3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds

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Abstract. We study 3-submersions from a QR-hypersurface of a quaternionic Kähler manifold onto an almost quaternionic hermitian manifold. We also prove the non-existence of quaternionic submersions between quaternionic Kähler manifolds which are not locally hyper-Kähler.

1. Introduction. In [WTS] B. Watson introduced the notion of 3-submersion, as a Riemannian submersion from an almost contact metric manifold onto an almost quaternionic manifold, which commutes with the structure tensors of type (1, 1). In [IMV1] and [IMV2], this concept has been extended in quaternionic setting. In this paper we study 3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds, we give an example and obtain some obstructions to the existence of quaternionic submersions.

The study of QR-submanifolds of a quaternionic Kähler manifold was initiated by A. Bejancu [BJC]. Among all submanifolds of a quaternionic Kähler manifold, QR-submanifolds have been intensively studied by several authors [AG, BEJ, BF, GS, KP, KPK, MNG, SHN1, SHN2]. In Section 2 we recall the definitions and basic properties of quaternionic manifolds and QR-submanifolds of a quaternionic Kähler manifold.

On the other hand, R. Güneş, B. Şahin and S. Keleş [GS] have shown that a QR-submanifold admits an almost contact 3-structure under some conditions. In Section 3 we see that on an orientable hypersurface of a quaternionic Kähler manifold there exists a natural almost contact metric 3-structure. This result will allow us to define the concept of QR 3-submersion. In Section 4 we obtain some properties for this kind of submersions and give an example. In the last section we prove the non-existence of quaternionic submersions between quaternionic Kähler non-locally hyper-Kähler manifolds.

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2. Preliminaries. Let $M$ be a differentiable manifold of dimension $n$ and assume that there is a rank 3-subbundle $\sigma$ of $\text{End}(TM)$ which is locally spanned by an almost hypercomplex structure, i.e. a triple $\{J_1, J_2, J_3\}$ of almost complex structures satisfying the quaternionic identities:

$$\begin{align*}
J_1^2 = -\text{Id}, & \quad \forall \alpha \in \{1, 2, 3\}, \\
J_1J_2 = -J_2J_1 = J_3.
\end{align*}$$

Then the bundle $\sigma$ is called an almost quaternionic structure on $M$ and $\{J_1, J_2, J_3\}$ is called a canonical local basis of $\sigma$. Moreover, $(M, g)$ is then said to be an almost quaternionic manifold. It is easy to see that any almost quaternionic manifold is of dimension $n = 4m$.

A Riemannian metric $g$ is adapted to the quaternionic structure $\sigma$ if

$$g(J_\alpha X, J_\alpha Y) = g(X, Y), \quad \forall \alpha \in \{1, 2, 3\},$$

for all vector fields $X, Y$ on $M$ and any local basis $\{J_1, J_2, J_3\}$ of $\sigma$. Moreover, $(M, \sigma, g)$ is then said to be a quaternionic Kähler manifold. Equivalently, there exists locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that

$$\begin{align*}
\nabla_X J_1 &= \omega_3(X)J_2 - \omega_2(X)J_3, \\
\nabla_X J_2 &= -\omega_3(X)J_1 + \omega_1(X)J_3, \\
\nabla_X J_3 &= \omega_2(X)J_1 - \omega_1(X)J_2,
\end{align*}$$

for any vector field $X$ on $M$. In particular, if $\omega_1 = \omega_2 = \omega_3 = 0$, then $(M, \sigma, g)$ is said to be a locally hyper-Kähler manifold.

We remark that any quaternionic Kähler manifold $M$ is an Einstein space, provided that $\dim M > 4$. Moreover, $M$ is irreducible (if $\text{Ric} \neq 0$) or locally hyper-Kähler manifold (if $\text{Ric} = 0$) (see [AL, BES, ISH, SLM]).

Let $(M, \sigma, g)$ be an almost quaternionic hermitian manifold. If $X \in T_pM$, $p \in M$, then the 4-plane $Q(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a quaternionic 4-plane. A 2-plane in $T_pM$ spanned by $\{X, Y\}$ is called half-quaternionic if $Q(X) = Q(Y)$.

The sectional curvature for a half-quaternionic 2-plane is called quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say $c$. It is well-known that a quaternionic Kähler manifold $(M, \sigma, g)$ is a quaternionic space form (denoted $M(c)$) if and only if its curvature tensor is

$$\begin{align*}
R(X, Y)Z &= \frac{c}{4}\left\{ g(Z, Y)X - g(X, Z)Y + \sum_{\alpha=1}^{3}[g(Z, J_\alpha Y)J_\alpha X \\
& \quad \quad - g(Z, J_\alpha X)J_\alpha Y + 2g(X, J_\alpha Y)J_\alpha Z] \right\}
\end{align*}$$

for all vector fields $X, Y, Z$ on $M$ and any local basis $\{J_1, J_2, J_3\}$ of $\sigma$. 

Let \((\overline{M}, \sigma, \overline{g})\) be a quaternionic Kähler manifold and let \(M\) be a real submanifold of \(\overline{M}\). Then \(M\) is said to be a \(QR\)-submanifold if there exists a vector subbundle \(D\) of the normal bundle \(TM\) such that:

1. \(J_\alpha(D_p) = D_p\) for all \(p \in M\) and \(\alpha \in \{1, 2, 3\}\);
2. \(J_\alpha(D^\perp_p) \subset T_pM\) for all \(p \in M\) and \(\alpha \in \{1, 2, 3\}\), where \(D^\perp\) is the complementary orthogonal bundle to \(D\) in \(TM\) (see \([BJC]\)).

3. QR-hypersurfaces and almost contact metric 3-structures.

Let \(M\) be an orientable hypersurface of a quaternionic Kähler manifold \((\overline{M}, \sigma, \overline{g})\) and \(\xi\) a unit normal field on \(M\). If we take \(D = 0\), then \(D^\perp = TM\) and we conclude that \(M\) is a QR-submanifold of \(\overline{M}\).

Let \(\{J_\alpha\}_{\alpha \in \{1, 2, 3\}}\) and \(\{J'_\alpha\}_{\alpha \in \{1, 2, 3\}}\) be two local bases defined on coordinate neighborhoods \(U\) and \(U'\) with \(U \cap U' \neq \emptyset\). Then, on \(U\),

\[\xi_\alpha = -J_\alpha \xi, \quad \forall \alpha \in \{1, 2, 3\},\]

defines tangent vector fields to \(M\) and similarly, on \(U'\),

\[\xi'_\alpha = -J'_\alpha \xi, \quad \forall \alpha \in \{1, 2, 3\},\]

defines tangent vector fields to \(M\).

Moreover, on \(U \cap U'\) we have

\[\xi'_\alpha = \sum_{\beta=1}^{3} c_{\alpha \beta} \xi_\beta, \quad \forall \alpha \in \{1, 2, 3\},\]

where \(C = (c_{\alpha \beta})_{\alpha, \beta \in \{1, 2, 3\}} \in SO(3)\). Thus, we obtain a distribution \(\mathcal{V}\) on \(M\), which is locally generated by \(\{\xi_\alpha\}_{\alpha \in \{1, 2, 3\}}\). Let \(\mathcal{H}\) be the orthogonal complementary distribution to \(\mathcal{V}\) with respect to the Riemannian metric \(g\) induced by \(\overline{g}\) on \(M\). We remark that for each \(p \in M\), \(\mathcal{H}_p\) is \(J_\alpha\)-invariant for all \(\alpha \in \{1, 2, 3\}\).

We recall that the distribution \(\mathcal{V}\) is integrable if and only if \(M\) is a mixed geodesic QR-hypersurface of \(\overline{M}\), i.e.

\[(3.1) \quad B(U, X) = 0, \quad \forall U \in \Gamma(\mathcal{V}), \forall X \in \Gamma(\mathcal{H}),\]

where \(B\) is the second fundamental form of \(M\) in \(\overline{M}\) (see \([BJC]\)).

DEFINITION 3.1 ([BLR]). Let \(M\) be a differentiable manifold equipped with a triple \((\phi, \xi, \eta)\), where \(\phi\) is a field of endomorphisms of tangent spaces, \(\xi\) is a vector field and \(\eta\) is a 1-form on \(M\). If

\[(3.2) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,\]

then we say that \((\phi, \xi, \eta)\) is an almost contact structure on \(M\).

DEFINITION 3.2 ([KUO]). Let \(M\) be a differentiable manifold which admits three almost contact structures \((\phi_\alpha, \xi_\alpha, \eta_\alpha), \alpha \in \{1, 2, 3\}\), satisfying the
following conditions:

\[ \eta_\alpha(\xi_\beta) = 0, \quad \forall \alpha \neq \beta, \]
\[ \phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \xi_\gamma, \]
\[ \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \eta_\gamma, \]
\[ \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta = \phi_\gamma, \]

where in (3.4)–(3.6), \((\alpha, \beta, \gamma)\) is an even permutation of \((1, 2, 3)\). Then the manifold \(M\) is said to have an almost contact 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}\).

**Definition 3.3** ([KUO]). Let \((M, g)\) be a Riemannian manifold endowed with an almost contact 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}\) such that

\[ \eta_\alpha(X) = g(X, \xi_\alpha), \quad \forall \alpha \in \{1,2,3\}, \]
\[ g(\phi_\alpha X, \phi_\alpha Y) = g(X, Y) - \eta_\alpha(X) \eta_\alpha(Y), \quad \forall \alpha \in \{1,2,3\}, \]

for all vector fields \(X, Y\) on \(M\). Then we say that \(M\) admits an almost contact metric 3-structure.

**Definition 3.4** ([BLR]). We say that an almost contact metric 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}\) on a Riemannian manifold \((M, g)\) is a 3-cosymplectic structure if

\[ (\nabla_X \phi_\alpha)(Y) = 0, \quad (\nabla_X \eta_\alpha)(Y) = 0, \quad \forall \alpha \in \{1,2,3\}. \]

Let \(M\) be an orientable hypersurface of a quaternionic Kähler manifold \(\overline{M}\). If \(S : TM \to \mathcal{H}\) is the canonical projection, then any local vector field \(X\) on \(M\) can be expressed as follows:

\[ X = SX + \sum_{\alpha=1}^{3} \eta_\alpha(X) \xi_\alpha, \]

where

\[ \eta_\alpha(X) = g(X, \xi_\alpha), \quad \forall \alpha \in \{1,2,3\}. \]

From (3.10) we have

\[ J_\alpha X = J_\alpha SX + \sum_{\beta=1}^{3} \eta_\beta(X) J_\alpha \xi_\beta, \quad \forall \alpha \in \{1,2,3\}. \]

From (3.12) we obtain the decomposition

\[ J_\alpha X = \phi_\alpha X + F_\alpha X, \]

where \(\phi_\alpha X\) is the tangential part of \(J_\alpha X\) given by

\[ \phi_\alpha X = J_\alpha SX + \eta_\beta(X) \xi_\gamma - \eta_\gamma(X) \xi_\beta, \]

and \(F_\alpha X\) is the normal part of \(J_\alpha X\) given by

\[ F_\alpha X = \eta_\alpha(X) \xi, \]

for all \(\alpha \in \{1,2,3\}\), where \((\alpha, \beta, \gamma)\) is an even permutation of \((1,2,3)\).
By straightforward computations, we can easily see that \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_\alpha\) defined by (3.11), (3.14) and (3.15) is an almost contact metric 3-structure on \(M\) and so we have the next result (see also [GS]).

**Proposition 3.5.** Any QR-hypersurface of a quaternionic Kähler manifold admits a natural almost contact metric 3-structure.

4. 3-submersions of QR-hypersurfaces

**Definition 4.1.** Let \(M\) be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold endowed with the natural almost contact metric 3-structure \((\phi_\alpha, \xi_\alpha, \eta_\alpha)_\alpha\in\{1,2,3\}\) given by Proposition 3.5, and let \((M', \sigma', g')\) be an almost quaternionic hermitian manifold. We say that a Riemannian submersion \(\pi : M \rightarrow M'\) is a QR 3-submersion if the following conditions are satisfied:

(i) \(\operatorname{Ker} \pi_* = \mathcal{V}\);
(ii) for each \(p \in M\), \(\sigma'_{\pi(p)}\) admits a canonical local basis \(\{J'_1, J'_2, J'_3\}\) such that \(\pi_* \phi_\alpha = J'_\alpha \pi_*\), \(\forall \alpha \in \{1, 2, 3\}\).

**Remark 4.2.** We recall that the sections of \(\mathcal{V}\), respectively \(\mathcal{H}\), are called vertical, respectively horizontal, vector fields. A Riemannian submersion \(\pi : M \rightarrow M'\) determines two \((1, 2)\) tensor fields \(T\) and \(A\) on \(M\) by the formulas

\[
T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \\
A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF,
\]

for any \(E, F \in \Gamma(TM)\), where \(v\) and \(h\) are the vertical and horizontal projections (see [KO, ON]).

We remark that for \(U, V \in \Gamma(\mathcal{V})\), \(T_U V\) coincides with the second fundamental form of the immersion of the fiber submanifolds, and for \(X, Y \in \Gamma(\mathcal{H})\), \(A_X Y = \frac{1}{2} v[X, Y]\), reflecting the complete integrability of the horizontal distribution \(\mathcal{H}\).

A horizontal vector field \(X\) on \(M\) is said to be basic if \(X\) is \(\pi\)-related to a vector field \(X'\) on \(M'\). It is clear that every vector field \(X'\) on \(M'\) has a unique horizontal lift \(X\) to \(M\), and \(X\) is basic.

**Remark 4.3.** If \(\pi : M \rightarrow M'\) is a Riemannian submersion and \(X, Y\) are basic vector fields on \(M\), \(\pi\)-related to \(X'\) and \(Y'\) on \(M'\), then we have the following properties (see [BES, FIP, ON]):

(i) \(h[X, Y]\) is a basic vector field and \(\pi_* h[X, Y] = [X', Y'] \circ \pi\);
(ii) \(h(\nabla_X Y)\) is a basic vector field \(\pi\)-related to \(\nabla'_{X'} Y'\), where \(\nabla\) and \(\nabla'\) are the Levi-Civita connections on \(M\) and \(M'\);
(iii) \([E, U] \in \Gamma(\mathcal{V})\) for all \(U \in \Gamma(\mathcal{V})\) and \(E \in \Gamma(TM)\).
Proposition 4.4. Let $M$ be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(M,\sigma,\bar{g})$ and let $(M',\sigma',\bar{g}')$ be an almost quaternionic hermitian manifold. If $\pi : M \to M'$ is a QR 3-submersion, then the distributions $\mathcal{V}$ and $\mathcal{H}$ are invariant by $\phi_\alpha$ for all $\alpha \in \{1,2,3\}$.

Proof. Let $V \in \Gamma(\mathcal{V})$. Then
$$\pi_*\phi_\alpha V = J'_\alpha \pi_* V = 0,$$
and so $\phi_\alpha(\mathcal{V}) \subset \mathcal{V}$.

On the other hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we derive from (3.8) that
$$g(\phi_\alpha X, V) = -g(X, \phi_\alpha V) = 0,$$
and thus $\phi_\alpha(\mathcal{H}) \subset \mathcal{H}$. □

Theorem 4.5. Let $\pi : M \to M'$ be a QR 3-submersion such that the canonical almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$ on $M$ is a 3-cosymplectic structure. Then $M'$ is locally hyper-Kähler.

Proof. For any local basic vector fields $X,Y$ on $M$, $\pi$-related to $X'$ and $Y'$ on $M'$, from (3.9) we have
$$(4.3) \quad \nabla_X \phi_\alpha Y - \phi_\alpha \nabla_X Y = 0, \quad \forall \alpha \in \{1,2,3\},$$
and from (4.3) we deduce
$$(4.4) \quad \pi_*(\nabla_X \phi_\alpha Y) - \pi_*\phi_\alpha \nabla_X Y = 0, \quad \forall \alpha \in \{1,2,3\}.$$Thus, since $Y$ is a basic vector field $\pi$-related to $Y'$, also $\phi_\alpha Y$ is basic and $\pi$-related to $J'_\alpha Y'$, and taking account of Definition 4.1 and Remark 4.3, we deduce from (4.4) that
$$(\nabla'_X, J'_\alpha) Y' = 0, \quad \forall \alpha \in \{1,2,3\},$$thus $(\nabla'_X, J'_\alpha) Y' = 0$, and so $M'$ is locally hyper-Kähler. □

Corollary 4.6. Let $M$ be a totally geodesic QR-hypersurface of a quaternionic Kähler manifold $(M,\sigma,\bar{g})$, and $(M',\sigma',\bar{g}')$ be an almost quaternionic hermitian manifold. If $\pi : M \to M'$ is a QR 3-submersion such that $\xi_1, \xi_2$ and $\xi_3$ are parallel in $M$, then $M'$ is locally hyper-Kähler.

Proof. In this case $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$ is a 3-cosymplectic structure on $M$ (see [GS]) and the proof is obvious from Theorem 4.5. □

Theorem 4.7. Let $M$ be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(M,\sigma,\bar{g})$, $(M',\sigma',\bar{g}')$ be an almost quaternionic hermitian manifold and $\pi : M \to M'$ be a QR 3-submersion. If the natural almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha \in \{1,2,3\}}$ on $M$ is 3-cosymplectic, then the fiber submanifolds are totally geodesic immersed and the horizontal distribution is integrable.
Proof. Since $M$ is 3-cosymplectic we have
\begin{equation}
\nabla_U \phi_\alpha V = \phi_\alpha \nabla_U V, \quad \forall \alpha \in \{1, 2, 3\},
\end{equation}
for all $U, V \in \Gamma(V)$. Taking the horizontal components, we obtain
\begin{equation}
T_U \phi_\alpha V = \phi_\alpha T_U V, \quad \forall \alpha \in \{1, 2, 3\},
\end{equation}
which immediately implies
\begin{equation}
T_U V = -T_{\phi_\alpha U} \phi_\alpha V, \quad \forall \alpha \in \{1, 2, 3\}.
\end{equation}
From (4.7), taking account of (3.6), we obtain $T = 0$. Similarly we obtain $A = 0$ and the proof is now complete, via Remark 4.2.

Let $M$ be an orientable submanifold of a Riemannian manifold $(M, g)$. We say that $M$ is a totally umbilical submanifold of $M$ if the second fundamental form $h$ of $M$ satisfies
\begin{equation}
h(E, F) = g(E, F)H, \quad \forall E, F \in \Gamma(TM),
\end{equation}
where $H$ is the mean curvature vector field on $M$. Moreover, if $H$ is non-zero and parallel in the normal bundle $TM^\perp$, then $M$ is called an extrinsic sphere.

By using the Gauss equation, (2.4) and the Gray-O’Neill equation (see [BES, FIP, MNG, ON]), we can easily prove the next result.

**Theorem 4.8.** Let $M$ be a QR extrinsic hypersphere of a flat quaternionic Kähler manifold $(M, \sigma, \overline{g})$ and let $(M', \sigma', g')$ be another quaternionic Kähler manifold. If $\pi : M \to M'$ is a QR 3-submersion, then $M'$ is a quaternionic space form.

**Example 4.9.** Let $S^{4m+3}$ be the standard hypersphere in $\mathbb{R}^{4m+4}$. Then the canonical mapping $\pi : S^{4m+3} \to P^m(\mathbb{H})$ is a QR 3-submersion.

### 5. Quaternionic submersions

**Definition 5.1 ([LMV1]).** Let $(M, \sigma, g)$ and $(N, \sigma', g')$ be two almost quaternionic hermitian manifolds. A map $f : M \to N$ is said to be $(\sigma, \sigma')$-holomorphic at a point $x \in M$ if for any $J \in \sigma_x$ there exists $J' \in \sigma'_{f(x)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that $f$ is $(\sigma, \sigma')$-holomorphic if it is $(\sigma, \sigma')$-holomorphic at each $x \in M$.

**Definition 5.2 ([LMV2]).** Let $(M, \sigma, g)$ and $(N, \sigma', g')$ be two almost quaternionic hermitian manifolds. A Riemannian submersion $\pi : M \to N$ which is a $(\sigma, \sigma')$-holomorphic map is called a quaternionic submersion.

**Theorem 5.3.** Let $\pi : (M, \sigma, g) \to (N, \sigma', g')$ be a quaternionic submersion such that $(M, \sigma, g)$ is a quaternionic Kähler manifold. Then $(N, \sigma', g')$ is a quaternionic Kähler manifold.

**Proof.** Let $X_*, Y_* \in \Gamma(TN)$ be such that $\pi_* X = X_*, \pi_* Y = Y_*$, where $X, Y \in \Gamma(TM)$. Then
(5.1) \((\nabla'_{X*}J'_\alpha)Y_*) = \nabla'_{X*}(J'_\alpha Y_*) - J'_\alpha(\nabla'_{X*}Y_*)\]
\[= \nabla'_{\pi_*X}(J'_\alpha \pi_* Y) - J'_\alpha(\nabla'_{\pi_*X} \pi_* Y)\]
\[= \nabla'_{\pi_*X}(\pi_*(J_\alpha Y)) - J'_\alpha \pi_* (h \nabla_X Y)\]
\[= \pi_*(h \nabla_X (J_\alpha Y)) - \pi_*(J_\alpha (h \nabla_X Y)) = \pi_*(\nabla_X J_\alpha Y).\]

Since \((M, \sigma, g)\) is a quaternionic Kähler manifold we have (2.3) and we can define 1-forms \(\omega'_1, \omega'_2, \omega'_3\) on \(N\) by
\[(5.2) \quad \omega'_\alpha(X_*) \circ \pi = \omega_\alpha(X), \quad \forall \alpha \in \{1, 2, 3\},\]
for any local vector field \(X_*\) on \(N\) and \(X\) a real basic vector field on \(M\) such that \(\pi_*X = X_*\).

From (2.3), (5.1) and (5.2) we deduce that for all \(\alpha \in \{1, 2, 3\},\)
\[(5.3) \quad (\nabla'_{X_*,J'_\alpha})Y_* = \omega'_{\alpha+2}(X_*) J'_{\alpha+1} Y_* - \omega'_{\alpha+1}(X_*) J'_{\alpha+2} Y_*\]
for any local vector fields \(X_*, Y_*\) on \(M'\), where the indices are taken from \(\{1, 2, 3\}\) modulo 3. Thus we conclude that \((N, \sigma', g')\) is a quaternionic Kähler manifold.

**Corollary 5.4.** Let \(\pi : (M, \sigma, g) \rightarrow (N, \sigma', g')\) be a quaternionic submersion such that \((M, \sigma, g)\) is a quaternionic Kähler manifold. Then both \((M, \sigma, g)\) and \((N, \sigma', g')\) are locally hyper-Kähler manifolds.

**Proof.** In this case the vertical and horizontal distributions are both integrable (see [IMV2]) and so we can easily conclude that \((M, \sigma, g)\) is a locally hyper-Kähler manifold. The assertion now follows from the above theorem.

**Corollary 5.5.** There are no quaternionic submersions between quaternionic Kähler manifolds which are not locally hyper-Kähler.

**Proof.** The assertion is obvious from the above corollary.

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**References**


3-submersions from QR-hypersurfaces


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