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Regularity for minimizers of functionals with variable growth and discontinuous coefficients

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Abstract. Based on the theory of variable exponent spaces, we study the regularity of local minimizers for a class of functionals with variable growth and discontinuous coefficients. Under suitable assumptions, we obtain local Hölder continuity of minimizers.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain. We study the local minimizers of the functional

$$J(u,E) = \int_E \frac{(A(x)\nabla u \cdot \nabla u)^{p(x)/2}}{p(x)}\,dx, \tag{1.1}$$

where $E \subset\subset \Omega$. The measurable coefficient matrix $A = (a_{ij}(x))_{N\times N}$ is bounded, i.e.

$$\|A\|_\infty = \sup_{x\in\Omega} \|A(x)\| < \infty, \tag{1.2}$$

where $\|A(x)\| = \max_{i,j} |a_{ij}(x)|$, and satisfies the uniform ellipticity condition: there exist positive constants Λ_1, Λ_2 such that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$,

$$\Lambda_1|\xi|^2 \leq A(x)\xi\cdot\xi \leq \Lambda_2|\xi|^2. \tag{1.3}$$

Here $A(x)$ is not necessarily symmetric or continuous.

In this paper, assume that

$$1 < p_- \leq p(x) \leq p_+ < \infty, \tag{1.4}$$

where $p_+ = \sup_{x\in\Omega} p(x)$, $p_- = \inf_{x\in\Omega} p(x)$. We will study the functional J in the framework of variable exponent function spaces, the definitions of which will be given in Section 2.

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Recall that $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a *local minimizer* of J if, for any $\phi \in W^{1,p(\cdot)}(\Omega)$ with $\operatorname{supp}\phi \subset\subset \Omega$, we have

$$J(u, \operatorname{supp}\phi) \le J(u+\phi, \operatorname{supp}\phi). \tag{1.5}$$

Then it follows that

$$\int_{\operatorname{supp}\phi} (A(x)\nabla u\cdot\nabla u)^{(p(x)-2)/2}(A(x)+A(x)^T)\nabla u\cdot\nabla\phi\,dx = 0,$$

i.e. u is a weak solution of

$$\operatorname{div}((A(x)\nabla u\cdot\nabla u)^{(p(x)-2)/2}(A(x)+A(x)^T)\nabla u) = 0, \tag{1.6}$$

where $A(x)^T$ is the transposed matrix of $A(x)$.

The regularity properties of local minimizers of the integral functional

$$F(u) = \int_\Omega f(x,\nabla u)\,dx$$

have been studied extensively, where $f : \Omega\times\mathbb{R}^N \to \mathbb{R}$ satisfies the condition

$$c_1|\xi|^{p(x)} - c_0 \le f(x,\xi) \le c_2|\xi|^{p(x)} - c_0 \tag{1.7}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. The simplest model is the functional $\int_\Omega |\nabla u|^{p(x)}\,dx$ which was investigated by Zhikov [Z87, Z93]. Acerbi and Fusco [AF] studied Hölder regularity of minimizers of F when $p(x)$ takes only two values in Ω. In [Z95], under the geometric condition

$$|p(x)-p(y)| \le \omega(|x-y|) \tag{1.8}$$

for any $x, y \in \Omega$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of $p(x)$ and satisfies

$$\limsup_{R\to 0} \omega(R)\ln(1/R) < \infty, \tag{1.9}$$

Zhikov proved an integrability result of minimizers for F under the hypothesis (1.7). In [FaZ], under assumptions (1.7)–(1.9), Fan and Zhao proved local $C^{0,\alpha}$ continuity of minimizers for some $0 < \alpha \le 1$. In [AM01], for $f(x,\xi)$ satisfying an additional continuity assumption with respect to x and with condition (1.9) reinforced to

$$\limsup_{R\to 0} \omega(R)\ln(1/R) = 0, \tag{1.10}$$

Acerbi and Mingione proved local $C^{0,\alpha}$ regularity of minimizers for any $0 < \alpha < 1$.

Motivated by their work, we prove here local Hölder continuity of minimizers for J under the assumption that the coefficient matrix $A(x)$ is of vanishing mean oscillation.

Denote by $|E|$ the Lebesgue measure of a measurable set E and define the mean value on a ball $B_R(x) \subset \Omega$ of a locally integrable function $v \in L^1_{\mathrm{loc}}(\Omega)$

by

$$v_{B_R(x)} = \fint_{B_R(x)} v\,dy = |B_R(x)|^{-1} \int_{B_R(x)} v\,dy.$$

Recall that a locally integrable function v is of *bounded mean oscillation* if

$$\fint_{B_R(x)} |v(y) - v_{B_R(x)}|\,dy$$

is uniformly bounded as $B_R(x)$ ranges over all balls contained in Ω. If, in addition, these averages tend to zero uniformly as $R \to 0$, we say that v is of *vanishing mean oscillation* and write $v \in \mathrm{VMO}(\Omega)$.

The main difficulty here is that the functional J exhibits variable growth condition and has more complicated nonlinearities. Thus, some techniques used in the constant exponent case cannot be carried out for the variable exponent case. In order to overcome this difficulty, combining with a localization technique associated to (1.1) and estimates in $L\log^\beta L$ spaces (for which we refer to [AM05]) we obtain the following regularity result:

THEOREM 1.1. *Suppose that A is of vanishing mean oscillation, i.e. $a_{ij} \in \mathrm{VMO}(\Omega)$. Let $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of J. Then, under assumptions* (1.4), (1.8) *and* (1.10), *$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $0 < \alpha < 1$.*

2. Preliminaries. In the studies of a class of nonlinear problems with variable exponential growth (see [AH, AS, ACX, CF, FaZ, FuZ, ZF, MR06, MR07, ZZ]), variable exponent spaces play an important role. Since they were thoroughly investigated by Kováčik and Rákosník [KR], variable exponent spaces have been used to model various phenomena. In [R], Růžička presented the mathematical theory for the application of variable exponent Sobolev spaces in electro-rheological fluids. As another application, Chen, Levine and Rao [CLR] suggested a model for image restoration based on a variable exponent Laplacian.

For the convenience of the reader, we recall some definitions and basic properties of variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a domain. For a deeper treatment of these spaces, we refer to [DHHR]. Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to [1,\infty]$. Denote

$$\rho_{p(\cdot)}(u) = \int_{\Omega\setminus\Omega_\infty} |u|^{p(x)}\,dx + \sup_{x\in\Omega_\infty} |u(x)|,$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ is the class of all functions u such that $\rho_{p(\cdot)}(tu) < \infty$ for some $t > 0$. We observe that $L^{p(\cdot)}(\Omega)$ is a

Banach space equipped with the norm

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(\lambda u) \le 1\}.$$

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the class of all functions $u \in L^{p(\cdot)}(\Omega)$ such that $|\nabla u| \in L^{p(\cdot)}(\Omega)$. We remark that $W^{1,p(\cdot)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

By $W_0^{1,p(\cdot)}(\Omega)$ we denote the subspace of $W^{1,p(\cdot)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We know that if $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p \in C(\overline{\Omega})$, then $\|u\|_{1,p(\cdot)}$ and $\|\nabla u\|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$. Under the condition (1.4), $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are reflexive.

Next, we present the following local L^q gradient estimate established in [AM05] which is useful in proving the main results in Section 3.

THEOREM 2.1. *Let $|f|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega)$ with $q > 1$ and $u \in W^{1,p(\cdot)}(\Omega)$ be a weak solution of the equation*

$$\operatorname{div}(a(x,\nabla u)) = \operatorname{div}(|f|^{p(x)-2}f) \quad \text{in } \Omega, \tag{2.1}$$

where $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies

$$|a(x,\xi)| \le c(1+|\xi|^2)^{(p(x)-1)/2} \quad \text{and} \quad c|\xi|^{p(x)} - c \le a(x,\xi)\xi$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$. Take $x_0 \in \Omega$. Then, under assumptions (1.4), (1.8) *and* (1.10), *there exist $0 < \sigma_0 < q-1$, $R_0 > 0$ with $B_{4R_0}(x_0) \subset\subset \Omega$ such that for any $0 < \sigma \le \sigma_0$, $R \le 2R_0$,*

$$\begin{aligned} &\Big(\fint_{B_R(x_0)} |\nabla u|^{p(x)(1+\sigma)}\,dx\Big)^{1/(1+\sigma)} \\ &\quad\le C \fint_{B_{2R}(x_0)} |\nabla u|^{p(x)}\,dx + C\Big(\fint_{B_{2R}(x_0)} |f|^{p(x)(1+\sigma)}\,dx + 1\Big)^{1/(1+\sigma)}, \end{aligned} \tag{2.2}$$

where $\sigma_0 = \sigma_0(p_+,p_-,N,K_0,R_0)$, $K_0 = \int_{B_{4R_0}(x_0)}(|\nabla u|^{p(x)}+1)\,dx$, $C = C(p_+,p_-,N)$.

REMARK. In fact, (2.2) was obtained on the cube $Q_R \subset \Omega$ in [AM05]. Here, we give a different version of local gradient estimate over the ball B_R which is easy to obtain following the discussion in [AM05].

3. Main results. In this section, by using the Morrey–Campanato integral characterization of Hölder continuity, we prove the local Hölder continuity of local minimizers for J.

Throughout this section, we assume that A is of vanishing mean oscillation. Let $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional J under assumptions (1.4), (1.8) and (1.10). In order to obtain Hölder continuity of local minimizers for J, we will use a localization technique.

Firstly, we prove a kind of reverse Hölder type inequalities for u. Note that u is a weak solution of (1.6); then it follows from (1.2) and (1.3) that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^N$,

$$|(A(x)\xi \cdot \xi)^{(p(x)-2)/2}(A(x) + A(x)^T)\xi| \le c|\xi|^{p(x)-1}$$

and

$$c|\xi|^{p(x)} \le (A(x)\xi \cdot \xi)^{(p(x)-2)/2}(A(x) + A(x)^T)\xi \cdot \xi,$$

where $c = c(p_+, p_-, \Lambda_1, \Lambda_2, \|A\|_\infty)$. Take $x_0 \in \Omega$. By Theorem 2.1, there exist $R_0, \sigma_0 > 0$ such that for any $0 < \sigma \le \sigma_0$, $R \le 2R_0$,

$$\Big(\fint_{B_R(x_0)} |\nabla u|^{p(x)(1+\sigma)}\,dx\Big)^{1/(1+\sigma)} \le C \fint_{B_{2R}(x_0)} |\nabla u|^{p(x)}\,dx, \tag{3.1}$$

where $\sigma_0 = \sigma_0(p_+, p_-, N, K_0, R_0)$, $C = C(p_+, p_-, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$, $K_0 = \int_{B_{4R_0}(x_0)}(|\nabla u|^{p(x)} + 1)\,dx$.

Take $0 < \sigma < \min\{\sigma_0, p_- - 1\}$. Then $|\nabla u|^{p(x)} \in L^{1+\sigma}_{\mathrm{loc}}(\Omega)$. For any $\varepsilon \in (0, \sigma)$, by (1.10), there exists $R_1 = R_1(\varepsilon) < R_0$ such that for any $R \le R_1$,

$$\omega(4R) < \min\{\varepsilon, \sigma/4\} \quad \text{and} \quad \omega(4R)\ln(1/4R) < \varepsilon. \tag{3.2}$$

For simplicity, we abbreviate $B_R = B_R(x_0)$ for any $R < R_1$. Denote $A_B = ((a_{ij})_B)_{N\times N}$. Then, by using (1.3), for any $\xi \in \mathbb{R}^N$,

$$\Lambda_1|\xi|^2 \le A_B\xi \cdot \xi \le \Lambda_2|\xi|^2. \tag{3.3}$$

As A is of vanishing mean oscillation, we assume that for any $R < R_1$,

$$\fint_{B_{2R}} \|A - A_{B_{2R}}\|\,dx < \varepsilon^{\frac{p_- -1}{p_-}\frac{\sigma}{1+\sigma}},$$

which implies

$$\Big(\fint_{B_{2R}} \|A - A_{B_{2R}}\|^{\frac{p_-}{p_- -1}\frac{1+\sigma}{\sigma}}\,dx\Big)^{\frac{p_- -1}{p_-}\frac{\sigma}{1+\sigma}} < c\varepsilon, \tag{3.4}$$

where $c = c(p_-, \sigma, \|A\|_\infty)$.

In the following, several technical results will be established. Firstly, we derive a useful local gradient estimate.

LEMMA 3.1. *Denote $p_2 = \sup_{x\in B_{2R}} p(x)$. Then*

$$\int_{B_{2R}} |\nabla u|^{p_2}\,dx \le c, \tag{3.5}$$

where $c = c(p_+, p_-, \sigma, K, N)$ and $K = \int_{B_{4R_0}} |\nabla u|^{p(x)}\,dx + 1$.

Proof. For any $x \in B_{2R}$, by using (1.8) we have

$$p_2 \le p(x) + \omega(4R) \le p(x)(1+\omega(4R)) \le p(x)(1+\sigma),$$

which implies $|\nabla u| \in L^{p_2}(B_{2R})$. It follows from (3.1) and (3.2) that

$$\begin{aligned}\int_{B_{2R}} |\nabla u|^{p_2}\,dx &\le \int_{B_{2R}} (|\nabla u|^{p(x)(1+\omega(4R)} + 1)\,dx \\ &\le c|B_{2R}|\Big\{\Big(\fint_{B_{4R}} |\nabla u|^{p(x)}\,dx\Big)^{1+\omega(4R)} + 1\Big\} \\ &\le c\Big\{|B_{2R}|\cdot|B_{4R}|^{-1-\omega(4R)}\Big(\int_{B_{4R}} |\nabla u|^{p(x)}\,dx\Big)^{1+\omega(4R)} + 1\Big\} \\ &\le c(R^{-N\omega(4R)}K^{1+\sigma} + 1),\end{aligned}$$

where $c = c(p_+, p_-, N)$. ∎

Let $v \in W^{1,p_2}(B_{2R})$ with $v - u \in W_0^{1,p_2}(B_{2R})$ be a minimizer of

$$\int_{B_{2R}} (A_{B_{2R}}\nabla v \cdot \nabla v)^{p_2/2}\,dx.$$

Then, for any $\phi \in W_0^{1,p_2}(B_{2R})$,

$$\begin{aligned}\int_{B_{2R}} (A_{B_{2R}}(\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}}\nabla v \cdot \nabla\phi \\ + A_{B_{2R}}(\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}}\nabla\phi \cdot \nabla v)\,dx = 0.\end{aligned} \tag{3.6}$$

Next, based on regularity results for a type of constant coefficient functionals, we give a comparison estimate associated to u and v in B_{2R}. In the proof, we will use estimates in $L\log^\beta L$ spaces (consult [AM05]) several times.

LEMMA 3.2. *We have*

$$\int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\,dx \le c\varepsilon \int_{B_{4R}} |\nabla u|^{p_2}\,dx + cR^N,$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$.

Proof. Choosing $\phi = u - v$ as a test function in (1.6) and (3.6) we get, respectively,

$$\begin{aligned}\int_{B_{2R}} ((A\nabla u \cdot \nabla u)^{(p(x)-2)/2} A\nabla u \cdot \nabla(u-v) \\ + (A\nabla u \cdot \nabla u)^{(p(x)-2)/2} A\nabla(u-v) \cdot \nabla u)\,dx = 0\end{aligned}$$

and

$$\int_{B_{2R}} (A_{B_{2R}}(\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}} \nabla v \cdot \nabla (u-v) \\ + A_{B_{2R}}(\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}} \nabla (u-v) \cdot \nabla v)\, dx = 0.$$

Then

$$\begin{aligned}
(3.7)\qquad & \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}} \nabla u \cdot \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A_{B_{2R}} \nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}} \nabla v \cdot \nabla (u-v))\, dx \\
& + \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}} \nabla (u-v) \cdot \nabla u\, dx \\
& - \int_{B_{2R}} (A_{B_{2R}} \nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}} \nabla (u-v) \cdot \nabla v\, dx \\
= & \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}} \nabla u \cdot \nabla (u-v)\, dx \\
& + \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}} \nabla (u-v) \cdot \nabla u\, dx \\
& - \int_{B_{2R}} (A \nabla u \cdot \nabla u)^{(p(x)-2)/2} A \nabla u \cdot \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A \nabla u \cdot \nabla u)^{(p(x)-2)/2} A \nabla (u-v) \cdot \nabla u\, dx \\
= & \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}} \nabla u \cdot \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p(x)-2)/2}) A_{B_{2R}} \nabla u \cdot \nabla (u-v)\, dx \\
& + \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p(x)-2)/2} A_{B_{2R}} \nabla u \cdot \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A \nabla u \cdot \nabla u)^{(p(x)-2)/2} A \nabla u \cdot \nabla (u-v)\, dx \\
& + \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p_2-2)/2} (\nabla u)^T A_{B_{2R}} \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}} \nabla (u-v)\, dx \\
& + \int_{B_{2R}} (A_{B_{2R}} \nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}} \cdot \nabla (u-v)\, dx \\
& - \int_{B_{2R}} (A \nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A \cdot \nabla (u-v)\, dx.
\end{aligned}$$

From the algebraic inequalities: for any ξ, $\eta \in \mathbb{R}^N$,

$$|\xi-\eta|^q \le c\big((A_{B_{2R}}\xi\cdot\xi)^{(q-2)/2}A_{B_{2R}}\xi - (A_{B_{2R}}\eta\cdot\eta)^{(q-2)/2}A_{B_{2R}}\eta\big)\cdot(\xi-\eta), \tag{3.8}$$

$$|\xi-\eta|^q \le c\big((A_{B_{2R}}\xi\cdot\xi)^{(q-2)/2}A_{B_{2R}}(\xi-\eta)\cdot\xi - (A_{B_{2R}}\eta\cdot\eta)^{(q-2)/2}A_{B_{2R}}(\xi-\eta)\cdot\eta\big) \tag{3.9}$$

if $2 \le q < \infty$, and

$$|\xi-\eta|^q \le \theta|\eta|^q + c\theta^{(q-2)/q}\big((A_{B_{2R}}\xi\cdot\xi)^{(q-2)/2}A_{B_{2R}}\xi - (A_{B_{2R}}\eta\cdot\eta)^{(q-2)/2}A_{B_{2R}}\eta\big)\cdot(\xi-\eta), \tag{3.10}$$

$$|\xi-\eta|^q \le c\theta^{(q-2)/q}\big((A_{B_{2R}}\xi\cdot\xi)^{(q-2)/2}A_{B_{2R}}(\xi-\eta)\cdot\xi - (A_{B_{2R}}\eta\cdot\eta)^{(q-2)/2}A_{B_{2R}}(\xi-\eta)\cdot\eta\big) + \theta|\eta|^q, \tag{3.11}$$

if $1 < q < 2$, where $\theta \in (0,1)$, we get:

(i) The case $p_2 \ge 2$. By (3.8) and (3.9), we have

$$\begin{aligned}\int_{B_{2R}} |\nabla u-\nabla v|^{p_2}\,dx &\le c\Big\{\int_{B_{2R}} (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u\cdot\nabla(u-v)\,dx \\ &\quad - \int_{B_{2R}} (A_{B_{2R}}\nabla v\cdot\nabla v)^{(p_2-2)/2}A_{B_{2R}}\nabla v\cdot\nabla(u-v)\,dx \\ &\quad + \int_{B_{2R}} (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla(u-v)\cdot\nabla u\,dx \\ &\quad - \int_{B_{2R}} (A_{B_{2R}}\nabla v\cdot\nabla v)^{(p_2-2)/2}A_{B_{2R}}\nabla(u-v)\cdot\nabla v\,dx\Big\}.\end{aligned}$$

Then, by (3.7) and the Young inequality, for any $\tau \in (0,1)$, we obtain

$$\begin{aligned}\int_{B_{2R}} |\nabla u-\nabla v|^{p_2}\,dx &\le \tau\int_{B_{2R}} |\nabla u-\nabla v|^{p_2}\,dx + c(\tau)\int_{B_{2R}} \big|(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u \\ &\qquad - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u\big|^{p_2/(p_2-1)}\,dx \\ &\quad + c(\tau)\int_{B_{2R}} \big|(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u \\ &\qquad - (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}A\nabla u\big|^{p_2/(p_2-1)}\,dx,\end{aligned}$$

$$+ c(\tau) \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} (\nabla u)^T A_{B_{2R}} - (A_{B_{2R}}\nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}}|^{p_2/(p_2-1)}\, dx$$
$$+ c(\tau) \int_{B_{2R}} \left|(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}} - (A\nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A\right|^{p_2/(p_2-1)}\, dx,$$

which implies

$$\begin{aligned}(3.12)\qquad \int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\, dx &\le c \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}}\nabla u - (A_{B_{2R}}\nabla u \cdot \nabla u)^{(p(x)-2)/2} A_{B_{2R}}\nabla u|^{p_2/p_2-1}\, dx \\ &+ c \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p(x)-2)/2} A_{B_{2R}}\nabla u - (A\nabla u \cdot \nabla u)^{(p(x)-2)/2} A\nabla u|^{p_2/p_2-1}\, dx \\ &+ c \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} (\nabla u)^T A_{B_{2R}} - (A_{B_{2R}}\nabla u \cdot\nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}}|^{p_2/(p_2-1)}\, dx \\ &+ c \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A_{B_{2R}} - (A\nabla u \cdot \nabla u)^{(p(x)-2)/2} (\nabla u)^T A|^{p_2/(p_2-1)}\, dx.\end{aligned}$$

(ii) The case $1 < p_2 < 2$. By (3.10) and (3.11), for $\theta \in (0,1)$, we have

$$\begin{aligned}\int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\, dx &\le \theta \int_{B_{2R}} |\nabla u|^{p_2}\, dx + c\theta^{(p_2-2)/p_2} \Big\{ \int_{B_{2R}} \big((A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}}\nabla u \\ &\quad - (A_{B_{2R}}\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}}\nabla v\big) \nabla(u-v)\, dx \\ &\quad + \int_{B_{2R}} ((A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}}\nabla(u-v) \cdot \nabla u \\ &\quad - (A_{B_{2R}}\nabla v \cdot \nabla v)^{(p_2-2)/2} A_{B_{2R}}\nabla(u-v) \cdot \nabla v)\, dx \Big\}.\end{aligned}$$

By (3.7) and the Young inequality, for any $\tau > 0$, we get

$$\begin{aligned}\int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\, dx &\le \theta \int_{B_{2R}} |\nabla u|^{p_2}\, dx + c\theta^{(p_2-2)/p_2} \tau \int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\, dx \\ &\quad + c\theta^{(p_2-2)/p_2} \tau^{-1/(p_2-1)} \Big\{ \int_{B_{2R}} |(A_{B_{2R}}\nabla u \cdot \nabla u)^{(p_2-2)/2} A_{B_{2R}}\nabla u\end{aligned}$$

$$
\begin{aligned}
&- (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u|^{p_2/(p_2-1)}\,dx\\
&+ \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u\\
&- (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}A\nabla u|^{p_2/(p_2-1)}\,dx\\
&+ \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}(\nabla u)^T A_{B_{2R}}\\
&- (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}}|^{p_2/(p_2-1)}\,dx\\
&+ \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}}\\
&- (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A|^{p_2/(p_2-1)}\,dx\Big\}.
\end{aligned}
$$

Take $\tau > 0$ such that $c\theta^{(p_2-2)/p_2}\tau = 1/2$ to obtain

$$
\begin{aligned}
(3.13)\qquad &\int\limits_{B_{2R}} |\nabla u - \nabla v|^{p_2}\,dx\\
&\le 2\theta \int\limits_{B_{2R}} |\nabla u|^{p_2}\,dx + c\theta^{(p_2-2)/(p_2-1)}\cdot\Big\{\int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u\\
&\qquad - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u|^{p_2/(p_2-1)}\,dx\\
&\qquad + \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u\\
&\qquad - (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}A\nabla u|^{p_2/(p_2-1)}\,dx\\
&\qquad + \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}(\nabla u)^T A_{B_{2R}}\\
&\qquad - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}}|^{p_2/(p_2-1)}\,dx\\
&\qquad + \int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}}\\
&\qquad - (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A|^{p_2/(p_2-1)}\,dx\Big\}.
\end{aligned}
$$

In the following, we consider

$$
\int\limits_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u|^{p_2/(p_2-1)}\,dx.
$$

For any $x \in B_{2R}$, by the Lagrange mean value theorem, there exists $\gamma \in (0,1)$ such that

$$
\begin{aligned}
&(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u\\
&\quad = \frac{p_2-p(x)}{2}\ln(A_{B_{2R}}\nabla u\cdot\nabla u)(A_{B_{2R}}\nabla u\cdot\nabla u)^{((p(x)-2)+\gamma(p_2-p(x)))/2}A_{B_{2R}}\nabla u.
\end{aligned}
$$

Formulas (1.8) and (3.3), lead to

$$\begin{aligned}(3.14)\qquad &\int_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u\\ &\qquad - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u|^{p_2/(p_2-1)}\,dx\\ &\le c\int_{B_{2R}} |\omega(4R)|\nabla u|^{p(x)-2+\gamma(p_2-p(x))+1}\\ &\qquad \times \big(|\ln \Lambda_1|+|\ln\Lambda_2|+\big|\ln|\nabla u|\big|\big)\big|^{p_2/(p_2-1)}\,dx,\end{aligned}$$

where $c=c(p_+,p_-,\Lambda_2,N,\|A\|_\infty)$. Note that

$$\begin{aligned}(3.15)\qquad &\int_{B_{2R}} \big||\nabla u|^{(p(x)-2)+\gamma(p_2-p(x))+1}\big(|\ln\Lambda_1|+|\ln\Lambda_2|+\big|\ln|\nabla u|\big|\big)\big|^{p_2/(p_2-1)}\,dx\\ &= \int_{\{x\in B_{2R}:|\nabla u(x)|>e\}} \big||\nabla u|^{p(x)-2+\gamma(p_2-p(x))+1}\\ &\qquad\times\big(|\ln\Lambda_1|+|\ln\Lambda_2|+\big|\ln|\nabla u|\big|\big)\big|^{p_2/(p_2-1)}\,dx\\ &\quad+\int_{\{x\in B_{2R}:|\nabla u(x)|\le e\}} \big||\nabla u|^{p(x)-2+\gamma(p_2-p(x))+1}\\ &\qquad\times\big(|\ln\Lambda_1|+|\ln\Lambda_2|+\big|\ln|\nabla u|\big|\big)\big|^{p_2/p_2-1}\,dx\\ &\le c\int_{B_{2R}} |\nabla u|^{p_2}\ln^{p_2/(p_2-1)}(e+|\nabla u|^{p_2})\,dx + c|B_{2R}|\\ &= c\int_{B_{2R}} |\nabla u|^{p_2}\ln^{p_2/(p_2-1)}\big(e+|\nabla u|^{p_2}\big\||\nabla u|^{p_2}\big\|_1^{-1}\big\||\nabla u|^{p_2}\big\|_1\big)\,dx + c|B_{2R}|\\ &\le c\int_{B_{2R}} |\nabla u|^{p_2}\ln^{p_2/(p_2-1)}\big(e+|\nabla u|^{p_2}\big\||\nabla u|^{p_2}\big\|_1^{-1}\big)\,dx\\ &\quad + c\int_{B_{2R}} |\nabla u|^{p_2}\ln^{p_2/(p_2-1)}\big(e+\big\||\nabla u|^{p_2}\big\|_1\big)\,dx + c|B_{2R}|,\end{aligned}$$

where $c=c(p_+,p_-,\Lambda_1,\Lambda_2)$, $\big\||\nabla u|^{p_2}\big\|_1 = \fint_{B_{2R}}|\nabla u|^{p_2}\,dx$.

For any $x\in B_{2R}$, we get

$$\begin{aligned}p_2(1+\sigma/4) &\le (p(x)+\omega(4R))(1+\sigma/4)\le p(x)(1+\sigma/4)+\omega(4R)p_-\\ &\le p(x)(1+\sigma/4+\omega(4R)).\end{aligned}$$

Then, from the estimate in $L\log^\beta L(\Omega)$: for any $f\in L\log^\beta L(\Omega)$,

$$\fint_\Omega |f|\ln^\beta\Big(e+\frac{|f|}{\|f\|_1}\Big)\,dx\le c\Big(\fint_\Omega |f|^\gamma\,dx\Big)^{1/\gamma},$$

we deduce

$$\int\limits_{B_{2R}} |\nabla u|^{p_2} \ln^{p_2/(p_2-1)} \big(e + |\nabla u|^{p_2} \big\| \, |\nabla u|^{p_2} \big\|_1^{-1}\big)\, dx$$
$$\le c|B_{2R}| \Big(\fint\limits_{B_{2R}} |\nabla u|^{p_2(1+\sigma/4)}\, dx \Big)^{4/(4+\sigma)}$$
$$\le c|B_{2R}| \Big(\fint\limits_{B_{2R}} |\nabla u|^{p(x)(1+\sigma/4+\omega(4R))}\, dx + 1 \Big)^{4/(4+\sigma)}.$$

Similarly to the proof of Lemma 3.1, we obtain

$$\int\limits_{B_{2R}} |\nabla u|^{p_2} \ln^{p_2/(p_2-1)} \big(e + |\nabla u|^{p_2} \big\| \, |\nabla u|^{p_2} \big\|_1^{-1}\big)\, dx \tag{3.16}$$
$$\le c|B_{2R}| \Big\{ \Big(\fint\limits_{B_{4R}} |\nabla u|^{p(x)}\, dx \Big)^{1+\frac{4}{4+\sigma}\omega(4R)} + 1 \Big\}$$
$$= c|B_{2R}| \Big\{ \fint\limits_{B_{4R}} |\nabla u|^{p(x)}\, dx \cdot |B_{4R}|^{-\frac{4}{4+\sigma}\omega(4R)} \Big(\int\limits_{B_{4R}} |\nabla u|^{p(x)}\, dx \Big)^{\frac{4}{4+\sigma}\omega(4R)} + 1 \Big\}$$
$$\le c|B_{2R}| \Big(R^{-\frac{4N}{4+\sigma}\omega(4R)} K^{\frac{4}{4+\sigma}\omega(4R)} \fint\limits_{B_{4R}} |\nabla u|^{p(x)}\, dx + 1 \Big)$$
$$\le c|B_{2R}| \Big(R^{-\frac{4N}{4+\sigma}\omega(4R)} K^{\frac{4}{4+\sigma}\omega(4R)} \fint\limits_{B_{4R}} |\nabla u|^{p_2}\, dx + R^{-\frac{4N}{4+\sigma}\omega(4R)} K^{\frac{4}{4+\sigma}\omega(4R)} + 1 \Big)$$
$$\le c R^{-\frac{4N}{4+\sigma}\omega(4R)} K^{\sigma} \int\limits_{B_{4R}} |\nabla u|^{p_2}\, dx + c(R^{-\frac{4N}{4+\sigma}\omega(4R)} K^{\sigma} + 1)|B_{2R}|$$
$$\le c \int\limits_{B_{4R}} |\nabla u|^{p_2}\, dx + c|B_{2R}|,$$

where $c = c(p_+, p_-, \sigma, K, N)$, and further

$$\int\limits_{B_{2R}} |\nabla u|^{p_2} \ln^{p_2/(p_2-1)} \big(e + \big\| \, |\nabla u|^{p_2} \big\|_1\big)\, dx \tag{3.17}$$
$$= \int\limits_{B_{2R}} |\nabla u|^{p_2}\, dx \cdot \ln^{p_2/(p_2-1)} \Big(|B_{2R}|^{-1} \cdot |B_{2R}| e + |B_{2R}|^{-1} \int\limits_{B_{2R}} |\nabla u|^{p_2}\, dx \Big)$$
$$\le c \Big\{ \int\limits_{B_{2R}} |\nabla u|^{p_2}\, dx \cdot \ln^{p_2/(p_2-1)} (|B_{2R}|^{-1}) + \int\limits_{B_{2R}} |\nabla u|^{p_2}\, dx \cdot \ln^{p_2/(p_2-1)} \Big(|B_{2R}| e + \int\limits_{B_{2R}} |\nabla u|^{p_2}\, dx \Big) \Big\}$$

$$\leq c \int_{B_{2R}} |\nabla u|^{p_2}\,dx \cdot \ln^{p_2/(p_2-1)}(|B_{2R}|^{-1}) + c\Big(|B_{2R}|e + \int_{B_{2R}} |\nabla u|^{p_2}\,dx\Big)^{1+\sigma}$$
$$\leq c \int_{B_{2R}} |\nabla u|^{p_2}\,dx \cdot \ln^{p_2/(p_2-1)}(|B_{2R}|^{-1}) + c(|B_{2R}|e + \int_{B_{2R}} |\nabla u|^{p_2}\,dx),$$

where $c = c(p_+, p_-, \sigma, K, N)$. Then, by using (3.14)–(3.17) we get

$$\begin{aligned}(3.18)\quad &\int_{B_{2R}} |(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}A_{B_{2R}}\nabla u \\ &\qquad - (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u|^{p_2/(p_2-1)}\,dx \\ &\leq c\omega(4R)^{p_2/(p_2-1)}\Big\{\int_{B_{4R}}|\nabla u|^{p_2}\,dx + \int_{B_{2R}}|\nabla u|^{p_2}\,dx\cdot\ln^{p_2/(p_2-1)}(|B_{2R}|^{-1}) + |B_{2R}|\Big\} \\ &\leq c\varepsilon^{p_2/(p_2-1)}\int_{B_{4R}}|\nabla u|^{p_2}\,dx + c\varepsilon^{p_2/(p_2-1)}|B_{2R}|,\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N)$.

Next, we consider

$$\int_{B_{2R}} \big|(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u - (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}A\nabla u\big|^{p_2/(p_2-1)}\,dx.$$

Note that

$$(p(x)-1)\frac{p_2}{p_2-1} \leq p(x).$$

By using the Hölder inequality and (3.1), (3.4), we have

$$\begin{aligned}(3.19)\quad &\int_{B_{2R}} \big|(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}A_{B_{2R}}\nabla u \\ &\qquad - (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}A\nabla u\big|^{p_2/(p_2-1)}\,dx \\ &\leq c\int_{B_{2R}} \big|\,\|A - A_{B_{2R}}\|\cdot|\nabla u|^{(p(x)-1)}\big|^{p_2/(p_2-1)}\,dx \\ &= c\int_{B_{2R}} \|A - A_{B_{2R}}\|^{p_2/(p_2-1)}|\nabla u|^{(p(x)-1)p_2/(p_2-1)}\,dx \\ &\leq c\int_{B_{2R}} \|A - A_{B_{2R}}\|^{p_2/(p_2-1)}(|\nabla u|^{p(x)}+1)\,dx \\ &\leq c|B_{2R}|\Big(\fint_{B_{2R}} \|A - A_{B_{2R}}\|^{\frac{p_2}{p_2-1}\frac{1+\sigma}{\sigma}}\,dx\Big)^{\sigma/(1+\sigma)} \\ &\qquad\times\Big(\fint_{B_{2R}} (|\nabla u|^{p(x)}+1)^{1+\sigma}\,dx\Big)^{1/(1+\sigma)}\end{aligned}$$

$$\le c\varepsilon^{p_2/(p_2-1)}|B_{2R}|\Big(\fint_{B_{4R}} |\nabla u|^{p(x)}\,dx + 1\Big)$$
$$\le c\varepsilon^{p_2/(p_2-1)}|B_{2R}|\Big(\fint_{B_{4R}} |\nabla u|^{p_2}\,dx + 1\Big)$$
$$\le c\varepsilon^{p_2/(p_2-1)}\Big(\int_{B_{4R}} |\nabla u|^{p_2}\,dx + |B_{2R}|\Big),$$

where $c = c(p_+, p_-, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$. Similarly we verify that

$$\begin{aligned}\int_{B_{2R}} \big|&(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p_2-2)/2}(\nabla u)^T A_{B_{2R}} \\ &- (A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}}\big|^{p_2/(p_2-1)}\,dx \\ &\le c\varepsilon^{p_2/(p_2-1)}\int_{B_{4R}} |\nabla u|^{p_2}\,dx + c\varepsilon^{p_2/(p_2-1)}|B_{2R}|\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\int_{B_{2R}} \big|&(A_{B_{2R}}\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A_{B_{2R}} \\ &- (A\nabla u\cdot\nabla u)^{(p(x)-2)/2}(\nabla u)^T A\big|^{p_2/(p_2-1)}\,dx \\ &\le c\varepsilon^{p_2/(p_2-1)}\Big(\int_{B_{4R}} |\nabla u|^{p_2}\,dx + |B_{2R}|\Big),\end{aligned} \tag{3.21}$$

where $c = c(p_+, p_-, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$.

It follows from (3.12)–(3.21) that

$$\begin{aligned}\int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\,dx &\le 2\theta \int_{B_{2R}} |\nabla u|^{p_2}\,dx \\ &+ c\theta^{(p_2-2)/(p_2-1)}\varepsilon^{p_2/(p_2-1)}\int_{B_{4R}} |\nabla u|^{p_2}\,dx + c\theta^{(p_2-2)/(p_2-1)}\varepsilon^{p_2/(p_2-1)}|B_{2R}|,\end{aligned}$$

where $c = c(p_+, p_-, \sigma, K, N, \Lambda_1, \Lambda_2, \|A\|_\infty)$. Take $\theta = \varepsilon$ to get

$$\int_{B_{2R}} |\nabla u - \nabla v|^{p_2}\,dx \le c\varepsilon \int_{B_{4R}} |\nabla u|^{p_2}\,dx + cR^N.$$

This completes the proof of Lemma 3.2. ∎

Finally, we complete the proof of the main result using a slight modification of a technical iteration lemma (see Lemma 3.2 in [AM01]) and Lemma 3.2.

Proof of Theorem 1.1. By using the regularity results of v ([FF]) and Lemma 3.2, we obtain

$$\begin{aligned}\int_{B_\rho} |\nabla u|^{p_2}\,dx &\le 2^{p_+}\int_{B_\rho} |\nabla v|^{p_2}\,dx + 2^{p_+}\int_{B_\rho} |\nabla u-\nabla v|^{p_2}\,dx\\ &\le c\left(\frac{\rho}{2R}\right)^N \int_{B_{2R}} |\nabla u|^{p_2}\,dx + c\varepsilon \int_{B_{4R}} |\nabla u|^{p_2}\,dx + cR^N\end{aligned}$$

for any $0<\rho<R$, where $c=c(p_+,p_-,\sigma,K,N,\Lambda_1,\Lambda_2,\|A\|_\infty)$.

Let $p_2(\rho)=\sup_{x\in B_{\rho/2}} p(x)$. Then

$$\begin{aligned}\int_{B_\rho} (|\nabla u|^{p_2(\rho)}+1)\,dx &\le 2\int_{B_\rho} (|\nabla u|^{p_2}+1)\,dx\\ &\le c((\rho/2R)^N+\varepsilon)\int_{B_{4R}} |\nabla u|^{p_2}\,dx + cR^N,\end{aligned}$$

which implies

$$\int_{B_\rho} |\nabla u|^{p_2(\rho)}\,dx \le c((\rho/R')^N+\varepsilon)\int_{B_{R'}} |\nabla u|^{p_2(R')}\,dx + cR^N,$$

for any $0<\rho<\frac{1}{8}R'\le\frac{1}{2}R_1$. By Lemma 3.2 in [AM01], for any $0<\tau<N$, there exists $\varepsilon_0(c,\tau,N)>0$ such that if $\varepsilon<\varepsilon_0$, we have

$$\int_{B_\rho} |\nabla u|^{p_2(\rho)}\,dx \le c(\rho/R')^{N-\tau}\Big(\int_{B_{R'}} |\nabla u|^{p_2(R')}\,dx + R^{N-\tau}\Big)$$

for any $0<\rho<\frac{1}{16}R'$.

Take $\varepsilon<\min\{\sigma,\varepsilon_0\}$. Then there exists $R_1>0$ such that for any $R\le R_1$, formula (3.2) is satisfied. Thus,

$$\int_{B_\rho} |\nabla u|^{p_2(\rho)}\,dx \le c(\rho/R_1)^{N-\tau}\Big(\int_{B_{R_1}} |\nabla u|^{p_2(R_1)}\,dx + R_1^{N-\tau}\Big) \le c\rho^{N-\tau} \tag{3.22}$$

for any $0<\rho<\frac{1}{16}R_1$. Therefore,

$$\int_{B_\rho} |\nabla u|^{p_-}\,dx \le \int_{B_\rho} |\nabla u|^{p_2(\rho)}\,dx + |B_\rho| \le c\rho^{N-\tau}. \tag{3.23}$$

As $p_-<N$, by using Morrey–Campanato's integral characterization of Hölder continuity together with a standard covering argument, we get the result. ∎

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