ANNALES
POLONICI MATHEMATICI
98.3 (2010)

3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds

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Abstract. We study 3-submersions from a QR-hypersurface of a quaternionic Kähler manifold onto an almost quaternionic hermitian manifold. We also prove the non-existence of quaternionic submersions between quaternionic Kähler manifolds which are not locally hyper-Kähler.

1. Introduction. In [WTS] B. Watson introduced the notion of 3-submersion, as a Riemannian submersion from an almost contact metric manifold onto an almost quaternionic manifold, which commutes with the structure tensors of type $(1,1)$. In [IMV1] and [IMV2], this concept has been extended in quaternionic setting. In this paper we study 3-submersions from QR-hypersurfaces of quaternionic Kähler manifolds, we give an example and obtain some obstructions to the existence of quaternionic submersions.

The study of QR-submanifolds of a quaternionic Kähler manifold was initiated by A. Bejancu [BJC]. Among all submanifolds of a quaternionic Kähler manifold, QR-submanifolds have been intensively studied by several authors [AG, BEJ, BF, GS, KP, KPK, MNG, SHN1, SHN2]. In Section 2 we recall the definitions and basic properties of quaternionic manifolds and QR-submanifolds of a quaternionic Kähler manifold.

On the other hand, R. Güneş, B. Şahin and S. Keleş [GS] have shown that a QR-submanifold admits an almost contact 3-structure under some conditions. In Section 3 we see that on an orientable hypersurface of a quaternionic Kähler manifold there exists a natural almost contact metric 3-structure. This result will allow us to define the concept of QR 3-submersion. In Section 4 we obtain some properties for this kind of submersions and give an example. In the last section we prove the non-existence of quaternionic submersions between quaternionic Kähler non-locally hyper-Kähler manifolds.

2010 *Mathematics Subject Classification*: Primary 53C15; Secondary 53C26.
Key words and phrases: Riemannian submersion, quaternionic Kähler manifold, QR-submanifold.

DOI: 10.4064/ap98-3-9

2. Preliminaries. Let M be a differentiable manifold of dimension n and assume that there is a rank 3-subbundle σ of $\mathrm{End}(TM)$ which is locally spanned by an almost hypercomplex structure, i.e. a triple $\{J_1, J_2, J_3\}$ of almost complex structures satisfying the quaternionic identities:

$$\begin{cases} J_\alpha^2 = -\mathrm{Id}, \quad \forall \alpha \in \{1,2,3\}, \\ J_1 J_2 = -J_2 J_1 = J_3. \end{cases} \tag{2.1}$$

Then the bundle σ is called an *almost quaternionic structure* on M and $\{J_1, J_2, J_3\}$ is called a *canonical local basis* of σ. Moreover, (M, g) is then said to be an *almost quaternionic manifold.* It is easy to see that any almost quaternionic manifold is of dimension $n = 4m$.

A Riemannian metric g is *adapted* to the quaternionic structure σ if

$$g(J_\alpha X, J_\alpha Y) = g(X, Y), \qquad \forall \alpha \in \{1,2,3\}, \tag{2.2}$$

for all vector fields X,Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ. Moreover, (M, σ, g) is then said to be an *almost quaternionic hermitian manifold.*

If the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g, then (M, σ, g) is said to be a *quaternionic Kähler manifold.* Equivalently, there exists locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that

$$\begin{cases} \nabla_X J_1 = \omega_3(X) J_2 - \omega_2(X) J_3, \\ \nabla_X J_2 = -\omega_3(X) J_1 + \omega_1(X) J_3, \\ \nabla_X J_3 = \omega_2(X) J_1 - \omega_1(X) J_2, \end{cases} \tag{2.3}$$

for any vector field X on M. In particular, if $\omega_1 = \omega_2 = \omega_3 = 0$, then (M, σ, g) is said to be a *locally hyper-Kähler manifold.*

We remark that any quaternionic Kähler manifold M is an Einstein space, provided that $\dim M > 4$. Moreover, M is irreducible (if $\mathrm{Ric} \neq 0$) or locally hyper-Kähler manifold (if $\mathrm{Ric} = 0$) (see [AL, BES, ISH, SLM]).

Let (M, σ, g) be an almost quaternionic hermitian manifold. If $X \in T_pM$, $p \in M$, then the 4-plane $Q(X)$ spanned by $\{X, J_1X, J_2X, J_3X\}$ is called a *quaternionic 4-plane.* A 2-plane in T_pM spanned by $\{X, Y\}$ is called *half-quaternionic* if $Q(X) = Q(Y)$.

The sectional curvature for a half-quaternionic 2-plane is called *quaternionic sectional curvature.* A quaternionic Kähler manifold is a *quaternionic space form* if its quaternionic sectional curvatures are equal to a constant, say c. It is well-known that a quaternionic Kähler manifold (M, σ, g) is a quaternionic space form (denoted $M(c)$) if and only if its curvature tensor is

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\Big\{ & g(Z,Y)X - g(X,Z)Y + \sum_{\alpha=1}^{3} [g(Z, J_\alpha Y) J_\alpha X \\ & - g(Z, J_\alpha X) J_\alpha Y + 2g(X, J_\alpha Y) J_\alpha Z]\Big\} \end{aligned} \tag{2.4}$$

for all vector fields X, Y, Z on M and any local basis $\{J_1, J_2, J_3\}$ of σ.

Let $(\overline{M}, \sigma, \overline{g})$ be a quaternionic Kähler manifold and let M be a real submanifold of $\overline{M}$. Then M is said to be a *QR-submanifold* if there exists a vector subbundle D of the normal bundle $TM^{\perp}$ such that:

(i) $J_\alpha(D_p) = D_p$ for all $p \in M$ and $\alpha \in \{1,2,3\}$;
(ii) $J_\alpha(D_p^{\perp}) \subset T_pM$ for all $p \in M$ and $\alpha \in \{1,2,3\}$, where $D^{\perp}$ is the complementary orthogonal bundle to D in $TM^{\perp}$ (see [BJC]).

3. QR-hypersurfaces and almost contact metric 3-structures. Let M be an orientable hypersurface of a quaternionic Kähler manifold $(\overline{M}, \sigma, \overline{g})$ and ξ a unit normal field on M. If we take $D = 0$, then $D^{\perp} = TM^{\perp}$ and we conclude that M is a QR-submanifold of $\overline{M}$.

Let $\{J_\alpha\}_{\alpha \in \{1,2,3\}}$ and $\{J'_\alpha\}_{\alpha \in \{1,2,3\}}$ be two local bases defined on coordinate neighborhoods $\overline{U}$ and $\overline{U}'$ with $\overline{U} \cap \overline{U}' \neq \emptyset$. Then, on $\overline{U}$,

$$\xi_\alpha = -J_\alpha \xi, \qquad \forall \alpha \in \{1,2,3\},$$

defines tangent vector fields to M and similarly, on $\overline{U}'$,

$$\xi'_\alpha = -J'_\alpha \xi, \qquad \forall \alpha \in \{1,2,3\},$$

defines tangent vector fields to M.

Moreover, on $\overline{U} \cap \overline{U}'$ we have

$$\xi'_\alpha = \sum_{\beta=1}^{3} c_{\alpha\beta} \xi_\beta, \qquad \forall \alpha \in \{1,2,3\},$$

where $C = (c_{\alpha\beta})_{\alpha,\beta \in \{1,2,3\}} \in SO(3)$. Thus, we obtain a distribution $\mathcal{V}$ on M, which is locally generated by $\{\xi_\alpha\}_{\alpha \in \{1,2,3\}}$. Let $\mathcal{H}$ be the orthogonal complementary distribution to $\mathcal{V}$ with respect to the Riemannian metric g induced by $\overline{g}$ on M. We remark that for each $p \in M$, $\mathcal{H}_p$ is J_α-invariant for all $\alpha \in \{1,2,3\}$.

We recall that the distribution $\mathcal{V}$ is integrable if and only if M is a mixed geodesic QR-hypersurface of $\overline{M}$, i.e.

$$B(U, X) = 0, \qquad \forall U \in \Gamma(\mathcal{V}), \forall X \in \Gamma(\mathcal{H}), \tag{3.1}$$

where B is the second fundamental form of M in $\overline{M}$ (see [BJC]).

DEFINITION 3.1 ([BLR]). Let M be a differentiable manifold equipped with a triple (ϕ, ξ, η), where ϕ is a field of endomorphisms of tangent spaces, ξ is a vector field and η is a 1-form on M. If

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{3.2}$$

then we say that (ϕ, ξ, η) is an *almost contact structure* on M.

DEFINITION 3.2 ([KUO]). Let M be a differentiable manifold which admits three almost contact structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1,2,3\}$, satisfying the following conditions:

$$\eta_\alpha(\xi_\beta) = 0, \quad \forall \alpha \neq \beta, \tag{3.3}$$

$$\phi_\alpha(\xi_\beta) = -\phi_\beta(\xi_\alpha) = \xi_\gamma, \tag{3.4}$$

$$\eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha = \eta_\gamma, \tag{3.5}$$

$$\phi_\alpha\phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta\phi_\alpha + \eta_\alpha \otimes \xi_\beta = \phi_\gamma, \tag{3.6}$$

where in (3.4)–(3.6), (α, β, γ) is an even permutation of $(1, 2, 3)$. Then the manifold M is said to have an *almost contact* 3*-structure* $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$.

Definition 3.3 ([KUO]). Let (M, g) be a Riemannian manifold endowed with an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ such that

$$\eta_\alpha(X) = g(X, \xi_\alpha), \quad \forall \alpha \in \{1, 2, 3\}, \tag{3.7}$$

$$g(\phi_\alpha X, \phi_\alpha Y) = g(X, Y) - \eta_\alpha(X)\eta_\alpha(Y), \quad \forall \alpha \in \{1, 2, 3\}, \tag{3.8}$$

for all vector fields X, Y on M. Then we say that M admits an *almost contact metric* 3*-structure.*

Definition 3.4 ([BLR]). We say that an almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ on a Riemannian manifold (M, g) is a 3*-cosymplectic structure* if

$$(\nabla_X \phi_\alpha)(Y) = 0, \quad (\nabla_X \eta_\alpha)(Y) = 0, \quad \forall \alpha \in \{1, 2, 3\}. \tag{3.9}$$

Let M be an orientable hypersurface of a quaternionic Kähler manifold $\overline{M}$. If $S : TM \to \mathcal{H}$ is the canonical projection, then any local vector field X on M can be expressed as follows:

$$X = SX + \sum_{\alpha=1}^{3} \eta_\alpha(X)\xi_\alpha, \tag{3.10}$$

where

$$\eta_\alpha(X) = g(X, \xi_\alpha), \quad \forall \alpha \in \{1, 2, 3\}. \tag{3.11}$$

From (3.10) we have

$$J_\alpha X = J_\alpha SX + \sum_{\beta=1}^{3} \eta_\beta(X) J_\alpha \xi_\beta, \quad \forall \alpha \in \{1, 2, 3\}. \tag{3.12}$$

From (3.12) we obtain the decomposition

$$J_\alpha X = \phi_\alpha X + F_\alpha X, \tag{3.13}$$

where $\phi_\alpha X$ is the tangential part of $J_\alpha X$ given by

$$\phi_\alpha X = J_\alpha SX + \eta_\beta(X)\xi_\gamma - \eta_\gamma(X)\xi_\beta, \tag{3.14}$$

and $F_\alpha X$ is the normal part of $J_\alpha X$ given by

$$F_\alpha X = \eta_\alpha(X)\xi, \tag{3.15}$$

for all $\alpha \in \{1, 2, 3\}$, where (α, β, γ) is an even permutation of $(1, 2, 3)$.

By straightforward computations, we can easily see that $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_\alpha$ defined by (3.11), (3.14) and (3.15) is an almost contact metric 3-structure on M and so we have the next result (see also [GS]).

Proposition 3.5. *Any QR-hypersurface of a quaternionic Kähler manifold admits a natural almost contact metric* 3*-structure.*

4. 3-submersions of QR-hypersurfaces

Definition 4.1. Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $\overline{M}$ endowed with the natural almost contact metric 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha\in\{1,2,3\}}$ given by Proposition 3.5, and let (M', σ', g') be an almost quaternionic hermitian manifold. We say that a Riemannian submersion $\pi : M \to M'$ is a *QR* 3*-submersion* if the following conditions are satisfied:

(i) $\operatorname{Ker} \pi_* = \mathcal{V}$;
(ii) for each $p \in M$, $\sigma'_{\pi(p)}$ admits a canonical local basis $\{J'_1, J'_2, J'_3\}$ such that

$$\pi_*\phi_\alpha = J'_\alpha \pi_*, \qquad \forall \alpha \in \{1, 2, 3\}.$$

Remark 4.2. We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called *vertical*, respectively *horizontal*, vector fields. A Riemannian submersion $\pi : M \to M'$ determines two $(1, 2)$ tensor fields T and A on M by the formulas

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{4.1}$$

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF, \tag{4.2}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [KO, ON]).

We remark that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fiber submanifolds, and for $X, Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2} v[X, Y]$, reflecting the complete integrability of the horizontal distribution $\mathcal{H}$.

A horizontal vector field X on M is said to be *basic* if X is π-related to a vector field X' on M'. It is clear that every vector field X' on M' has a unique horizontal lift X to M, and X is basic.

Remark 4.3. If $\pi : M \to M'$ is a Riemannian submersion and X, Y are basic vector fields on M, π-related to X' and Y' on M', then we have the following properties (see [BES, FIP, ON]):

(i) $h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;
(ii) $h(\nabla_X Y)$ is a basic vector field π-related to $\nabla'_{X'} Y'$, where ∇ and ∇' are the Levi-Civita connections on M and M';
(iii) $[E, U] \in \Gamma(\mathcal{V})$ for all $U \in \Gamma(\mathcal{V})$ and $E \in \Gamma(TM)$.

Proposition 4.4. *Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M},\overline{\sigma},\overline{g})$ and let (M',σ',g') be an almost quaternionic hermitian manifold. If $\pi : M \to M'$ is a QR 3-submersion, then the distributions $\mathcal{V}$ and $\mathcal{H}$ are invariant by ϕ_α for all $\alpha \in \{1,2,3\}$.*

Proof. Let $V \in \Gamma(\mathcal{V})$. Then

$$\pi_*\phi_\alpha V = J'_\alpha \pi_* V = 0,$$

and so $\phi_\alpha(\mathcal{V}) \subset \mathcal{V}$.

On the other hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we derive from (3.8) that

$$g(\phi_\alpha X, V) = -g(X, \phi_\alpha V) = 0,$$

and thus $\phi_\alpha(\mathcal{H}) \subset \mathcal{H}$. ∎

Theorem 4.5. *Let $\pi : M \to M'$ be a QR 3-submersion such that the canonical almost contact 3-structure $(\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha\in\{1,2,3\}}$ on M is a 3-cosymplectic structure. Then M' is locally hyper-Kähler.*

Proof. For any local basic vector fields X, Y on M, π-related to X' and Y' on M', from (3.9) we have

$$\nabla_X \phi_\alpha Y - \phi_\alpha \nabla_X Y = 0, \qquad \forall \alpha \in \{1,2,3\}. \tag{4.3}$$

and from (4.3) we deduce

$$\pi_*(\nabla_X \phi_\alpha Y) - \pi_* \phi_\alpha \nabla_X Y = 0, \qquad \forall \alpha \in \{1,2,3\}. \tag{4.4}$$

Thus, since Y is a basic vector field π-related to Y', also $\phi_\alpha Y$ is basic and π-related to $J'_\alpha Y'$, and taking account of Definition 4.1 and Remark 4.3, we deduce from (4.4) that

$$\nabla'_{X'} J'_\alpha Y' - J'_\alpha \nabla'_{X'} Y' = 0, \qquad \forall \alpha \in \{1,2,3\},$$

thus $(\nabla'_{X'} J'_\alpha) Y' = 0$, and so M' is locally hyper-Kähler. ∎

Corollary 4.6. *Let M be a totally geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M},\overline{\sigma},\overline{g})$, and (M',σ',g') be an almost quaternionic hermitian manifold. If $\pi : M \to M'$ is a QR 3-submersion such that ξ_1, ξ_2 and ξ_3 are parallel in M, then M' is locally hyper-Kähler.*

Proof. In this case $(\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha\in\{1,2,3\}}$ is a 3-cosymplectic structure on M (see [GS]) and the proof is obvious from Theorem 4.5. ∎

Theorem 4.7. *Let M be a mixed geodesic QR-hypersurface of a quaternionic Kähler manifold $(\overline{M},\overline{\sigma},\overline{g})$, (M',σ',g') be an almost quaternionic hermitian manifold and $\pi : M \to M'$ be a QR 3-submersion. If the natural almost contact metric 3-structure $(\phi_\alpha,\xi_\alpha,\eta_\alpha)_{\alpha\in\{1,2,3\}}$ on M is 3-cosymplectic, then the fiber submanifolds are totally geodesic immersed and the horizontal distribution is integrable.*

Proof. Since M is 3-cosymplectic we have

$$\nabla_U \phi_\alpha V = \phi_\alpha \nabla_U V, \quad \forall \alpha \in \{1,2,3\}, \tag{4.5}$$

for all $U, V \in \Gamma(\mathcal{V})$. Taking the horizontal components, we obtain

$$T_U \phi_\alpha V = \phi_\alpha T_U V, \quad \forall \alpha \in \{1,2,3\}, \tag{4.6}$$

which immediately implies

$$T_U V = -T_{\phi_\alpha U} \phi_\alpha V, \quad \forall \alpha \in \{1,2,3\}. \tag{4.7}$$

From (4.7), taking account of (3.6), we obtain $T = 0$. Similarly we obtain $A = 0$ and the proof is now complete, via Remark 4.2. ∎

Let M be an orientable submanifold of a Riemannian manifold $(\overline{M}, \overline{g})$. We say that M is a *totally umbilical submanifold* of $\overline{M}$ if the second fundamental form h of M satisfies

$$h(E,F) = g(E,F)H, \quad \forall E, F \in \Gamma(TM), \tag{4.8}$$

where H is the mean curvature vector field on M. Moreover, if H is non-zero and parallel in the normal bundle $TM^\perp$, then M is called an *extrinsic sphere.*

By using the Gauss equation, (2.4) and the Gray-O'Neill equation (see [BES, FIP, MNG, ON]), we can easily prove the next result.

THEOREM 4.8. *Let M be a QR extrinsic hypersphere of a flat quaternionic Kähler manifold $(\overline{M}, \overline{\sigma}, \overline{g})$ and let (M', σ', g') be another quaternionic Kähler manifold. If $\pi : M \to M'$ is a QR 3-submersion, then M' is a quaternionic space form.*

EXAMPLE 4.9. Let S^{4m+3} be the standard hypersphere in $\mathbb{R}^{4m+4}$. Then the canonical mapping $\pi : S^{4m+3} \to P^m(\mathbb{H})$ is a QR 3-submersion.

5. Quaternionic submersions

DEFINITION 5.1 ([IMV1]). Let (M, σ, g) and (N, σ', g') be two almost quaternionic hermitian manifolds. A map $f : M \to N$ is said to be (σ, σ')-*holomorphic at a point* $x \in M$ if for any $J \in \sigma_x$ there exists $J' \in \sigma'_{f(x)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that f is (σ, σ')-holomorphic if it is (σ, σ')-holomorphic at each $x \in M$.

DEFINITION 5.2 ([IMV2]). Let (M, σ, g) and (N, σ', g') be two almost quaternionic hermitian manifolds. A Riemannian submersion $\pi : M \to N$ which is a (σ, σ')-holomorphic map is called a *quaternionic submersion.*

THEOREM 5.3. *Let $\pi : (M, \sigma, g) \to (N, \sigma', g')$ be a quaternionic submersion such that (M, σ, g) is a quaternionic Kähler manifold. Then (N, σ', g') is a quaternionic Kähler manifold.*

Proof. Let $X_*, Y_* \in \Gamma(TN)$ be such that $\pi_* X = X_*$, $\pi_* Y = Y_*$, where $X, Y \in \Gamma(TM)$. Then

$$(5.1)\quad (\nabla'_{X_*}J'_\alpha)Y_* = \nabla'_{X_*}(J'_\alpha Y_*) - J'_\alpha(\nabla'_{X_*}Y_*)$$
$$= \nabla'_{\pi_*X}(J'_\alpha \pi_* Y) - J'_\alpha(\nabla'_{\pi_*X}\pi_*Y)$$
$$= \nabla'_{\pi_*X}(\pi_*(J_\alpha Y)) - J'_\alpha \pi_*(h\nabla_X Y)$$
$$= \pi_*(h\nabla_X(J_\alpha Y)) - \pi_*(J_\alpha(h\nabla_X Y)) = \pi_*((\nabla_X J_\alpha)Y).$$

Since (M,σ,g) is a quaternionic Kähler manifold we have (2.3) and we can define 1-forms $\omega'_1, \omega'_2, \omega'_3$ on N by

$$(5.2)\qquad \omega'_\alpha(X_*)\circ\pi = \omega_\alpha(X), \quad \forall\alpha\in\{1,2,3\},$$

for any local vector field X_* on N and X a real basic vector field on M such that $\pi_*X = X_*$.

From (2.3), (5.1) and (5.2) we deduce that for all $\alpha\in\{1,2,3\}$,

$$(5.3)\qquad (\nabla'_{X_*}J'_\alpha)Y_* = \omega'_{\alpha+2}(X_*)J'_{\alpha+1}Y_* - \omega'_{\alpha+1}(X_*)J'_{\alpha+2}Y_*$$

for any local vector fields X_*, Y_* on M', where the indices are taken from $\{1,2,3\}$ modulo 3. Thus we conclude that (N,σ',g') is a quaternionic Kähler manifold. ∎

COROLLARY 5.4. *Let $\pi : (M,\sigma,g)\to(N,\sigma',g')$ be a quaternionic submersion such that (M,σ,g) is a quaternionic Kähler manifold. Then both (M,σ,g) and (N,σ',g') are locally hyper-Kähler manifolds.*

Proof. In this case the vertical and horizontal distributions are both integrable (see [IMV2]) and so we can easily conclude that (M,σ,g) is a locally hyper-Kähler manifold. The assertion now follows from the above theorem. ∎

COROLLARY 5.5. *There are no quaternionic submersions between quaternionic Kähler manifolds which are not locally hyper-Kähler.*

Proof. The assertion is obvious from the above corollary. ∎

Acknowledgments. The author expresses his gratitude to the referee for carefully reading the manuscript and giving useful comments. This research was partly supported by CNCSIS-UEFISCSU, project PNII-IDEI code 8/2008, contract no. 525/2009.

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Received 3.6.2009
and in final form 15.11.2009 (2114)