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An Osserman-type condition on $g.f.f$-manifolds with Lorentz metric

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Abstract. A condition of Osserman type, called the φ-null Osserman condition, is introduced and studied in the context of Lorentz globally framed f-manifolds. An explicit example shows the naturality of this condition in the setting of Lorentz $\mathcal{S}$-manifolds. We prove that a Lorentz $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman, extending a well-known result in the case of Lorentz Sasaki space forms. Then we state a characterization of a particular class of φ-null Osserman $\mathcal{S}$-manifolds. Finally, some examples are examined.

1. Introduction. The study of the behaviour of the Jacobi operators is an important topic in Riemannian and, more generally, in semi-Riemannian geometry. More precisely, let (M, g) be a Riemannian manifold with curvature tensor R and consider a point p in M. For any unit vector $X \in T_pM$, the symmetric endomorphism $R_X = R_p(\cdot, X)X : X^\perp \to X^\perp$ is called the *Jacobi operator* with respect to X. If the eigenvalues of R_X are independent of the choices of X and p, one says that (M, g) is an *Osserman manifold* [21].

The Osserman conjecture states that an Osserman manifold is either flat or locally a rank-one symmetric space, and some progress towards this conjecture was made in [7–9, 17–19]. Osserman manifolds were also studied in the Lorentzian context [3, 11, 12], where a complete solution is available.

Recently, Atindogbe and Duggal [1] have introduced and studied suitable operators of Jacobi type associated with a semi-Riemannian degenerate metric.

In [12] the authors defined the Jacobi operator $\bar{R}_u$, u being a null (or lightlike) vector tangent to a Lorentz manifold M. Given a unit timelike vector z tangent to M, they introduced and investigated the so-called null Osserman condition with respect to z (see also [13]).

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Obviously, Lorentz almost contact manifolds can be studied in this context. In particular, a Lorentz Sasaki space form, whose characteristic vector field ξ is timelike, turns out to be globally null Osserman with respect to ξ [13]. This result does not hold in the context of Lorentz globally framed f-manifolds $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $s \geq 2$, as we will see with a counterexample.

This motivates the introduction of a more general condition of Osserman type, which we call the *φ-null Osserman condition*.

The main results of this paper give the links between the φ-null Osserman condition and the behaviour of the φ-sectional curvature in Lorentz $\mathcal{S}$-manifolds. After a preliminary section, where we gather some facts about $g.f.f$-manifolds, needed in the rest of the paper, in Section 3 we discuss the relationship between the null Osserman condition and the Lorentz $\mathcal{S}$-structures, giving an example of a Lorentz $\mathcal{S}$-space form which does not satisfy the null Osserman conditions. We endow the compact Lie group $U(2)$ with a Lorentz $\mathcal{S}$-structure of rank 2. This manifold is an $\mathcal{S}$-space form with two characteristic vector fields ξ_1 and ξ_2, ξ_1 timelike, that does not satisfy the null Osserman condition with respect to ξ_1.

In Section 4 we introduce the notion of φ-null Osserman manifold, and we state that a Lorentz $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman with respect to the timelike characteristic vector field. We prove, in Section 5, an algebraic characterization of the Riemannian curvature tensor field in a particular class of φ-null Osserman Lorentz $\mathcal{S}$-manifolds. Moreover, we look at the behaviour of the φ-sectional curvature when the Jacobi operator has a single eigenvalue. In particular, it is interesting to note that the existence of the only eigenvalue 1 of the Jacobi operator is related to the φ-sectional flatness of the manifold. Finally in the case of 4-dimensional φ-null Osserman manifolds we find a compact example, using the Lie group $U(2)$, and also a non-compact one.

All manifolds, tensor fields and maps are assumed to be smooth, moreover we suppose all manifolds are connected. We will use the Einstein convention omitting the sum symbol for repeated indices. Following the notations of S. Kobayashi and K. Nomizu [16] for the curvature tensor R we have $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ and $R(X,Y,Z,W) = g(R(Z,W)Y,X)$ for any $X,Y,Z,W \in \mathfrak{X}(M)$. The *sectional curvature* $K_p(\pi)$ at p of a non-degenerate 2-plane $\pi = \operatorname{span}\{X,Y\}$ is given by

$$K_p(\pi) = K_p(X,Y) = \frac{R_p(X,Y,X,Y)}{\varDelta(\pi)} = \frac{g_p(R_p(X,Y)Y,X)}{\varDelta(\pi)},$$

where $\varDelta(\pi) = g(X,X)g(Y,Y) - g(X,Y)^2 \neq 0$.

2. Preliminaries. Following [2, 5, 22] we recall some definitions. An *almost contact manifold* is a $(2n+1)$-dimensional manifold M endowed with an almost contact structure, i.e. M^{2n+1} has a $(1,1)$-tensor field f such that $\operatorname{rank}(f) = 2n$, a 1-form η and a vector field ξ satisfying $f^2(X) = -X + \eta(X)\xi$ and $\eta(\xi) = 1$. Moreover, if g is a semi-Riemannian metric on M^{2n+1} such that, for any $X, Y \in \mathfrak{X}(M^{2n+1})$,

$$g(fX, fY) = g(X, Y) - \varepsilon\eta(X)\eta(Y),$$

where $\varepsilon = \pm 1$ according to the causal character of ξ, then M^{2n+1} is called an *indefinite almost contact manifold*. Such a manifold is said to be an *indefinite contact manifold* if $d\eta = \Phi$, Φ being defined by $\Phi(X, Y) = g(X, fY)$. Furthermore, if the structure (f, ξ, η) is *normal*, i.e. $N = [f, f] + 2d\eta \otimes \xi = 0$, then the indefinite contact structure is called an *indefinite Sasaki structure* and, in this case, the manifold $(M^{2n+1}, f, \xi, \eta, g)$ is called *indefinite Sasaki.*

In the Riemannian case a generalization of these structures was studied by Blair [2] and by Goldberg and Yano [15]. In [5] we studied such structures in semi-Riemannian context.

A manifold M is called a *globally framed f-manifold* (briefly *$g.f.f$-manifold*) if it is endowed with a nowhere-vanishing $(1,1)$-tensor field φ of constant rank such that $\ker\varphi$ is parallelizable, i.e. there exist global vector fields ξ_α, $\alpha \in \{1, \dots, s\}$, and 1-forms η^α, satisfying

$$\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta.$$

A $g.f.f$-manifold $(M^{2n+s}, \varphi, \xi_\alpha, \eta^\alpha)$, $\alpha \in \{1, \dots, s\}$, is said to be an *indefinite $g.f.f$-manifold* if g is a semi-Riemannian metric satisfying the compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y)$$

for any vector fields X, Y, where $\varepsilon_\alpha = \pm 1$ according to whether ξ_α is spacelike or timelike. Then, for any $\alpha \in \{1, \dots, s\}$ and $X \in \mathfrak{X}(M^{2n+s})$, one has $\eta^\alpha(X) = \varepsilon_\alpha g(X, \xi_\alpha)$.

An indefinite $g.f.f$-manifold is an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\eta^\alpha = \Phi$ for any $\alpha \in \{1, \dots, s\}$, where $\Phi(X, Y) = g(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(M^{2n+s})$. The normality condition is expressed by the vanishing of the tensor field $N_\varphi + 2d\eta^\alpha \otimes \xi_\alpha$, N_φ being the Nijenhuis torsion of φ.

Note that, for $s = 1$, we recover the notion of indefinite Sasaki manifold.

We recall that $\nabla_X\xi_\alpha = -\varepsilon_\alpha\varphi X$ and $\ker\varphi$ is an integrable flat distribution since $\nabla_{\xi_\alpha}\xi_\beta = 0$ for any $\alpha, \beta \in \{1, \dots, s\}$. Anyway, an indefinite $\mathcal{S}$-manifold is never flat and it is never a real space form since, for example, $K(X, \xi_\alpha) = \varepsilon_\alpha$ for any non-lightlike $X \in \operatorname{Im}\varphi_p$.

For more details we refer the reader to [5], where we describe three examples of non-compact indefinite $\mathcal{S}$-manifolds. More precisely, we construct two different indefinite $\mathcal{S}$-structures with metrics of index $\nu = 2$ on $\mathbb{R}^6$ and an indefinite $\mathcal{S}$-structure with Lorentz metric on $\mathbb{R}^4$. Moreover, in [6] we give explicit examples of compact indefinite $g.f.f$-manifolds and indefinite $\mathcal{S}$-manifolds.

We also remark that every $g.f.f$-manifold is subject to the following topological condition: it has to be either non-compact or compact with vanishing Euler characteristic, since it never admits vanishing vector fields. This implies that it always admits Lorentz metrics.

Let us fix some notation connected with the curvature tensor field. As usual, a 2-plane $\pi = \operatorname{span}\{X, \varphi X\}$ in T_pM, with $p \in M$ and $X \in \operatorname{Im}\varphi_p$, is said to be a *$\varphi$-plane* and the sectional curvature at p of such a plane, with X a non-lightlike vector, is called the *φ-sectional curvature* at p and is denoted by $H_p(X)$.

An indefinite $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be an *indefinite $\mathcal{S}$-space form* if the φ-sectional curvature $H_p(X)$ is constant, for any point and any φ-plane. In particular, in [5] it is proved that an indefinite $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is an indefinite $\mathcal{S}$-space form with $H_p(X) = c$ if and only if the Riemannian $(0,4)$-type curvature tensor field R is given by

$$\begin{aligned} R(X,Y,Z,W) &\\ = &-\frac{c+3\varepsilon}{4}\{g(\varphi Y,\varphi Z)g(\varphi X,\varphi W) - g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\}\\ &- \frac{c-\varepsilon}{4}\big\{\Phi(W,X)\Phi(Z,Y) - \Phi(Z,X)\Phi(W,Y)\\ &+ 2\Phi(X,Y)\Phi(W,Z)\big\} - \big\{\widetilde{\eta}(W)\widetilde{\eta}(X)g(\varphi Z,\varphi Y)\\ &- \widetilde{\eta}(W)\widetilde{\eta}(Y)g(\varphi Z,\varphi X) + \widetilde{\eta}(Y)\widetilde{\eta}(Z)g(\varphi W,\varphi X)\\ &- \widetilde{\eta}(Z)\widetilde{\eta}(X)g(\varphi W,\varphi Y)\big\} \end{aligned} \tag{2.1}$$

for any vector fields X, Y, Z and W on M, where $\varepsilon = \sum_{\alpha=1}^s \varepsilon_\alpha$, $\widetilde{\xi} = \sum_{\alpha=1}^s \xi_\alpha$ and $\widetilde{\eta} = \varepsilon_\alpha \eta^\alpha$.

In regard to the curvature tensor of an indefinite $\mathcal{S}$-manifold, it is important to recall the following formulas, for any $X, Y, Z, W \in \operatorname{Im}\varphi$ and any $\alpha, \beta, \gamma, \delta \in \{1, \dots, s\}$:

$$\begin{aligned} &R(X,\xi_\alpha,X,Y) = 0, \qquad R(\xi_\alpha,X,\xi_\beta,Y) = \varepsilon_\alpha\varepsilon_\beta g(X,Y),\\ &R(\xi_\alpha,X,\xi_\beta,\xi_\gamma) = 0, \qquad R(\xi_\alpha,\xi_\delta,\xi_\beta,\xi_\gamma) = 0,\\ &R(X,Y,\varphi Z,W) + R(X,Y,Z,\varphi W) = \varepsilon P(X,Y;Z,W), \end{aligned} \tag{2.2}$$

where $P(X,Y;Z,W) = \Phi(X,Z)g(Y,W) - \Phi(X,W)g(Y,Z) - \Phi(Y,Z)g(X,W) + \Phi(Y,W)g(X,Z)$.

Finally, we recall some useful properties of a curvature-like algebraic tensor. Let (V, g) be a pseudo-Euclidean real vector space of index ν, $0 < \nu < \dim V$. A multilinear map $F : V^4 \to \mathbb{R}$ is called a *curvature-like map* (or *curvature-like algebraic tensor*) if it satisfies the following conditions:

$$F(y,x,z,w) = -F(x,y,z,w),$$
$$F(z,w,x,y) = F(x,y,z,w),$$
$$F(x,y,z,w) + F(x,z,w,y) + F(x,w,y,z) = 0.$$

For any non-degenerate 2-plane $\pi = \mathrm{span}\{z,w\}$ in V it is possible to define the number

$$k(z,w) = \frac{F(z,w,z,w)}{\Delta(\pi)}.$$

If $k(z,w)$ is constant for any non-degenerate 2-plane and $k(z,w) = k$ then one gets $F(x,y,z,w) = k\,(g(x,z)g(y,w) - g(y,z)g(x,w))$. Now arguments similar to those in [20, Proposition 28, p. 229] can be used to prove the following result.

Lemma 2.1. *Let (V,g) be a Lorentz real vector space and $F : V^4 \to \mathbb{R}$ a curvature-like map. Then the following conditions are equivalent.*

(a) $F(x,y,z,w) = k\,(g(x,z)g(y,w) - g(y,z)g(x,w))$,
(b) $F(x,y,y,x) = 0$ *for any degenerate plane* $\pi = \mathrm{span}\{x,y\}$ *in* V.

3. Null Osserman condition and Lorentz $\mathcal{S}$-manifolds. Let (M,g) be a Lorentz manifold and $p \in M$. Consider $S_p^{\pm}(M) = \{z \in T_pM \mid g_p(z,z) = \pm 1\}$, and let $R_z : z^{\perp} \to z^{\perp}$ be the Jacobi operator with respect to $z \in S_p^{\pm}(M)$.

It is well-known that a Lorentz manifold has constant sectional curvature at a point p if and only if it satisfies the Osserman condition at p, that is, the eigenvalues of R_z are independent of $z \in S_p^{+}(M)$, or equivalently $z \in S_p^{-}(M)$ (see [13]).

Consequently, no Lorentz $\mathcal{S}$-manifold can satisfy the Osserman condition since such a manifold cannot have constant sectional curvature, as remarked in Section 2.

In [12] the authors introduce another Osserman condition, named the null Osserman condition. Namely, let (M,g) be a Lorentz manifold, $p \in M$ and u a null vector in T_pM. Then the orthogonal complement $u^{\perp}$ of u is a degenerate vector space since $\mathrm{span}\{u\} \subset u^{\perp}$. Therefore, one can consider the quotient space $\bar{u}^{\perp} = u^{\perp}/\mathrm{span}\{u\}$ and the canonical projection $\pi : u^{\perp} \to \bar{u}^{\perp}$. It is possible to define a positive definite inner product $\bar{g}$ on $\bar{u}^{\perp}$ putting $\bar{g}(\bar{x},\bar{y}) = g(x,y)$, where, for any $x,y \in u^{\perp}$, $\bar{x} = \pi(x)$ and $\bar{y} = \pi(y)$.

From now on, bar-objects will be geometrical objects related to $\bar{u}^{\perp}$. Let u be a null vector in T_pM; the Jacobi operator with respect to u can be defined by the linear map $\bar{R}_u : \bar{u}^{\perp} \to \bar{u}^{\perp}$ such that $\bar{R}_u\bar{x} = \pi(R(x,u)u)$ ([12] and [13, Definition 3.2.1]).

Clearly, $\bar{R}_u$ is self-adjoint with respect to $\bar{g}$, hence $\bar{R}_u$ is diagonalizable.

In Lorentzian geometry it is well-known that a null vector u and a timelike vector z are never orthogonal. Hence, in a Lorentz manifold (M, g), the *null congruence set* determined by a timelike vector $z \in T_pM$ at p is defined by

$$N(z) = \{u \in T_pM \mid g(u, u) = 0,\ g(u, z) = -1\}.$$

A Lorentz manifold (M, g) is called *null Osserman* with respect to a unit timelike vector $z \in T_pM$ at a point p if the characteristic polynomial of $\bar{R}_u$ is independent of $u \in N(z)$.

Another set associated to a unit timelike vector z in T_pM is the *celestial sphere* $S(z)$ of z given by

$$S(z) = \{x \in z^{\perp} \mid g(x, x) = 1\}.$$

According to a result in [13], using the celestial sphere of z, one can obtain all the elements of $N(z)$. In fact, one has

$$\forall u \in N(z)\ \exists! x \in S(z) \quad u = z + x.$$

It is very natural to use this definition in the context of Lorentz contact manifolds. Lorentz Sasaki space forms are globally null Osserman with respect to the timelike characteristic vector field, as stated in [13].

In a Lorentz $\mathcal{S}$-space form an easy example shows that the null Osserman condition with respect to a timelike characteristic vector does not hold. Indeed, considering the 4-dimensional manifold $U(2)$ and the Lie algebra $\mathfrak{u}(2)$, we denote by ξ_1, ξ_2, X, Y the left-invariant vector fields on $U(2)$ determined, in the same order, by the basis $\{\imath E_{11}, -\imath E_{22}, E_{12} - E_{21}, \imath(E_{12} + E_{21})\}$ of $\mathfrak{u}(2)$, where $(E_{ij})_{i,j\in\{1,2\}}$ is the canonical basis of $\mathfrak{gl}(2, \mathbb{C})$. Let us consider the left-invariant 1-forms η^1 and η^2 determined by the dual 1-forms of $\imath E_{11}$ and $-\imath E_{22}$, respectively, and the left-invariant tensor field φ such that $\varphi(X) = Y$, $\varphi(Y) = -X$ and $\varphi(\xi_1) = \varphi(\xi_2) = 0$. The manifold $U(2)$ is connected and compact with Euler number $\chi(U(2)) = 0$, thus we can define a left-invariant Lorentz metric g such that the vector fields ξ_1, ξ_2, X and Y form an orthonormal basis with $g(\xi_1, \xi_1) = -1$. Such a structure on $U(2)$ has been constructed in the Riemannian context [10] and then adapted to the Lorentzian case [6].

This structure is a normal indefinite $g.f.f$-structure and its associated Sasaki 2-form Φ satisfies $\Phi = d\eta^{\alpha}$ for any $\alpha \in \{1, 2\}$, so that it turns out to be a Lorentz $\mathcal{S}$-structure on $U(2)$. Moreover, one sees at once that $U(2)$ has constant φ-sectional curvature 4. We see that $U(2)$ does not satisfy the null Osserman condition with respect to $(\xi_1)_p$, for any $p \in U(2)$. In fact, fixing $p \in U(2)$ and putting

$$u_1 = X_p + (\xi_1)_p, \qquad u_2 = Y_p + (\xi_1)_p, \qquad u_3 = (\xi_2)_p + (\xi_1)_p,$$

one has $u_1, u_2, u_3 \in N((\xi_1)_p)$. By (2.1), we have

$$R(Y_p, u_1)u_1 = Y_p + 3g(Y_p, \varphi u_1)\varphi u_1 + \widetilde{\eta}(u_1)\widetilde{\eta}(u_1)Y_p = 5Y_p,$$
$$R((\xi_2)_p, u_1)u_1 = \sum_{\alpha=1}^{2}(\xi_\alpha)_p + X_p = (\xi_2)_p + u_1.$$

Analogously, for u_2, we obtain

$$R(X_p, u_2)u_2 = X_p + 3X_p + X_p = 5X_p,$$
$$R((\xi_2)_p, u_2)u_2 = \sum_{\alpha=1}^{2}(\xi_\alpha)_p + Y_p = (\xi_2)_p + u_2.$$

For any $z \in u_3^{\perp}$, we have $R(z, u_3)u_3 = -\widetilde{\eta}(u_3)\widetilde{\eta}(u_3)\varphi^2 z = 0$, since $\widetilde{\eta}(u_3) = 0$.

Then it is evident that the eigenvalues of $\bar{R}_{u_1}$ and $\bar{R}_{u_2}$ are 5 and 1, whereas $\bar{R}_{u_3} = 0$.

4. The φ-Null Osserman condition. The example of $U(2)$ inspired us to introduce a new Osserman condition that will be applied to Lorentz $g.f.f$-manifolds.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, be a Lorentz $g.f.f$-manifold. It is easy to check that the timelike vector field must be a characteristic vector field. Without loss of generality we can assume that ξ_1 is the timelike vector field.

If $s \geq 2$ as in the example of $U(2)$, then the flatness of $\ker\varphi$ influences the behaviour of the Jacobi operators $\bar{R}_{u_\alpha}$ with $u_\alpha = (\xi_1)_p + (\xi_\alpha)_p$, for any $\alpha \in \{2, \dots, s\}$ and $p \in M$. Since the matter is related to the null vector $u_\alpha = (\xi_1)_p + (\xi_\alpha)_p$, we give the following Osserman condition.

Given a point p of M, the set

$$S_\varphi((\xi_1)_p) = S((\xi_1)_p) \cap \operatorname{Im}\varphi_p$$

is called the *φ-celestial sphere* of $(\xi_1)_p$ at p. We define the *φ-null congruence set* $N_\varphi((\xi_1)_p)$, analogous to the null congruence set, putting

$$N_\varphi((\xi_1)_p) = \{u \in T_pM \mid u = (\xi_1)_p + x,\ x \in S_\varphi((\xi_1)_p)\}.$$

Now we are ready to state the definition of the φ-null Osserman condition with respect to the timelike vector $(\xi_1)_p$ at a point $p \in M$.

Definition 4.1. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentz $g.f.f$-manifold, $\dim M = 2n + s$, $n, s \geq 1$, with timelike vector field ξ_1 and consider $p \in M$. The manifold M is called *φ-null Osserman* with respect to $(\xi_1)_p$ at a point $p \in M$ if the characteristic polynomial of $\bar{R}_u$ is independent of $u \in N_\varphi((\xi_1)_p)$, that is, the eigenvalues of $\bar{R}_u$ are independent of $u \in N_\varphi((\xi_1)_p)$.

Remark 4.2. If $(M, \varphi, \xi, \eta, g)$ is a Lorentz almost contact manifold, then it can be considered as a Lorentz $g.f.f$-manifold with $s = 1$. Obviously, $S((\xi)_p) = S_\varphi((\xi)_p)$ and $N((\xi)_p) = N_\varphi((\xi)_p)$, for any $p \in M$. It follows that the null Osserman condition with respect to ξ_p at a point p coincides with the φ-null Osserman condition at the same point.

It is clear that $U(2)$ satisfies the φ-null Osserman condition with respect to $(\xi_1)_p$ at a point $p \in U(2)$. In fact, consider an arbitrary unit vector z of $\operatorname{Im}\varphi_p$ and put $z = aX_p + bY_p$. Setting $u_4 = z + (\xi_1)_p$, we have $u_4 \in N_\varphi((\xi_1)_p)$ and $u_4^\perp = \operatorname{span}\{X_p + a(\xi_1)_p, Y_p + b(\xi_1)_p, (\xi_2)_p\} = \operatorname{span}\{\varphi u_4, u_4, (\xi_2)_p\}$. Then

$$\begin{aligned} &R(\varphi u_4, u_4)u_4 = \varphi u_4 + 3\varphi u_4 + \varphi u_4 = 5\varphi u_4, \\ &R((\xi_2)_p, u_4)u_4 = \sum_{\alpha=1}^{2}(\xi_\alpha)_p - \varphi^2 u_4 = (\xi_2)_p + (\xi_1)_p + z = (\xi_2)_p + u_4. \end{aligned} \tag{4.1}$$

It follows that the eigenvalues of $\bar{R}_u$ are 5 and 1, for any $u = z + (\xi_1)_p$ in $N_\varphi((\xi_1)_p)$ with $z \in \operatorname{Im}\varphi_p$ and $g(z,z) = 1$, hence the eigenvalues of $\bar{R}_u$ are independent of the choice of $u \in N_\varphi((\xi_1)_p)$.

Remark 4.3. It is evident that $u \in N_\varphi((\xi_1)_p)$ if and only if $-u \in N_\varphi(-(\xi_1)_p)$, since $S_\varphi((\xi_1)_p) = S_\varphi(-(\xi_1)_p)$ and $-x, x \in S_\varphi((\xi_1)_p)$. Furthermore, $\bar{u}^\perp = -\bar{u}^\perp$ and $\bar{R}_u = \bar{R}_{-u}$, thus, for any $p \in M$, the φ-null Osserman condition with respect to $(\xi_1)_p$ is equivalent to the φ-null Osserman condition with respect to $-(\xi_1)_p$.

In [13] the null Osserman condition was extended to the whole manifold by giving first the definition of a pointwise null Osserman manifold with respect to a timelike line subbundle L of the tangent bundle, and then of a globally null Osserman manifold with respect to L. Namely, a Lorentz manifold (M, g), $\dim M \geq 3$, is called *pointwise null Osserman with respect to L* if it is null Osserman with respect to each timelike unit $z \in L$, and *globally null Osserman with respect to L* if it is pointwise null Osserman with respect to L and moreover the common characteristic polynomial of all the $\bar{R}_u$'s, for $u \in N(z)$, is independent of the unit $z \in L$.

Looking for a similar extension of the φ-null Osserman condition to the whole Lorentz $g.f.f$-manifold, it is natural to consider the timelike line bundle $L = \operatorname{span}\{\xi_1\}$; hence, considering Remark 4.3, we give the following definition.

Definition 4.4. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentz $g.f.f$-manifold, $\dim M = 2n+s$, $n, s \geq 1$, with timelike vector field ξ_1. Then M is said to be a *globally φ-null Osserman manifold* with respect to ξ_1 if it is φ-null Osserman with respect to $(\xi_1)_p$ for any $p \in M$, and the common characteristic polynomial of all the $\bar{R}_u$'s, for $u \in N_\varphi((\xi_1)_p)$, is independent of the p choice in M.

Looking again at the example of $U(2)$ one can see at once that it is a globally φ-null Osserman manifold with respect to ξ_1. In fact, it is clear that the eigenvalues of $\bar{R}_u$ are independent of the point p.

In the next theorem we prove, more generally, that each Lorentz $\mathcal{S}$-space form satisfies the φ-null Osserman condition.

THEOREM 4.5. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\dim M = 2n + s$, be a Lorentz $\mathcal{S}$-manifold with ξ_1 timelike and constant φ-sectional curvature, and let $p \in M$. Then M satisfies the φ-null Osserman condition with respect to the timelike characteristic vector at p.*

Proof. Let $p \in M$. Denoting by c the φ-sectional curvature, (2.1) holds with $\varepsilon = s - 2$.

Let $u \in N_\varphi((\xi_1)_p)$, $u = (\xi_1)_p + x_1$ with $x_1 \in S_\varphi((\xi_1)_p)$, and consider $x \in u^\perp$. We have

$$g(\varphi u, \varphi u) = g(u, u) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(u)\eta^\alpha(u) = \eta^1(u)\eta^1(u) = 1, \tag{4.2}$$

$$g(\varphi x, \varphi u) = g(x, u) - \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha(x)\eta^\alpha(u) = \eta^1(x). \tag{4.3}$$

By (2.1), (4.2) and (4.3) we compute $R(x, u, u, w)$ for any $w \in T_pM$ to obtain

$$\begin{aligned} R_p(x, u, u, w) & \\ = & -\frac{c + 3(s-2)}{4}\{g(\varphi x, \varphi w) - \eta^1(x)g(\varphi u, \varphi w)\} \\ & -\tfrac{3}{4}(c - s + 2)g(x, \varphi u)g(w, \varphi u) \\ & -\{\widetilde{\eta}(w)\widetilde{\eta}(x) + \widetilde{\eta}(w)\eta^1(x) + g(\varphi w, \varphi x) + \widetilde{\eta}(x)g(\varphi w, \varphi u)\}. \end{aligned} \tag{4.4}$$

Let us consider an orthonormal base $\{x_1, \varphi x_1, x_3, \dots, x_{2n}\}$ of $\operatorname{Im} \varphi_p$. It induces the bases $\mathfrak{B} = \{u, \varphi x_1, (\xi_2)_p, \dots, (\xi_s)_p, x_3, \dots, x_{2n}\}$ of $u^\perp$ and $\overline{\mathfrak{B}} = \{\overline{\varphi x}_1, (\bar{\xi}_2)_p, \dots, (\bar{\xi}_s)_p, \bar{x}_3, \dots, \bar{x}_{2n}\}$ of $\bar{u}^\perp$. For brevity, we denote them by $\mathfrak{B} = \{e_i\}_{1 \le i \le m}$ and $\bar{\mathfrak{B}} = \{\bar{e}_i\}_{1 \le i \le m-1}$, where $m = 2n + s - 1$. In general, for any $x \in u^\perp$,

$$\bar{R}_u(\bar{x}) = -\sum_{i=1}^{m-1} R_p(x, u, u, e_i)\bar{e}_i. \tag{4.5}$$

By (4.4) and (4.5) we get

$$\bar{R}_u(\overline{\varphi x}_1) = \left\{\frac{c + 3(s-2)}{4} + \frac{3}{4}(c - s + 2)\right\}\overline{\varphi x}_1 + \overline{\varphi x}_1 = (c+1)\overline{\varphi x}_1,$$

$$\bar{R}_u(\bar{x}_j) = \frac{c + 3(s-2)}{4}\bar{x}_j + \bar{x}_j = \frac{c + 3s - 2}{4}\bar{x}_j, \quad \forall j \in \{2, \dots, 2n\},$$

$$\bar{R}_u((\bar{\xi}_\beta)_p) = \sum_{\gamma=2}^{s} \widetilde{\eta}((\xi_\beta)_p)\widetilde{\eta}((\xi_\gamma)_p)(\bar{\xi}_\gamma)_p = \sum_{\gamma=2}^{s}(\bar{\xi}_\gamma)_p, \quad \forall \beta \in \{2, \dots, s\}.$$

It follows that the representation matrix of $\bar{R}_u$ with respect to $\bar{\mathfrak{B}}$ is independent of the choice of $u \in N_\varphi((\xi_1)_p)$. In particular, it is easy to compute that the other eigenvalues are 0 and $s-1$, with eigenvectors $\bar{x}_\alpha = (\bar{\xi_2})_p - (\bar{\xi_\alpha})_p$, $\alpha \in \{3, \dots, s\}$, and $\bar{x} = \sum_{\beta=2}^{s} (\bar{\xi_\beta})_p$, respectively. ∎

By the above proof we note that, as for $U(2)$, each Lorentz $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ with $\dim M = 2n + s$ and constant φ-sectional curvature is globally φ-null Osserman with respect to ξ_1.

From now on, since the Osserman conditions are formulated pointwise, to simplify the notation we omit any reference to the point.

5. The φ-null Osserman condition on Lorentz $\mathcal{S}$-manifolds with additional assumptions. In this section we proceed with the study of φ-null Osserman manifolds; we will find an expression for the curvature tensor field of a φ-null Osserman Lorentz $\mathcal{S}$-manifold with two characteristic vector fields, using a suitable expression for null vectors. An analogous statement can be found in different contexts [13]. In the first part of this section we collect the technical issues needed for the main result, which will be provided in the second subsection.

5.1. Technical results. In [14] the authors gave an explicit construction of a complex structure on a $(4m + 2)$-dimensional globally Osserman manifold with exactly two distinct eigenvalues of the Jacobi operators with multiplicities 1 and $4m$ (see also [13]). We will use such a construction, adapting it when the manifold satisfies the φ-null Osserman condition at a point.

Following [12, 13] we recall that if (M, g) is a Lorentz manifold and u is a null vector of T_pM, then a non-degenerate subspace $W \subset u^\perp$ such that $\dim W = \dim \bar{u}^\perp$ is called a *geometric realization* of $\bar{u}^\perp$. Moreover, let $\pi|_W : (W, g) \to (\bar{u}^\perp, g)$ be an isometry, where we use the same letter g to denote the non-degenerate metrics on W and $\bar{u}^\perp$ for simplicity. A vector $x \in W$ is said to be a *geometrically realized eigenvector* of $\bar{R}_u$ in W corresponding to an eigenvalue λ if $\pi|_W(x) = \bar{x}$ is an eigenvector of $\bar{R}_u$ with eigenvalue λ (see [13]).

Remark 5.1. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a $(2n + s)$-dimensional φ-null Osserman Lorentz $\mathcal{S}$-manifold at a point $p \in M$ and $u \in N_\varphi(\xi_1)$. We suppose that the Jacobi operator $\bar{R}_u$, restricted to $u^\perp \cap \operatorname{Im}\varphi$, has exactly two eigenvalues, c_1 and c_2, with multiplicities 1 and $2n - 2$.

Since $u = \xi_1 + x$ with $x \in S_\varphi(\xi_1)$, using (2.2), it is easy to see that the eigenvalues and the eigenvectors of the Jacobi operator $\bar{R}_u$ are connected with those of $R_x|_{x^\perp \cap \operatorname{Im}\varphi}$. Namely, one can prove that $v \in x^\perp \cap \operatorname{Im}\varphi$ is an eigenvector of R_x related to the eigenvalue λ if and only if it is a geometrically realized eigenvector of $\bar{R}_u$ related to the eigenvalue $\lambda + 1$ (see [4]).

Now, fix $p \in M$ and, following [14], identify $S_\varphi(\xi_1) \cong S^{2n-1}$. For any $x \in S^{2n-1}$ we consider the operator $R_x : x^\perp \cap \operatorname{Im}\varphi \to x^\perp \cap \operatorname{Im}\varphi$ and the line bundle over the sphere S^{2n-1} defined by the eigenspace corresponding to the eigenvalue $c_1 - 1$ of R_x. Since any line bundle over a sphere is trivial, we have a map $J : S_\varphi(\xi_1) \to S_\varphi(\xi_1)$ such that $Jx = v_x$ for any $x \in S_\varphi(\xi_1)$, where v is a global unit section of the line bundle. To simplify the writing, we put $\lambda = c_1 - 1$ and $\mu = c_2 - 1$. Then the proofs of the following sequence of claims proceed along the same lines as the proofs in [14], which the reader is referred to for details.

Claim (a). *The map J satisfies $J^2(x) = -x$ and $J(-x) = -J(x)$ for any $x \in S_\varphi(\xi_1)$.*

Considering the 2-plane $V_x = \operatorname{span}\{x, Jx\}$, if w is a unit vector in V_x, then there exists $\theta \in [0, 2\pi[$ such that $w = \cos(\theta)x + \sin(\theta)Jx$. Defining $z(w) = -\sin(\theta)x + \cos(\theta)Jx$, one proves that $z(w)$ is an eigenvector of R_w corresponding to λ, then $z(w) = \pm Jw$. Using this last formula, it follows that $J^2(x) = -x$ and $J(-x) = -J(x)$ for any $x \in S_\varphi(\xi_1)$.

Claim (b). *$J : \operatorname{Im}\varphi \to \operatorname{Im}\varphi$ is linear.*

The map J is extended to $\operatorname{Im}\varphi$ putting $J(ax) = aJ(x)$, where $a \in \mathbb{R}$. Assuming that $J(\cos(\theta)x+\sin(\theta)y) = \cos(\theta)Jx+\sin(\theta)Jy$ for all angles θ and any unit vectors x, y such that $y \perp V_x$, we obtain $J(x'+y') = J(x') + J(y')$, for any x', y' such that $y' \perp V_{x'}$, which implies the claim.

Claim (c). *$J(\cos(\theta)x + \sin(\theta)y) = \cos(\theta)Jx + \sin(\theta)Jy$ for all angles θ and any unit vectors x, y such that $y \perp V_x$.*

Define $J' = \pm J$ and consider $A_\theta = \cos(\theta)x + \sin(\theta)y$, $B_\theta = \cos(\theta)Jx + \sin(\theta)J'y$. Assuming that B_θ is an eigenvector of R_{A_θ}, one has $B_\theta = \pm JA_\theta$ for any angle θ. For $\theta = 0$ one has $B_\theta = JA_\theta$; then the plus sign occurs. For $\theta = \pi/2$ it follows that $J'y = B_\theta = JA_\theta = Jy$, i.e. $J' = J$, which implies the claim.

Claim (d). *$R_{A_\theta}(B_\theta) = \lambda B_\theta$.*

The above formula is equivalent to $R(B_\theta, A_\theta, A_\theta, B_\theta) = -\lambda$. Expanding this last formula in terms of x, Jx, y and $J'y$ one finds

$$R(Jx, x, y, J'y) + R(J'y, x, y, Jx) = \mu - \lambda.$$

After proving the following two technical lemmas, one obtains (d).

Lemma 5.2 ([14]).

(1) *$R(z, v)w = -R(z, w)v$ when v, w and z are unit vectors such that $v \perp w$ and $w, v \perp V_z$.*
(2) *$R(z, v)w = 0$ when v, w and z are unit vectors such that $z \perp V_v$ and $z, v \perp V_w$.*

(3) $2R(x, y, J'y, Jx) = R(Jx, x, y, J'y)$.
(4) $2R(J'y, x, y, Jx) = R(Jx, x, y, J'y)$.

LEMMA 5.3 ([14]). *The curvature tensor satisfies*

$$R(Jx, x, y, J'y) = \pm\frac{2(\mu-\lambda)}{3}.$$

Now we give some remarks about a null vector of a Lorentz $\mathcal{S}$-manifold with two characteristic vector fields and next we prove a lemma.

REMARK 5.4. Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, be a Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1 and u a null vector in T_pM, $p \in M$. Since $TM = \operatorname{Im}\varphi \oplus \ker\varphi$, one can write

$$u = \lambda x + a\xi_1 + b\xi_2,$$

where $x \in \operatorname{Im}\varphi$ such that $g(x, x) = 1$. Since u is a null vector, we have $\lambda^2 + b^2 = a^2$, so there exists $\theta \in [0, 2\pi[$ such that

$$u = a(\cos(\theta)x + \xi_1 + \sin(\theta)\xi_2).$$

We take $a = 1$, since it is not restrictive, hence

$$u = \cos(\theta)x + \xi_1 + \sin(\theta)\xi_2, \tag{5.1}$$

For $\cos(\theta) \neq 0$ consider the vector $w = \tan(\theta)\xi_1 + (1/\cos(\theta))\xi_2$. It is easy to check that w is a unit vector orthogonal to u, therefore

$$u^\perp = \operatorname{span}\{u, \varphi x, x_2, \varphi x_2, \dots, x_n, \varphi x_n, w\}.$$

Any $y \in u^\perp$ can be written as

$$y = \rho u + \nu y' + \kappa w, \tag{5.2}$$

where $y' \in \operatorname{span}\{\varphi x, x_2, \varphi x_2, \dots, x_n, \varphi x_n\} \subset \operatorname{Im}\varphi_p \cap u^\perp$ and $\rho, \kappa, \nu \in \mathbb{R}$.

We define two $(1, 3)$-type tensors S^* and S_* putting

$$S^*(x, y)v = \widetilde{\eta}(y)\widetilde{\eta}(v)x - \widetilde{\eta}(x)\widetilde{\eta}(v)y + g(y, v)\widetilde{\eta}(x)\widetilde{\xi} - g(x, v)\widetilde{\eta}(y)\widetilde{\xi},$$
$$S_*(x, y)v = -g(\varphi y, \varphi v)\varphi^2 x + g(\varphi x, \varphi v)\varphi^2 y.$$

REMARK 5.5. If $u \in N_\varphi(\xi_1)$ and $y \in \operatorname{Im}\varphi \cap u^\perp$, then

$$g(S^*(u, y)u, y) - g(S_*(u, y)u, y) = 0.$$

The following lemma gives an expression for a curvature-like map F that vanishes on a particular type of degenerate 2-plane and has a suitable behaviour with respect to the characteristic vector fields.

LEMMA 5.6. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, be a Lorentz g.f.f-manifold with timelike vector field ξ_1. Let $p \in M$ and let $F : (T_pM)^4 \to \mathbb{R}$ be a curvature-like map such that, for any $x, y, v \in \operatorname{Im}\varphi$ and any $\alpha, \beta, \gamma \in \{1, 2\}$,*

$$\begin{aligned} &F(x, \xi_\alpha, y, v) = 0, && F(\xi_\alpha, x, \xi_\beta, y) = \varepsilon_\alpha\varepsilon_\beta g(x, y), \\ &F(\xi_\alpha, x, \xi_\beta, \xi_\gamma) = 0, && F(\xi_1, \xi_2, \xi_1, \xi_2) = 0. \end{aligned} \tag{5.3}$$

Then the following statements are equivalent:

(a) *F vanishes on any degenerate 2-plane $\pi = \operatorname{span}\{u, y\}$ with $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$,*

(b) *$F(x, y, v, z) = g(S_*(x, y)v, z) - g(S^*(x, y)v, z)$ for any $x, y, v, z \in T_pM$.*

Proof. An easy computation, using Remark 5.5, shows that (b)⇒(a).

Conversely, fix $p \in M$ and consider the curvature-like map H such that, for any $x, y, z, v \in T_pM$,

$$H(x, y, v, z) = F(x, y, z, w) - g(S_*(x, y)v, z) + g(S^*(x, y)v, z). \tag{5.4}$$

Condition (a) and Remark 5.5 imply that H vanishes on any degenerate 2-plane $\operatorname{span}\{u, y\}$ with $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$. We start by proving that H vanishes on any degenerate 2-plane. To see this, let u be a null vector of T_pM, as in (5.1), such that $\cos\theta \neq 0$. By the hypotheses and using (5.2), for any $y \in u^\perp$ we have

$$\begin{aligned}
&g(S_*(u, y)u, y)\\
&\quad = (\rho g(\varphi u, \varphi u) + \nu g(\varphi u, \varphi y'))^2 - g(\varphi u, \varphi u)(\rho^2 g(\varphi u, \varphi u) + \nu^2 g(\varphi y', \varphi y'))\\
&\quad = \rho^2 g(\varphi u, \varphi u)^2 - \rho^2 g(\varphi u, \varphi u)^2 - \nu^2 g(\varphi y', \varphi y') g(\varphi u, \varphi u)\\
&\quad = -\nu^2 g(y', y') g(\varphi u, \varphi u),
\end{aligned}$$

$$\begin{aligned}
g(S^*(u, y)u, y) &= -\widetilde{\eta}(u)\widetilde{\eta}(u) g(y, y),\\
F(u, y, u, y) &= \nu^2 F(u, y', u, y') + 2\kappa\nu F(u, y', u, w) + \kappa^2 F(u, w, u, w)\\
&= \nu^2 \cos^2(\theta) F(x, y', x, y') + (1 - \sin\theta)^2 (\nu^2 g(y', y') + \kappa^2)\\
&= \nu^2 g(\varphi u, \varphi u) F(x, y', x, y') + \widetilde{\eta}(u)\widetilde{\eta}(u) g(y, y)\\
&= \nu^2 g(\varphi u, \varphi u) F(u', y', u', y') - \nu^2 g(\varphi u, \varphi u) g(y', y')\\
&\qquad + \widetilde{\eta}(u)\widetilde{\eta}(u) g(y, y),
\end{aligned}$$

where $u' = x + \xi_1$ which belongs to $N_\varphi(\xi_1)$. Hence

$$H(u, y, u, y) = \nu^2 g(\varphi u, \varphi u) F(u', y', u', y') \tag{5.5}$$

with $u' = x + \xi_1$ and $y \in u^\perp \cap \operatorname{Im}\varphi$.

If $\cos\theta = 0$, then $u = \xi_1 \pm \xi_2$ and $u^\perp = \operatorname{span}\{u\} \oplus \operatorname{Im}\varphi$. By direct computation, it is easy to check that

$$H(u, y, u, y) = 0 \tag{5.6}$$

for any $y \in u^\perp$.

Equations (5.5) and (5.6) clearly imply that H vanishes on any degenerate 2-plane. Applying Lemma 2.1 to H one has

$$\begin{aligned}
F(x, y, v, z) = k(g(x, v)g(y, z) - g(y, v)g(x, z)) + g(S_*(x, y)v, z)&\\
- g(S^*(x, y)v, z).&
\end{aligned} \tag{5.7}$$

By definition of k, using the hypotheses and (5.4), we deduce

$$k = \frac{H(\xi_\alpha, x, \xi_\alpha, x)}{\varepsilon_\alpha g(x,x)} = \frac{F(\xi_\alpha, x, \xi_\alpha, x) - g(x,x)}{\varepsilon_\alpha g(x,x)} = 0.$$

Substituting this in (5.7), we obtain our assertion. ∎

5.2. Main results. Now, we consider the following two standard tensor fields of type $(1,3)$, evaluating them at the point p:

$$\begin{aligned} R^0(x,y)v &= g(\pi^I(y), \pi^I(v))\pi^I(x) - g(\pi^I(x), \pi^I(v))\pi^I(y), \\ R^J(x,y)v &= g\big(J(\pi^I(y)), \pi^I(v)\big)J(\pi^I(x)) - g\big(J(\pi^I(x)), \pi^I(v)\big)J(\pi^I(y)) \\ &\quad + 2g\big(\pi^I(x), J(\pi^I(y))\big)J(\pi^I(v)), \end{aligned}$$

where $\pi^I : T_pM \to \operatorname{Im}\varphi$, is the projection on $\operatorname{Im}\varphi$, and J is an almost Hermitian structure on $\operatorname{Im}\varphi$.

It is useful to note that R^J and R^0 vanish on triplets containing a characteristic vector and that, for any $x, y, v \in T_pM$, they are orthogonal to ξ_1 and ξ_2.

Now we are ready to prove the following result.

Theorem 5.7. *Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1,2\}$ and $n > 1$, be a $(2n+2)$-dimensional Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1. The following three statements are equivalent:*

(a) *M is φ-null Osserman with respect to ξ_1 and for any $u \in N_\varphi(\xi_1)$ the Jacobi operator $\bar{R}_u|_{\operatorname{Im}\varphi\cap u^\perp}$ has exactly two distinct eigenvalues c_1 and c_2 with multiplicities 1 and $2(n-1)$, respectively.*

(b) *There exist an almost Hermitian structure J on $\operatorname{Im}\varphi_p$ and $c_1, c_2 \in \mathbb{R}$ such that, for any $x, y, v \in T_pM$,*

$$R(x,y)v = S^*(x,y)v - S_*(x,y)v + c_2R^0(x,y)v + \frac{c_1 - c_2}{3}R^J(x,y)v.$$

(c) (i) *For any $v \in \operatorname{span}\{\xi_1\}$ and $x \in \xi_1^\perp$,*

$$R(x,v)v = (\eta^1(v))^2(x - \widetilde{\eta}(x)\xi_2).$$

(ii) *There exist an almost Hermitian structure J on $\operatorname{Im}\varphi_p$ and c_1, c_2 in $\mathbb{R}$ such that, for any $v, y, x \in \xi_1^\perp$,*

$$\begin{aligned} R(x,y)v = \eta^2(v)&(\eta^2(y)x - \eta^2(x)y) \\ &+ (g(y,v)\eta^2(x) - g(x,v)\eta^2(y))\widetilde{\xi} \\ &+ g(\varphi y, \varphi v)\varphi^2 x - g(\varphi x, \varphi v)\varphi^2 y + c_2R^0(x,y)v \\ &+ \frac{c_1 - c_2}{3}R^J(x,y)v. \end{aligned}$$

Proof. We begin by proving (a)⇒(b). Under the assumption (a), by Remark 5.1 we know that $\operatorname{Im}\varphi_p$ is endowed with an almost Hermitian structure

J such that Jx is an eigenvector of $\bar{R}_u$ relative to the eigenvalue c_1. To prove (b), we consider the curvature-like map F on T_pM given by

$$F(x,y,v,z) = R(x,y,v,z) + \mu g(R^0(x,y)v,z) + \tau g(R^J(x,y)v,z), \tag{5.8}$$

where $\mu, \tau \in \mathbb{R}$.

We want to apply Lemma 5.6 to F. Concerning the hypotheses of Lemma 5.6, we see at once that F satisfies (5.3) since $F = R$ if one of its four arguments is a characteristic vector and moreover since (2.2) hold. Thus we must only compute $F(u,y,u,y)$ for any degenerate vector $u \in N_\varphi(\xi_1)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$.

Namely, considering a null vector $u \in N_\varphi(\xi_1)$ and a vector $y \in u^\perp \cap \operatorname{Im}\varphi$, we find suitable values of μ and τ in $\mathbb{R}$ for which F vanishes on the degenerate 2-plane $\pi = \operatorname{span}\{u,y\}$.

Putting $y_1 = Jx_1 \in u^\perp$, one computes

$$\begin{aligned} F(y_1,u,u,y_1) &= -g(R(y_1,u)u,y_1) + \mu g(R^0(y_1,u)u,y_1) \\ &\quad + \tau g(R^J(y_1,u)u,y_1) = -c_1 + \mu + 3\tau. \end{aligned} \tag{5.9}$$

Analogously, if y_2 and y_2' are orthonormal eigenvectors of $\bar{R}_u$ with respect to the eigenvalue c_2, then

$$\begin{aligned} F(y_2,u,u,y_2) &= -g(R(y_2,u)u,y_2) + \mu g(R^0(y_2,u)u,y_2) \\ &\quad + \tau g(R^J(y_2,u)u,y_2) = -c_2 + \mu, \end{aligned} \tag{5.10}$$

$$\begin{aligned} F(y_2,u,u,y_2') &= -g(R(y_2,u)u,y_2') + \mu g(R^0(y_2,u)u,y_2') \\ &\quad + \tau g(R^J(y_2,u)u,y_2') = 0, \end{aligned} \tag{5.11}$$

$$\begin{aligned} F(y_2,u,u,y_1) &= -g(R(y_2,u)u,y_1) + \mu g(R^0(y_2,u)u,y_1) \\ &\quad + \tau g(R^J(y_2,u)u,y_1) = 0. \end{aligned} \tag{5.12}$$

Now, imposing $F = 0$, we get

$$\mu = c_2 \quad \text{and} \quad \tau = (c_1 - c_2)/3. \tag{5.13}$$

Therefore, since a vector y in $u^\perp \cap \operatorname{Im}\varphi$ can be written as $y = ay_1 + b_j y_2^j$, where y_1 and y_2^j are eigenvectors of $\bar{R}_u$ in $u^\perp \cap \xi_1^\perp$ corresponding to c_1 and c_2, respectively, by (5.9)–(5.12) we have

$$\begin{aligned} F(y,u,u,y) &= a^2 F(y_1,u,u,y_1) + ab_j F(y_1,u,u,y_2^j) + ab_k F(y_2^k,u,u,y_1) \\ &\quad + b_k b_j F(y_2^k,u,u,y_2^j) = 0. \end{aligned}$$

Therefore, applying Lemma 5.6, we obtain $F(x,y,v,z) = g(S_*(x,y)v,z) - g(S^*(x,y)v,z)$ for any $x,y,v,z \in T_pM$. Then, by (5.8) and (5.13), we get

$$\begin{aligned} R(x,y,v,z) &= g(S_*(x,y)v,z) - g(S^*(x,y)v,z) - c_2 g(R^0(x,y)v,z) \\ &\quad - \frac{c_1 - c_2}{3} g(R^J(x,y)v,z). \end{aligned}$$

Thus

$$R(x,y)v = -S_*(x,y)v + S^*(x,y)v + c_2 R^0(x,y)v + \frac{c_1 - c_2}{3} R^J(x,y)v.$$

The proof (b)⇒(c) is straightforward. In fact, for any $v \in \operatorname{span}\{\xi_1\}$ and $x \in \xi_1^\perp$,

$$R(x,v)v = S^*(x,v)v = (\eta^1(v))^2(x+\widetilde{\eta}(x)\xi_1+\varepsilon_1\widetilde{\eta}(x)\widetilde{\xi}) = (\eta^1(v))^2(x-\widetilde{\eta}(x)\xi_2),$$

which implies (c)(i).

For any $v, y, x \in \xi_1^\perp$, by (b) one gets

$$\begin{aligned} R(x,y)v &= \eta^2(y)\eta^2(v)x - \eta^2(x)\eta^2(v)y + (g(y,v)\eta^2(x) - g(x,v)\eta^2(y))\widetilde{\xi} \\ &\quad + \left(-S_* + c_2 R^0 + \frac{c_1 - c_2}{3} R^J\right)(x,y)v, \end{aligned}$$

which is (c)(ii).

Finally, we prove (c)⇒(a). Consider $u \in N(\xi_1)$ with $u = \xi_1 + x_1$ and put $y_1 = Jx_1$. One has

$$R(y_1,u)u = R(y_1,\xi_1)\xi_1 + R(y_1,x_1)\xi_1 + R(y_1,\xi_1)x_1 + R(y_1,x_1)x_1.$$

Using (c), we have

$$R(y_1,\xi_1)\xi_1 = y_1 \quad \text{and} \quad R(y_1,x_1)x_1 = (c_1 - 1)y_1.$$

By (ii), for any $v \in \xi_1^\perp$, it is clear that

$$\begin{aligned} g(R(y_1,x_1)\xi_1, v) &= -g(R(y_1,x_1)v, \xi_1) = 0, \\ g(R(y_1,\xi_1)x_1, v) &= g(R(x_1,v)y_1, \xi_1) = 0. \end{aligned}$$

On the other hand, if $v = \xi_1$, then

$$g(R(y_1,x_1)\xi_1,\xi_1) = 0, \qquad g(R(y_1,\xi_1)x_1,\xi_1) = -g(y_1,x_1) = 0.$$

Hence, $\bar{R}_u(\overline{y}_1) = c_1\overline{y}_1$.

Analogously, considering $y_2 \in (\operatorname{span}\{x_1, y_1\})^\perp \cap \operatorname{Im}\varphi$, we have

$$R(y_2,u)u = R(y_2,\xi_1)\xi_1 + R(y_2,x_1)\xi_1 + R(y_2,\xi_1)x_1 + R(y_2,x_1)x_1.$$

As for y_1, using (c), it is easy to check that $R(x_1,v)y_2 = 0$ and $R(y_2,x_1)v = 0$. Moreover, applying (i), we get

$$R(y_2,\xi_1)\xi_1 = y_2.$$

The relation (ii) implies

$$R(y_2,x_1)x_1 = (c_2 - 1)y_2.$$

Therefore $\bar{R}_u(\overline{y}_2) = c_2\overline{y}_2$.

Finally, to prove the φ-null Osserman condition, we have to check that no eigenvalue depends on $u \in N_\varphi(\xi_1)$. In fact, by (c) we find

$$R(\xi_2,\xi_1)\xi_1 = 0, \qquad R(\xi_2,x_1)x_1 = g(x_1,x_1)\widetilde{\xi} = \xi_1 + \xi_2.$$

It is easy to see that, for any $v \in \xi_1^\perp$,

$$\begin{aligned} g(R(\xi_2,\xi_1)x_1,v)+g(R(\xi_2,x_1)\xi_1,v) &= g(R(\xi_2,v)x_1,\xi_1)-2g(R(\xi_2,x_1)v,\xi_1) \\ &= 2g(x_1,v)-g(x_1,v)=g(x_1,v). \end{aligned}$$

Moreover, since

$$g(R(\xi_2,\xi_1)x_1,\xi_1)+g(R(\xi_2,x_1)\xi_1,\xi_1)=-g(R(\xi_2,\xi_1)\xi_1,x_1)=0,$$

one obtains $R(\xi_2,\xi_1)x_1+R(\xi_2,x_1)\xi_1=x_1$. Then $R(\xi_2,u)u=\xi_2+\xi_1+x_1=\xi_2+u$, so $\bar R_u(\bar\xi_2)=\bar\xi_2$. This proves (a). ∎

Remark 5.8. Since R has to satisfy the last formula in (2.2), for any $x,y,v,z\in \operatorname{Im}\varphi$ one gets

$$(1-c_2)P(x,y;v,z)+\frac{c_1-c_2}{3}\big(g(R^J(x,y)\varphi v,z)+g(R^J(x,y)v,\varphi z)\big)=0. \tag{5.14}$$

If φx_1 realizes geometrically an eigenvector of $\bar R_u$, with $u=\xi_1+x_1\in N_\varphi(\xi_1)$, related to the eigenvalue c_1, then $\varphi=\pm J$ and (5.14) yields $c_1-4c_2+3=0$, according to the case of Lorentz $\mathcal{S}$-space forms.

By Theorem 4.5, it is a simple matter to prove the following result in the particular case of the Jacobi operator with exactly one eigenvalue.

Proposition 5.9. *Let $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,2\}$ and $n>1$, be a $(2n+2)$-dimensional Lorentz $\mathcal{S}$-manifold with timelike vector field ξ_1. Then M is φ-null Osserman with respect to ξ_1, and the Jacobi operator $\bar R_u|_{u^\perp\cap \operatorname{Im}\varphi}$ has a single eigenvalue λ, if and only if it is a Lorentz $\mathcal{S}$-space form with φ-sectional curvature $c=0$. Moreover, $\lambda=1$.*

Now we deal with the case $n=1$, which is a special case because it is clear that any 4-dimensional Lorentz $g.f.f$-manifold is φ-null Osserman with respect to ξ_1. More precisely, for any $u=\xi_1+x_1\in N_\varphi(\xi_1)$ the only eigenvector of the Jacobi operator $\bar R_u|_{u^\perp\cap\operatorname{Im}\varphi}$ is realized geometrically by φx_1 in $u^\perp\cap\xi_1^\perp$. Unlike the result of Proposition 5.9, the eigenvalue of the Jacobi operator need not be 1, as in the case of $U(2)$. When the only eigenvalue is 1, the φ-sectional curvature is zero.

In order to clarify this statement we give an example. Let $\mathbb{R}^4$ be endowed with the Lorentz $\mathcal{S}$-structure, constructed as follows [5]. Denoting the standard coordinates by $\{x,y,z^1,z^2\}$, we define on $\mathbb{R}^4$ two vector fields and two 1-forms putting

$$\xi_\alpha=\frac{\partial}{\partial z^\alpha},\qquad \eta^\alpha=dz^\alpha+y\,dx,$$

for any $\alpha\in\{1,2\}$. The tensor fields φ and g are given in the standard basis by

$$F := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & y & 0 & 0 \end{pmatrix}, \qquad G := \begin{pmatrix} 1/2 & 0 & -y & y \\ 0 & 1/2 & 0 & 0 \\ -y & 0 & -1 & 0 \\ y & 0 & 0 & 1 \end{pmatrix},$$

respectively. It is easy to check that $(\mathbb{R}^4, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2\}$, is a Lorentz $\mathcal{S}$-manifold with different causal type of the characteristic vector fields. Moreover, it is a Lorentz space form with φ-sectional curvature $c = 0$. Therefore, by (2.1),

$$\begin{aligned} R(X, Y, V) = \widetilde{\eta}(X) g(\varphi V, \varphi Y) \sum_{\alpha=1}^{2} \xi_\alpha - \widetilde{\eta}(Y) g(\varphi V, \varphi X) \sum_{\alpha=1}^{2} \xi_\alpha \\ - \widetilde{\eta}(Y)\widetilde{\eta}(V)\varphi^2 X + \widetilde{\eta}(V)\widetilde{\eta}(X)\varphi^2 Y \end{aligned}$$

for any $X, Y, V \in \mathfrak{X}(\mathbb{R}^4)$. Since $\operatorname{Im}\varphi = \langle X, Y\rangle$, where $X = \sqrt{2}(\frac{\partial}{\partial x} - y\xi_1 - y\xi_2)$ and $Y = \sqrt{2}\frac{\partial}{\partial y}$, one has

$$\bar{R}_u \varphi Z = \varphi Z, \qquad \bar{R}_u \xi_2 = \xi_2,$$

for any $Z = aX + bY$ and $u = \xi_1 + Z$, where $a^2 + b^2 = 1$. Then the only eigenvalue of $\bar{R}_u$, where $u \in N_\varphi(\xi_1)$, is 1.

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