recall that $\mathcal{L}_M$ denotes the set of all functions $\psi \in$ Lip$_2(I, Z)$ such that $M_\psi \leq M$, where $M_\psi$ is given by (1) and $M$ is a fixed constant.

**Theorem 2.** Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$. Assume that $(Y, \| \cdot \|)$ is a normed linear space and $(Z, \| \cdot \|)$ is a Banach space. If the composition operator $N$ generated by $h : I \times Y \to Z$ maps $A(I, Y)$ into $\mathcal{L}_M$, then there exists an additive mapping $a : Y \to Z$ and a mapping $b \in \mathcal{L}_M$ such that
\[ h(t, y) = a(y) + b(t), \quad t \in I, y \in Y. \]

Moreover, if the operator $N$ is continuous, then $a$ is a continuous linear mapping.

**Proof.** Take $r, t \in I$ with $r < t$ and $y, \overline{y} \in Y$. Define an affine function by setting
\[ \varphi(u) = y + \frac{\overline{y} - y}{t - r}(u - r), \quad u \in I. \]

Since $N\varphi \in \mathcal{L}_M$, we obtain
\[ \|(t - s)(N\varphi)(r) + (s - r)(N\varphi)(t) - (t - r)(N\varphi)(s)\| \leq M(t - r)(t - s)(s - r) \]

(continues on the next page)
for all \( s \in (r, t) \). Choosing \( s = (1/2)(r + t) \) and taking into account the relations
\[
\varphi(r) = y, \quad \varphi(t) = \bar{y}, \quad \varphi\left(\frac{r + t}{2}\right) = \frac{y + \bar{y}}{2}
\]
we get
\[
(9) \quad \left\| h(r, y) + h(t, \bar{y}) - 2h\left(\frac{r + t}{2}, \frac{y + \bar{y}}{2}\right) \right\| \leq \frac{1}{2} M(t - r)^2.
\]
Letting \( r \to t \) and making use of the continuity of \( h(\cdot, y) \) we deduce that
\[
h(t, y) + h(t, \bar{y}) - 2h\left(t, \frac{y + \bar{y}}{2}\right) = 0, \quad t \in I, \ y, \bar{y} \in Y,
\]
so \( h(t, \cdot) \) satisfies the Jensen functional equation in a normed linear space \( Y \). Hence there exist functions \( a : I \times Y \to Z \) and \( b : I \to Z \) such that
\[
(10) \quad h(t, y) = a(t, y) + b(t)
\]
and \( a(t, \cdot) : Y \to Z \) is an additive mapping (cf., e.g., [1, Theorem 1, p. 315]). We conclude from (10) that \( b = h(\cdot, 0) \), hence \( b \in L_M \) and finally \( a(\cdot, y) \in L_M \) for each \( y \in Y \).

Combining (10) and (9) we get
\[
\left\| a(r, y) + b(r) + a(t, \bar{y}) + b(t) - 2a\left(\frac{r + t}{2}, \frac{y + \bar{y}}{2}\right) - 2b\left(\frac{r + t}{2}\right) \right\| \leq \frac{1}{2} M(t - r)^2
\]
for every \( r, t \in I \) and \( y, \bar{y} \in Y \). Take \( n y \) and \( n \bar{y} \), \( n \in \mathbb{N} \), instead of \( y \) and \( \bar{y} \), respectively. Next, since \( a(t, ny) = na(t, y) \), dividing both sides of the resulting inequality by \( n \) and letting \( n \to \infty \), we conclude that
\[
a(r, y) + a(t, \bar{y}) = 2a\left(\frac{r + t}{2}, \frac{y + \bar{y}}{2}\right)
\]
for all \( r, t \in I \) and \( y, \bar{y} \in Y \), which means that the function \( (r, y) \mapsto a(r, y) \) is Jensen. Since \( a(0, 0) = 0 \), the function \( a \) is additive with respect to the pair of variables \((t, y) \in I \times Y \). We observe that
\[
a(t, y) = a((t, 0) + (0, y)) = a(t, 0) + a(0, y) = a(0, y),
\]
since \( a(t, 0) = 0 \) for all \( t \in I \). Thus \( a(y) := a(0, y) \) does not depend on the first variable \( t \) and
\[
(11) \quad h(t, y) = a(y) + b(t), \quad t \in I, \ y \in Y.
\]
This finishes the proof of the first part of Theorem 2.

It remains to prove that \( a \) is continuous if so is \( N \). But this follows from Lemma 4.

An easy verification shows that the inverse result is valid.
Theorem 3. Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$, $Y$ be a normed linear space and $Z$ be a Banach space. If $a : Y \to Z$ is a continuous linear map and $b \in \text{Lip}_2(I, Z)$, then the composition operator $N$ generated by $h(t, y) = a(y) + b(t)$, $t \in I$, $y \in Y$, is continuous and maps the space $\mathcal{A}(I, Y)$ into $\mathcal{L}M$, where $M = M_b$.

4. Lipschitzian composition operators. The generator of a Lipschitzian composition operator has a form slightly different from that in Theorem 2.

Theorem 4. Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$, $Y$ be a normed linear space and $Z$ be a Banach space. If the composition operator $N$ generated by $h : I \times Y \to Z$ maps $\mathcal{A}(I, Y)$ into $\text{Lip}_2(I, Z)$ and satisfies the Lipschitz condition, i.e., there exists a positive constant $L$ such that
\begin{equation}
\| N\varphi - N\psi \|_2 \leq L\| \varphi - \psi \|_2, \quad \varphi, \psi \in \mathcal{A}(I, Y),
\end{equation}
then there exist functions $a : I \times Y \to Z$ and $b : I \to Z$ such that for each $y \in Y$ and $t \in I$, $a(\cdot, y), b \in \text{Lip}_2(I, Z)$ and $a(t, \cdot)$ is a continuous linear map of $Y$ into $Z$ and
\[ h(t, y) = a(t, y) + b(t), \quad t \in I, \ y \in Y. \]

In particular, $N$ is affine.

Proof. We mimic the first part of the proof of Theorem 2. By Lemma 4 the generator $h$ of $N$ is continuous with respect to each variable. Making use of (12) and the definition of the norm $\| \cdot \|_2$ we infer that
\begin{equation}
\left\| \left[ r, s, t; h(\cdot, \varphi(\cdot)) - h(\cdot, \psi(\cdot)) \right] \right\| \leq L\| \varphi - \psi \|_2
\end{equation}
for all $r, s, t \in I, r < s < t$. Take arbitrary $r, t \in I$ with $r < t$ and define the functions
\[ \varphi(u) = y + \frac{\overline{y} - y}{t - r}(u - r), \quad \psi(u) = 0, \quad u \in I. \]

Of course,
\[ \varphi(r) = y, \quad \varphi(t) = \overline{y}, \quad \varphi\left( \frac{r + t}{2} \right) = \frac{y + \overline{y}}{2} \]
and
\[ \varphi(0) = \frac{ty - r\overline{y}}{t - r}, \quad \varphi'(0) = \frac{y - \overline{y}}{t - r}. \]

Setting $s = (r + t)/2$ in (13) we obtain
\begin{align*}
\left\| h(t, \overline{y}) - h(t, 0) - 2h\left( \frac{r + t}{2}, \frac{y + \overline{y}}{2} \right) + 2h\left( \frac{r + t}{2}, 0 \right) + h(r, y) - h(r, 0) \right\| \\
&\leq \frac{1}{2}L(t - r)(\| ty - r\overline{y} \| + \| y - \overline{y} \|).
\end{align*}
Letting $t$ tend to $r$ and making use of the continuity of $h$ with respect to the first variable we hence get

\[ h(r, \overline{y}) + h(r, y) = 2h \left( r, \frac{y + \overline{y}}{2} \right), \]

which shows that, for every fixed $r \in I$, the function $h(r, \cdot)$ satisfies the Jensen functional equation in the normed linear space $Y$. As in the proof of Theorem 2, there exist $a : I \times Y \to Z$ and $b : I \to Z$ such that

\[ h(r, y) = a(r, y) + b(r), \quad r \in I, \ y \in Y, \]

where $a(r, \cdot)$ is an additive map for every $r \in I$. Now the remainder is clear. ■

To obtain a converse to the last theorem we will require that $I$ is a compact interval such that $0 \in I$.

As an application of the uniform boundedness principle one obtains the following lemma.

**Lemma 5.** Let $I = [0, 1]$ and let $Y, Z$ be Banach spaces. If $a : I \times Y \to Z$ is such that $a(\cdot, y) \in \text{Lip}_2(I, Z)$ for $y \in Y$ and each $a(t, \cdot)$ ($t \in I$) is linear and continuous, then $a'_i(t, \cdot)$ is also linear and continuous.

**Theorem 5.** Let $I = [0, 1]$ and let $Y, Z$ be Banach spaces. If $a : I \times Y \to Z$ and $b : I \to Z$ are such that $a(\cdot, y), b \in \text{Lip}_2(I, Z)$ and $a(t, \cdot)$ is a continuous linear map of $Y$ into $Z$ and

\[ h(t, y) = a(t, y) + b(t), \quad (t, y) \in I \times Y, \]

then the operator $N$, $(N\varphi)(t) = h(t, \varphi(t))$, maps $A(I, Y)$ into $\text{Lip}_2(I, Z)$ and

\[ \|N\varphi_1 - N\varphi_2\|_2 \leq L\|\varphi_1 - \varphi_2\|_2 \quad \text{for some } L > 0. \]

**Proof.** Without loss of generality we may assume that $b \equiv 0$. In that case $N$ is linear. Take $\varphi(t) = ct + d$, $t \in I = [0, 1]$, where $c, d \in Y$. We have

\[ (N\varphi)(t) = a(t, \varphi(t)) = a(t, ct + d) = ta(t, c) + a(t, d). \]

Of course, the function $N\varphi$ is differentiable and

\[ (N\varphi)'(t) = a(t, c) + ta'_i(t, c) + a'_r(t, d), \quad t \in I. \]

The function $a'_i(\cdot, d)$ is Lipschitz (cf. Lemma 1). Since the product of two bounded Lipschitz functions is Lipschitz as well and $a(\cdot, c)$ has a bounded derivative in $I$, we see that $N(\varphi) \in \text{Lip}_2(I, Z)$ (cf. Lemma 2).

Further, from the compactness of $I$, the continuity of $a(\cdot, y)$ and $a'_i(\cdot, y)$ for each $y \in Y$ and the uniform boundedness principle we conclude that $\|a(t, \cdot)\|, \|a'_i(t, \cdot)\| \leq K$ for all $t \in I$ and some constant $K > 0$. Hence

\[ \frac{\|a(t, y) - a(r, y)\|}{|t - r|} \leq \sup_{s \in I} \|a'_i(s, y)\| \leq K\|y\| \]

which establishes the result.
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for all \( y \in Y \) and \( 0 \leq r < t \leq 1 \). For \( y \in Y \) and \( 0 \leq r < s < t \leq 1 \) we have \( \|[r, s, t; a(\cdot, y)]\| \leq M_{a(\cdot, y)} < \infty \). Again by the uniform boundedness principle we can find a constant \( L > 0 \) such that

\[
\|[r, s, t; a(\cdot, y)]\| \leq L \|y\|
\]

for all \( y \in Y \) and \( 0 \leq r < s < t \leq 1 \).

Let \( \varphi \in \mathcal{A}(I, Y) \), \( \varphi(t) = ct + d \), and \( \psi(t) = (N\varphi)(t), \ t \in I \). Since

\[
[r, s, t; \psi] = t[r, s, t; a(\cdot, c)] + \frac{a(r, c) - a(s, c)}{s - r} + [r, s, t; a(\cdot, d)]
\]

for all \( 0 \leq r < s < t \leq 1 \), by (15) and (16) we have

\[
M_\psi \leq K \|c\| + L(\|c\| + \|d\|).
\]

Consequently, \( \|N\varphi\|_2 \leq (2K + L)\|\varphi\|_2 \).

5. Hölder composition operators. The following result deals with composition operators mapping \( \mathcal{A}(I, Y) \) into \( \text{Lip}_2(I, Z) \) satisfying the Hölder condition.

**Theorem 6.** Let \( I \subset \mathbb{R} \) be an interval such that \( 0 \in I, Y \) be a normed linear space and \( Z \) be a Banach space. The composition operator \( N \) generated by \( h : I \times Y \to Z \) mapping \( \mathcal{A}(I, Y) \) into \( \text{Lip}_2(I, Z) \) satisfies the Hölder condition, i.e., there exist positive constants \( L \) and \( \alpha < 1 \) for which

\[
\|N\varphi - N\psi\|_2 \leq L\|\varphi - \psi\|^{\alpha}, \quad \varphi, \psi \in \mathcal{A}(I, Y),
\]

if and only if \( N \) is a constant map, that is, there exists \( b \in \text{Lip}_2(I, Z) \) such that

\[
h(t, y) = b(t), \quad t \in I, \ y \in Y.
\]

**Proof.** The “if” part is clear. We will prove the “only if” part. As in the proof of Theorem 4, inequality (17) gives

\[
\left\| h(t, y) - h(t, 0) - 2h\left(\frac{r + t}{2}, \frac{y + \overline{y}}{2}\right) \right. \\
\left. + 2h\left(\frac{r + t}{2}, 0\right) + h(r, y) - h(r, 0) \right\|
\]

\[
\leq \frac{1}{2}L(t - r)^{1 - \alpha}(\|ty - r\overline{y}\| + \|y - \overline{y}\|)^\alpha
\]

for all \( r, t \in I \) with \( r < t \) and all \( y, \overline{y} \in Y \). Analysis similar to that in the proof of Theorem 4 shows that

\[
h(r, y) = a(r, y) + b(r), \quad r \in I, \ y \in Y,
\]

where \( a(t, \cdot) \) is a continuous linear map and \( a(\cdot, y), b \in \text{Lip}_2(I, Z) \). Combining (19) and (18) we obtain
\[
\left\| a(t, \overline{y}) - 2a\left(\frac{r + t}{2}, \frac{y + \overline{y}}{2}\right) + a(r, y) \right\| \leq \frac{1}{2}L(t-r)^{2-\alpha}(\|ty-r\overline{y}\| + \|y-\overline{y}\|)^{\alpha}.
\]

Now replacing \( y \) and \( \overline{y} \) by \( n y \) and \( n\overline{y} \), \( n \in \mathbb{N} \), respectively, then applying the additivity of \( a(t, \cdot) \), and finally dividing by \( n \) we deduce that

\[
\left\| a(t, \overline{y}) - 2a\left(\frac{r + t}{2}, \frac{y + \overline{y}}{2}\right) + a(r, y) \right\| \leq \frac{1}{2}L(t-r)^{2-\alpha}n^{\alpha-1}(\|ty-r\overline{y}\| + \|y-\overline{y}\|)^{\alpha}.
\]

Letting \( n \to \infty \) we can assert that

\[
a(r, y) + a(t, \overline{y}) = 2a\left(\frac{r + t}{2}, \frac{y + \overline{y}}{2}\right), \quad r, t \in I, \ y, \overline{y} \in Y,
\]

which means that the mapping \((t, y) \mapsto a(t, y)\) satisfies the Jensen functional equation in \( I \times Y \). As in Theorem 2, we have

\[
h(t, y) = a(y) + b(t), \quad t \in I, \ y \in Y
\]

with \( a(y) = a(0, y) \). Since \( a \) is linear and satisfies the Hölder inequality with \( \alpha < 1 \), it follows that \( a \equiv 0 \). \( \blacksquare \)

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