ANNALES
POLONICI MATHEMATICI
98.3 (2010)

On the value distribution of differential polynomials of meromorphic functions

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Abstract. Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$, let $k \in \mathbb{N}$ and $\varphi = Re^P$, where $R \not\equiv 0$ is a rational function and P is a polynomial, and let $a_0, a_1, \ldots, a_{k-1}$ be holomorphic functions on $\mathbb{C}$. If all zeros of f have multiplicity at least k except possibly finitely many, and $f = 0 \Leftrightarrow f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f = 0$, then $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f - \varphi$ has infinitely many zeros.

1. Introduction. Let f and g be meromorphic functions on $\mathbb{C}$, and let a, b be two complex numbers. If $g = b$ whenever $f = a$, we write $f = a \Rightarrow g = b$. If $f = a \Rightarrow g = b$ and $g = b \Rightarrow f = a$, we write $f = a \Leftrightarrow g = b$. The *order* $\rho(f)$ (see [8, 14]) of the meromorphic function f is defined as

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}.$$

In 1959, Hayman [7] proved the following result, which is known as Hayman's Alternative.

Theorem A. *Let f be a transcendental meromorphic function on $\mathbb{C}$. Then either f assumes every finite value infinitely often, or every derivative of f assumes every finite nonzero value infinitely often.*

This result has undergone various extensions (see [1, 2, 4, 5, 9, 11, 12, 13], etc.). In 2001, Fang [5] proved the following result for functions of infinite order.

Theorem B. *Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$. If $f = 0 \Leftrightarrow f' = 0$, then $f' - b(z)$ has infinitely many zeros for any $b(z) \in S$, where $S = \{az^n : a \in \mathbb{C} \setminus \{0\}, n = 0, 1, 2, \ldots\}$.*

In 2005, the first author [12] proved

2010 *Mathematics Subject Classification*: Primary 30D35.
Key words and phrases: meromorphic function, Zalcman's lemma, infinite order, zero.

DOI: 10.4064/ap98-3-7

Theorem C. *Let f be a transcendental meromorphic function on $\mathbb{C}$, and let R ($\not\equiv 0$) be a rational function and $k \in \mathbb{N}$. Suppose that all zeros of f have multiplicity at least k except possibly finitely many, and $f = 0 \Leftrightarrow f^{(k)} = 0$. Then $f^{(k)} - R$ has infinitely many zeros.*

A natural problem arises: *Can the rational function R in Theorem C be replaced by a more general meromorphic function?* In this paper, for the case of f with infinite order, we prove the following result.

Theorem 1. *Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$, let $k \in \mathbb{N}$ and $\varphi = Re^P$, where $R \not\equiv 0$ is a rational function and P is a polynomial, and let $a_0, a_1, \dots, a_{k-1}$ be holomorphic functions on $\mathbb{C}$. Set*

$$L[f] := f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f. \tag{$*$}$$

Suppose that all zeros of f have multiplicity at least k except possibly finitely many, and $f = 0 \Leftrightarrow L[f] = 0$. Then $L[f] - \varphi$ has infinitely many zeros.

Remark 1. Obviously, the assumption "all zeros of f have multiplicity at least k, and $f = 0 \Leftrightarrow L[f] = 0$" is equivalent to "all zeros of f have multiplicity at least $k + 1$, and $f = 0 \Leftarrow L[f] = 0$".

Theorem 2. *Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$, let $k \in \mathbb{N}$ and $\varphi = Re^P$, where $R \not\equiv 0$ is a rational function and P is a polynomial, and let $a_0, a_1, \dots, a_{k-1}$ be holomorphic functions on $\mathbb{C}$. If f has only finitely many zeros, then $L[f] - \varphi$ has infinitely many zeros, where $L[f]$ is defined in ($*$).*

From Theorems 1 and 2, we get

Corollary 1. *Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$, let $k \in \mathbb{N}$ and $\varphi = Re^P$, where $R \not\equiv 0$ is a rational function and P is a polynomial. Suppose that all zeros of f have multiplicity at least k except possibly finitely many, and $f = 0 \Leftrightarrow f^{(k)} = 0$. Then $f^{(k)} - \varphi$ has infinitely many zeros.*

Corollary 2. *Let f be a transcendental meromorphic function of infinite order on $\mathbb{C}$, let $k \in \mathbb{N}$ and $\varphi = Re^P$, where $R \not\equiv 0$ is a rational function and P is a polynomial. If f has only finitely many zeros, then $f^{(k)} - \varphi$ has infinitely many zeros.*

Remark 2. As Hayman's inequality [7, 8] for small functions is still unknown, Theorem 2 and Corollary 2 are not direct consequences of Hayman's inequality.

2. Some lemmas. The following three lemmas are due to Liu, Nevo and Pang [9].

Lemma 1. *Let k be a positive integer and let $\{f_n\}$ be a family of functions meromorphic on $\Delta = \{z : |z| < 1\}$, all of whose zeros have multiplicity at least $k+1$. If $a_n \to a$, $|a| < 1$, and $f_n^{\#}(a_n) \to \infty$, then there exist a subsequence of $\{f_n\}$ (which we still write as $\{f_n\}$), a sequence of points $z_n \in D$, $z_n \to z_0$, $|z_0| < 1$, and a sequence of positive numbers $\rho_n \to 0$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, such that $g^{\#}(\zeta) \le g^{\#}(0) = k+1$, and $\rho_n \le M/\sqrt[k+1]{f_n^{\#}(a_n)}$, where M is independent of n.

Here, as usual, $g^{\#}(\zeta) = |g'(\zeta)|/(1 + |g(\zeta)|^2)$ is the spherical derivative of g. The above lemma is in fact another version of Zalcman's Lemma (see [3, 10, 11, 15, 16], etc.). The main difference here is the estimate of ρ_n in the vicinity of some point of nonnormality. Moreover, by using the Ahlfors–Shimizu characteristic function, we can deduce (as in [10] or [11]) that the limit function g in Lemma 1 has order at most 2 since $g^{\#}(\zeta) \le g^{\#}(0) = k+1$.

Lemma 2. *Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^{\#}(z_n) > |z_n|^N$ if n is sufficiently large.*

Lemma 3. *Let $R(z) \not\equiv 0$ be a rational function. Then there exists $k > 0$ such that $|zR'(z)| \le k|R(z)|$ for large enough z.*

The next lemma is due to Fang [5] and Fang–Zalcman [6].

Lemma 4. *Let f be a meromorphic function of finite order on $\mathbb{C}$, b a nonzero complex number, and k a positive integer. If all zeros of f have multiplicity at least k, $f = 0 \Leftrightarrow f^{(k)} = 0$, and $f^{(k)} \ne b$, then f is a constant.*

3. Proofs of theorems

Proof of Theorem 1. Suppose that $L[f](z) - \varphi(z)$ has finitely many zeros. Then, for large z, we have

$$\frac{L[f](z)}{\varphi(z)} \ne 1. \tag{1}$$

Set

$$F(z) = f(z)/\varphi(z). \tag{2}$$

Obviously, the order of F is equal to that of f, and so F is of infinite order. By Lemma 2, there exist points $z_n \to \infty$ such that for every $N > 0$ and sufficiently large n we have $F^{\#}(z_n) > |z_n|^N$. Noting that $\varphi(z)$ has only finitely many zeros and poles, we find that all zeros of $F(z+z_n)$ (for large n) in Δ have multiplicity at least $k+1$.

Then, by Lemma 1, there exist a subsequence of $\{F(z+z_n)\}$ (without loss of generality, we may still write it as $F(z+z_n)$), a sequence of points $z_n' \to z_0$ and $|z_0|<1$, and a sequence of positive numbers $\rho_n \to 0$ such that $\rho_n \le M/\sqrt[k+1]{F^{\#}(z_n)}$ and

$$g_n(\zeta)=\frac{F(z_n+z_n'+\rho_n\zeta)}{\rho_n^k}\to g(\zeta) \tag{3}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, and M is independent of n. Moreover, g is of order at most 2. By Hurwitz's theorem, all zeros of g have multiplicity at least $k+1$.

By simple calculation, for $0\le i\le k$, we have

$$F^{(i)}(z)=\frac{f^{(i)}(z)}{\varphi(z)}-\sum_{j=1}^{i}\binom{i}{j}F^{(i-j)}(z)\frac{\varphi^{(j)}(z)}{\varphi(z)}. \tag{4}$$

Obviously, $\varphi^{(j)}(z)=\sum_{m=0}^{j}\binom{j}{m}R^{(m)}(z)(e^{P(z)})^{(j-m)}$, so that $\varphi^{(j)}(z)/\varphi(z)$ is a polynomial of $R^{(m)}(z)/R(z)$ and $P^{(m)}(z)$ $(m=1,\dots,j)$. Now we rewrite (4) as

$$F^{(i)}(z)=\frac{f^{(i)}(z)}{\varphi(z)}-\sum_{j=1}^{i}Q_j(z)F^{(i-j)}(z), \tag{5}$$

where $Q_j(z)$ is a polynomial of $R^{(m)}(z)/R(z)$ and $P^{(m)}(z)$ $(m=1,\dots,j)$ for $j=1,\dots,i$.

Thus, from (3) and (5), we have

$$\begin{aligned}\rho_n^{k-i}g_n^{(i)}(\zeta)&=F^{(i)}(z_n+z_n'+\rho_n\zeta)\\&=\frac{f^{(i)}(z_n+z_n'+\rho_n\zeta)}{\varphi(z_n+z_n'+\rho_n\zeta)}-\sum_{j=1}^{i}Q_j(z_n+z_n'+\rho_n\zeta)F^{(i-j)}(z_n+z_n'+\rho_n\zeta)\\&=\frac{f^{(i)}(z_n+z_n'+\rho_n\zeta)}{\varphi(z_n+z_n'+\rho_n\zeta)}-\sum_{j=1}^{i}\rho_n^jQ_j(z_n+z_n'+\rho_n\zeta)\,\frac{F^{(i-j)}(z_n+z_n'+\rho_n\zeta)}{\rho_n^j}\end{aligned}$$

for $i=0,1,\dots,k$.

Now we show that on each compact subset of $\mathbb{C}$,

$$\lim_{n\to\infty}\rho_n^jQ_j(z_n+z_n'+\rho_n\zeta)=0 \qquad \text{for } 1\le j\le i\le k. \tag{6}$$

First, by Lemma 3, we get

$$\lim_{n\to\infty}\frac{R^{(m)}(z_n+z_n'+\rho_n\zeta)}{R(z_n+z_n'+\rho_n\zeta)}=0 \qquad (1\le m\le j). \tag{7}$$

On the other hand, for large n, we have

$$P^{(m)}(z_n + z_n' + \rho_n\zeta) = O(z_n^p), \tag{8}$$

where $p = \max\{\deg P - m, 0\}$ and $1 \le m \le j$. Noting that, for sufficiently large n and every $N > 0$,

$$\rho_n \le \frac{M}{\sqrt[k+1]{F^{\#}(z_n)}} < M|z_n|^{-N/(k+1)},$$

for any given $\alpha > 0$ we have

$$\rho_n^\alpha |z_n|^p < M^\alpha |z_n|^{p-\alpha N/(k+1)} \to 0,$$

since we can choose N so large that $p - \alpha N/(k+1) < 0$. This and (8) imply that, for any given $\alpha > 0$,

$$\lim_{n\to\infty} \rho_n^\alpha P^{(m)}(z_n + z_n' + \rho_n\zeta) = 0 \quad \text{for } 1 \le m \le j, \tag{9}$$

Recalling that $Q_j(z)$ is a polynomial of $R^{(m)}(z)/R(z)$ and $P^{(m)}(z)$ ($m = 1, \ldots, j$), from (7) and (9) we obtain (6).

We note that $F^{(i-j)}(z_n + z_n' + \rho_n\zeta)/\rho_n^j$ is locally bounded on $\mathbb{C}$ minus the set of poles of $g(\zeta)$ since $F(z_n + z_n' + \rho_n\zeta)/\rho_n^k \to g(\zeta)$. Then, on every compact subset of $\mathbb{C}$ which contains no poles of $g(\zeta)$, we have

$$\frac{f^{(k)}(z_n + z_n' + \rho_n\zeta)}{\varphi(z_n + z_n' + \rho_n\zeta)} \to g^{(k)}(\zeta),$$

and

$$\frac{f^{(i)}(z_n + z_n' + \rho_n\zeta)}{\varphi(z_n + z_n' + \rho_n\zeta)} \to 0,$$

for $i = 0, 1, \ldots, k-1$, and thus

$$\frac{L[f](z_n + z_n' + \rho_n\zeta)}{\varphi(z_n + z_n' + \rho_n\zeta)} \to g^{(k)}(\zeta), \tag{10}$$

since $a_0, \ldots, a_{k-1}$ are holomorphic.

We claim

(i) $g(\zeta) = 0 \Leftrightarrow g^{(k)}(\zeta) = 0$;
(ii) $g^{(k)} \ne 1$ on $\mathbb{C}$.

Obviously, $g(\zeta) = 0 \Rightarrow g^{(k)}(\zeta) = 0$. Now suppose $g^{(k)}(\zeta_0) = 0$. Since all zeros of $g(\zeta)$ have multiplicity at least $k+1$, we know that $g^{(k)}(\zeta) \not\equiv 0$. Hurwitz's theorem implies that there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$L[f](z_n + z_n' + \rho_n\zeta_n) = 0.$$

It follows that $f(z_n + z_n' + \rho_n\zeta_n) = 0$. Hence $g(\zeta_0) = \lim_{n\to\infty} g_n(\zeta_n) = 0$. So $g^{(k)}(\zeta) = 0 \Rightarrow g(\zeta) = 0$. This proves (i).

Next we prove (ii). From (1) and (10), Hurwitz's theorem shows that on $\mathbb{C}$ minus the poles of g, the derivative $g^{(k)}$ is either identically 1, or never equal to 1. Clearly, the same alternative also holds on the whole $\mathbb{C}$. If $g^{(k)}(\zeta) \equiv 1$, then g is a polynomial of degree k. But this contradicts the fact all zeros of g have multiplicity at least $k+1$. So we get (ii).

Thus by Lemma 4, g must be a constant, contradiction. This completes the proof of Theorem 1. ∎

Proof of Theorem 2. Since f has only finitely many zeros, by applying Hurwitz's theorem, we deduce from (3) that $g \neq 0$. Then, by using the same argument as in the proof of Theorem 1, we can prove Theorem 2. Here we omit the details. ∎

Acknowledgments. We wish to thank the referee for his/her valuable comments and suggestions.

The first author was supported by NSFC (Grant No. 10871094).

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Received 17.9.2009
and in final form 12.11.2009 (2081)