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On a homology of algebras with unit

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Abstract. We present a very general construction of a chain complex for an arbitrary (even non-associative and non-commutative) algebra with unit and with any topology over a field with a suitable topology. We prove that for the algebra of smooth functions on a smooth manifold with the weak topology the homology vector spaces of this chain complex coincide with the classical singular homology groups of the manifold with real coefficients. We also show that for an associative and commutative algebra with unit endowed with the discrete topology this chain complex is dual to the de Rham complex.

1. Introduction. Investigating how the exterior derivative acts on linear natural operators lifting differential forms on smooth manifolds to product preserving bundle functors leads to a chain complex of any Weil algebra (see [2] and [3]). But the same construction may be carried out in a more general situation, namely for any (even non-associative and non-commutative) algebra with unit and with any topology over a field with a suitable topology, for instance, over the field of real or complex numbers with the usual topology. In our paper we present this general construction. We also show that, for the algebra of smooth functions on a smooth manifold with the weak topology over the field of real numbers with the usual topology, the homology vector spaces of this chain complex coincide with the classical differentiable singular homology groups of the manifold with real coefficients. This fact is precisely formulated in Theorem 3.9, which is the main result of the paper.

Whereas the injectivity of the isomorphism in Theorem 3.9 can be deduced from the de Rham theorem, its surjectivity seems to be much harder to prove. For this reason our proof is based on sheaf theory. In Theorem 3.2 we construct a fine resolution of the constant sheaf and in Theorem 3.8 we show the completeness of the presheaves with which the sheaves of this resolution are associated. For the notions and theorems of sheaf theory which we use we refer the reader to [7].

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We finish the paper with Theorem 4.1 showing that our construction with the discrete topology on the algebra yields the chain complex dual to the well known de Rham complex in the special case when the de Rham complex is defined, that is, when the algebra is associative and commutative. From this theorem we also conclude that the topology on the algebra with unit plays a vital role in the construction considered.

The construction described here is relatively simple and very general (and can be easily generalized even more in one way or another). We have proved that the chain complex, which it gives in a special but important case, has good homology vector spaces, so we may hope that it can also be useful when applied to some other algebras with unit which appear in different branches of mathematics. Our paper is intended as a motivation for further studies of the subject, in particular, for studying the André–Quillen homology of commutative algebras in the context of our construction.

2. The general construction. Let A be a vector space over a field F, and let $A\times A\to A$ with $(a,b)\mapsto ab$ be a bilinear map. Suppose that there is $1\in A$ such that $1a=a$ and $a1=a$ for every $a\in A$. Moreover, we have some topologies on A and F, the latter being such that the maps $F\times F\to F$ with $(\alpha,\beta)\mapsto\alpha+\beta$ and $F\to F$ with $\beta\mapsto\alpha\beta$ for every $\alpha\in F$ are continuous.

We shall construct a chain complex

$$C_0A \overset{\partial_1}{\longleftarrow} C_1A \overset{\partial_2}{\longleftarrow} C_2A \overset{\partial_3}{\longleftarrow} \cdots \tag{2.1}$$

of vector spaces over F.

For each $p\in\mathbb{N}$ we define C_pA to be the vector space over F consisting of all continuous $(p+1)$-linear maps $f : A\times\cdots\times A\to F$ which are skew-symmetric in the last p variables and satisfy

$$\begin{aligned}(2.2)\qquad & f(a,b_1,\ldots,b_{q-1},cd,b_{q+1},\ldots,b_p)\\ &= f(ac,b_1,\ldots,b_{q-1},d,b_{q+1},\ldots,b_p)+f(da,b_1,\ldots,b_{q-1},c,b_{q+1},\ldots,b_p)\end{aligned}$$

for every $q\in\{1,\ldots,p\}$ and all $a,b_1,\ldots,b_{q-1},b_{q+1},\ldots,b_p,c,d\in A$.

For every $p\ge 1$ and every $f\in C_pA$ we define $\partial_pf : A\times\cdots\times A\to F$ by the formula

$$(\partial_pf)(a,b_1,\ldots,b_{p-1}) = f(1,a,b_1,\ldots,b_{p-1}). \tag{2.3}$$

We claim that $\partial_pf\in C_{p-1}A$. Obviously, ∂_pf is continuous, p-linear and skew-symmetric in the last $p-1$ variables. Furthermore, if $p\ge 2$, then

$$\begin{aligned}(\partial_pf)(a,cd,b_2,\ldots,b_{p-1}) &= f(1,a,cd,b_2,\ldots,b_{p-1})\\ &= f(c,a,d,b_2,\ldots,b_{p-1})+f(d,a,c,b_2,\ldots,b_{p-1})\\ &= f(a,c,d,b_2,\ldots,b_{p-1})+f(c,a,d,b_2,\ldots,b_{p-1})\\ &\quad + f(d,a,c,b_2,\ldots,b_{p-1})+f(a,d,c,b_2,\ldots,b_{p-1})\end{aligned}$$

$$= f(1, ac, d, b_2, \ldots, b_{p-1}) + f(1, da, c, b_2, \ldots, b_{p-1})$$
$$= (\partial_p f)(ac, d, b_2, \ldots, b_{p-1}) + (\partial_p f)(da, c, b_2, \ldots, b_{p-1})$$

for all $a, b_2, \ldots, b_{p-1}, c, d \in A$, as desired. Of course, $\partial_p : C_p A \to C_{p-1} A$ is linear.

What is left is to check that $\partial_{p-1}\partial_p = 0$ for every $p \geq 2$. We have

$$\begin{aligned}(\partial_{p-1}(\partial_p f))(a, b_1, \ldots, b_{p-2}) &= (\partial_p f)(1, a, b_1, \ldots, b_{p-2}) \\ &= f(1, 1, a, b_1, \ldots, b_{p-2})\end{aligned}$$

for every $f \in C_p A$ and all $a, b_1, \ldots, b_{p-2} \in A$. But $f(1, 1, a, b_1, \ldots, b_{p-2}) = 0$, because

$$\begin{aligned}f(1, 1, a, b_1, \ldots, b_{p-2}) &= f(1, 11, a, b_1, \ldots, b_{p-2}) \\ &= f(11, 1, a, b_1, \ldots, b_{p-2}) + f(11, 1, a, b_1, \ldots, b_{p-2}) \\ &= f(1, 1, a, b_1, \ldots, b_{p-2}) + f(1, 1, a, b_1, \ldots, b_{p-2}).\end{aligned}$$

Thus our construction is complete. As usual, we can define the homology vector spaces $H_p A = \ker \partial_p / \operatorname{im} \partial_{p+1}$ for every $p \geq 1$ and $H_0 A = C_0 A / \operatorname{im} \partial_1$.

3. The case of a smooth manifold. Let M be an n-dimensional manifold of class $\mathcal{C}^\infty$, which is Hausdorff and satisfies the second axiom of countability. Of course, the set $\mathcal{C}^\infty(M, \mathbb{R})$ of all functions $M \to \mathbb{R}$ of class $\mathcal{C}^\infty$ is an $\mathbb{R}$-algebra with unit and $\mathbb{R}$ has the usual topology. We will consider $\mathcal{C}^\infty(M, \mathbb{R})$ as a topological space with the topology called $\mathcal{C}^\infty$-*compact-open* or *weak*. This topology is defined by the subbase consisting of the sets

$$\left\{\psi \in \mathcal{C}^\infty(M, \mathbb{R}) : \forall_{x \in K} \left| \frac{\partial^{|\alpha|}(\psi \circ \xi^{-1})}{\partial \xi^\alpha}(\xi(x)) - \frac{\partial^{|\alpha|}(\varphi \circ \xi^{-1})}{\partial \xi^\alpha}(\xi(x)) \right| < \varepsilon \right\},$$

where $\varphi \in \mathcal{C}^\infty(M, \mathbb{R})$, (U, ξ) is a chart on M, $K \subset U$ is compact, $\alpha \in \mathbb{N}^n$ and $\varepsilon > 0$ (see for instance [5]).

Thus we have everything we need to construct the chain complex

$$C_0(\mathcal{C}^\infty(M, \mathbb{R})) \xleftarrow{\partial_{1,M}} C_1(\mathcal{C}^\infty(M, \mathbb{R})) \xleftarrow{\partial_{2,M}} C_2(\mathcal{C}^\infty(M, \mathbb{R})) \xleftarrow{\partial_{3,M}} \cdots \tag{3.1}$$

in the manner described in Section 2. Our goal is to prove that the homology vector spaces $H_p(\mathcal{C}^\infty(M, \mathbb{R}))$ of this chain complex coincide with those of the classical singular homology theory with real coefficients.

For each open subset U of M we have the chain complex

$$C_0(\mathcal{C}^\infty(U, \mathbb{R})) \xleftarrow{\partial_{1,U}} C_1(\mathcal{C}^\infty(U, \mathbb{R})) \xleftarrow{\partial_{2,U}} C_2(\mathcal{C}^\infty(U, \mathbb{R})) \xleftarrow{\partial_{3,U}} \cdots$$

constructed in the manner described in Section 2 for the $\mathbb{R}$-algebra $\mathcal{C}^\infty(U, \mathbb{R})$ with the C^∞-compact-open topology. We also define $\partial_{0,U} : C_0(\mathcal{C}^\infty(U, \mathbb{R})) \to \mathbb{R}$ by the formula

$$\partial_{0,U}(f) = f(1). \tag{3.2}$$

Since $f(1,1)=0$ for every $f \in C_{1,U}(\mathcal{C}^\infty(U,\mathbb{R}))$, we see that $\partial_{0,U}\partial_{1,U}=0$ and we obtain the chain complex

$$\mathbb{R} \xleftarrow{\partial_{0,U}} C_0(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{1,U}} C_1(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{2,U}} C_2(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{3,U}} \cdots$$

of real vector spaces (sometimes it will be convenient to write $C_{-1}(\mathcal{C}^\infty(U,\mathbb{R}))$ instead of $\mathbb{R}$ in this complex). For every $p \geq 0$ and all open subsets U and V of M such that $U \subset V$ we have the linear map

$$j_{p,U,V} : C_p(\mathcal{C}^\infty(U,\mathbb{R})) \to C_p(\mathcal{C}^\infty(V,\mathbb{R}))$$

given by the formula

$$(j_{p,U,V}(f))(\varphi,\varphi_1,\ldots,\varphi_p) = f(\varphi|_U,\varphi_1|_U,\ldots,\varphi_p|_U)$$

for every $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$ and all $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(V,\mathbb{R})$ (note that $j_{p,U,V}(f)$ is continuous, because the map $\mathcal{C}^\infty(V,\mathbb{R}) \to \mathcal{C}^\infty(U,\mathbb{R})$ with $\varphi \mapsto \varphi|_U$ is continuous). We also put $j_{-1,U,V} = \mathrm{id}_{\mathbb{R}}$. Of course, if $p \geq -1$ and U,V,W are open subsets of M such that $U \subset V \subset W$, then $j_{p,V,W}j_{p,U,V} = j_{p,U,W}$. In addition, $\partial_{p,V}j_{p,U,V} = j_{p-1,U,V}\partial_{p,U}$ for every $p \geq 0$ and all open subsets U and V of M such that $U \subset V$.

For any real vector space P we will denote by P^* the dual vector space (i.e. P^* consists of all linear maps $P \to \mathbb{R}$) and for any linear map $r : P \to Q$ between real vector spaces we will denote by r^* the dual map (i.e. $r^* : Q^* \to P^*$ with $\beta \mapsto \beta \circ r$). The above remarks imply that for each $p \geq -1$ the vector spaces $C_p(\mathcal{C}^\infty(U,\mathbb{R}))^*$, where U is an open subset of M, and the maps $j^*_{p,U,V}$, where U and V are open subsets of M such that $U \subset V$, form a presheaf of real vector spaces on the topological space M. We will denote by $\mathcal{C}^p(M)$ the sheaf associated with this presheaf. Since $\mathcal{C}^{-1}(M)$ is nothing but the constant sheaf on M with stalk $\mathbb{R}$, we will write $\mathcal{R}$ rather than $\mathcal{C}^{-1}(M)$. The above remarks also imply that for each $p \geq 0$ the maps $\partial^*_{p,U}$, where U is an open subset of M, form a presheaf homomorphism. We will denote by ∂^p the sheaf homomorphism $\mathcal{C}^{p-1}(M) \to \mathcal{C}^p(M)$ associated with this presheaf homomorphism. Therefore we obtain the sequence

$$0 \to \mathcal{R} \xrightarrow{\partial^0} \mathcal{C}^0(M) \xrightarrow{\partial^1} \mathcal{C}^1(M) \xrightarrow{\partial^2} \mathcal{C}^2(M) \xrightarrow{\partial^3} \cdots \tag{3.3}$$

of sheaves of real vector spaces on M and their homomorphisms such that $\partial^{p+1}\partial^p = 0$ for every $p \geq 0$.

Our first task is to show that this sequence is a fine resolution of $\mathcal{R}$. To this end we need the following analogue of the Poincaré lemma.

LEMMA 3.1. *If* (U,ξ) *is a chart on* M *such that* $\xi(U)$ *is star-shaped, then the sequence*

$$(3.4)\qquad 0 \xleftarrow{\partial_{-1,U}} \mathbb{R} \xleftarrow{\partial_{0,U}} C_0(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{1,U}} C_1(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{2,U}} C_2(\mathcal{C}^\infty(U,\mathbb{R})) \xleftarrow{\partial_{3,U}} \cdots$$

is exact.

Proof. Without loss of generality we may assume that U is an open subset of $\mathbb{R}^n$ and that it is star-shaped with respect to 0.

For every $p \geq 0$, every $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$ and all $\varphi, \varphi_1, \ldots, \varphi_{p+1} \in \mathcal{C}^\infty(U,\mathbb{R})$ we define

$$(3.5)\qquad (h_{p,U}f)(\varphi,\varphi_1,\ldots,\varphi_{p+1}) = \sum_{k=1}^{p+1} \sum_{j_1,\ldots,j_{p+1}=1}^{n} (-1)^{k-1} \times f\biggl(x^{j_k}\int_0^1 t^p\varphi(tx)\frac{\partial\varphi_1}{\partial x^{j_1}}(tx)\ldots\frac{\partial\varphi_{p+1}}{\partial x^{j_{p+1}}}(tx)\,dt, x^{j_1},\ldots,\widehat{x^{j_k}},\ldots,x^{j_{p+1}}\biggr).$$

Here, of course, $x^1,\ldots,x^n$ denote the standard coordinates on $\mathbb{R}^n$ and each x^j is treated as the function $U \to \mathbb{R}$ with $x \mapsto x^j$; the integral is also treated as a function $U \to \mathbb{R}$ of the variable x. The Leibniz rule makes it obvious that $h_{p,U}(f)$ satisfies (2.2). That $h_{p,U}(f)$ is continuous is a consequence of the continuity of the maps

$$\mathcal{C}^\infty(U,\mathbb{R}) \to \mathcal{C}^\infty(U,\mathbb{R}),\qquad \psi \mapsto \frac{\partial\psi}{\partial x^j},$$
$$\mathcal{C}^\infty(U,\mathbb{R}) \times \mathcal{C}^\infty(U,\mathbb{R}) \to \mathcal{C}^\infty(U,\mathbb{R}),\qquad (\chi,\psi) \mapsto \chi\psi,$$
$$\mathcal{C}^\infty(U,\mathbb{R}) \to \mathcal{C}^\infty(U,\mathbb{R}),\qquad \psi \mapsto \int_0^1 t^q\psi(tx)\,dt,$$

for $q \geq 0$. Here, as above, the integral is treated as the function

$$U \to \mathbb{R},\qquad x \mapsto \int_0^1 t^q\psi(tx)\,dt.$$

Finally, transposing φ_l and φ_{l+1} for $l \in \{1,\ldots,p\}$ in (3.5) we see that $h_{p,U}(f)$ is skew-symmetric in the last $p+1$ variables, because f is skew-symmetric in the last p variables. Therefore we have the linear map $h_{p,U} : C_p(\mathcal{C}^\infty(U,\mathbb{R})) \to C_{p+1}(\mathcal{C}^\infty(U,\mathbb{R}))$ for every $p \geq 0$. We also define $h_{-1,U} : \mathbb{R} \to C_0(\mathcal{C}^\infty(U,\mathbb{R}))$ by the formula

$$(3.6)\qquad (h_{-1,U}f)(\varphi) = f\varphi(0)$$

for every $f \in \mathbb{R}$ and every $\varphi \in C_0(\mathcal{C}^\infty(U,\mathbb{R}))$, as well as $h_{-2,U} : 0 \to \mathbb{R}$ by the formula

$$(3.7)\qquad h_{-2,U}(0) = 0.$$

The lemma will be proved once we show that the $h_{p,U}$, where $p \geq -2$, form a homotopy operator between the identity map of (3.4) and the zero map of (3.4), that is,

$$h_{p-1,U}\partial_{p,U} f + \partial_{p+1,U} h_{p,U} f = f \tag{3.8}$$

for every $p \geq -1$ and every $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$.

According to (3.2), (3.6), (3.7), we have

$$h_{-2,U}\partial_{-1,U} f + \partial_{0,U} h_{-1,U} f = (h_{-1,U} f)(1) = f$$

for every $f \in C_{-1}(\mathcal{C}^\infty(U,\mathbb{R}))$, which shows (3.8) in the case $p = -1$. According to (2.3), (3.2), (3.5), (3.6), we have

$$\begin{aligned}(h_{-1,U}\partial_{0,U} f + \partial_{1,U} h_{0,U} f)(\varphi) &= (\partial_{0,U} f)\varphi(0) + (h_{0,U} f)(1,\varphi)\\ &= f(1)\varphi(0) + \sum_{j=1}^{n} f\Big(x^j \int_0^1 \frac{\partial\varphi}{\partial x^j}(tx)\,dt\Big)\\ &= f(1)\varphi(0) + f(\varphi(x) - \varphi(0)) = f(\varphi)\end{aligned}$$

for every $f \in C_0(\mathcal{C}^\infty(U,\mathbb{R}))$ and every $\varphi \in \mathcal{C}^\infty(U,\mathbb{R})$, which shows (3.8) in the case $p = 0$. Suppose now that $p \geq 1$.

It is well known that the set of polynomials $\mathbb{R}[x^1,\ldots,x^n]$ (treated as functions $U \to \mathbb{R}$) is dense in $\mathcal{C}^\infty(U,\mathbb{R})$ (one can prove this fact using the Bernstein polynomials (see for instance [1]) or the Tonelli polynomials (see for instance [4])). Thus it suffices to show that both the sides of (3.8) agree on each $(p+1)$-tuple of monomials $(x^\alpha, x^{\alpha_1},\ldots,x^{\alpha_p})$, where $\alpha,\alpha_1,\ldots,\alpha_p \in \mathbb{N}^n$ (as usual, $x^\beta = (x^1)^{\beta^1}\ldots(x^n)^{\beta^n}$ for every $\beta \in \mathbb{N}^n$).

Let $e_1,\ldots,e_n$ denote the standard basis of the real vector space $\mathbb{R}^n$, and let $|\beta| = \beta^1 + \cdots + \beta^n$ for every $\beta \in \mathbb{N}^n$. By induction on $|\alpha_1 + \cdots + \alpha_p|$ it is easy to check that (2.2) implies

$$\begin{aligned}&f(x^\alpha, x^{\alpha_1},\ldots,x^{\alpha_p})\\ &\quad= \sum_{j_1,\ldots,j_p=1}^{n} \alpha_1^{j_1}\ldots\alpha_p^{j_p} f(x^{\alpha+\alpha_1+\cdots+\alpha_p-e_{j_1}-\cdots-e_{j_p}}, x^{j_1},\ldots,x^{j_p}).\end{aligned} \tag{3.9}$$

From (2.3), (3.5), (3.9) it follows that

$$\begin{aligned}&(h_{p-1,U}\partial_{p,U} f)(x^\alpha, x^{\alpha_1},\ldots,x^{\alpha_p})\\ &\quad= \sum_{k=1}^{p}(-1)^{k-1}\sum_{j_1,\ldots,j_p=1}^{n} \tfrac{\alpha_1^{j_1}\ldots\alpha_p^{j_p}}{|\alpha+\alpha_1+\cdots+\alpha_p|}\\ &\qquad\times(\partial_{p,U} f)(x^{\alpha+\alpha_1+\cdots+\alpha_p-e_{j_1}-\cdots-\widehat{e_{j_k}}-\cdots-e_{j_p}}, x^{j_1},\ldots,\widehat{x^{j_k}},\ldots,x^{j_p})\end{aligned} \tag{3.10}$$

$$= \sum_{k=1}^{p}(-1)^{k-1} \sum_{j_1,\dots,j_p=1}^{n} \tfrac{\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(1, x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p}}, x^{j_1},\dots,\widehat{x^{j_k}},\dots,x^{j_p})$$
$$= \sum_{k=1}^{p}(-1)^{k-1} \sum_{j,j_1,\dots,j_p=1}^{n} \tfrac{(\alpha+\alpha_1+\dots+\alpha_p)^j\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p}-e_j}, x^j, x^{j_1},\dots,\widehat{x^{j_k}},\dots,x^{j_p})$$

(we have written $(\alpha+\alpha_1+\dots+\alpha_p)^j$ instead of $(\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p})^j$ in the last line, because f is skew-symmetric in the last p variables). Exchanging the positions of j and j_k and using the skew-symmetry of f in the last p variables yields

$$\text{(3.11)} \qquad \sum_{j,j_1,\dots,j_p=1}^{n} \tfrac{\alpha_l^j\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p}-e_j}, x^j, x^{j_1},\dots,\widehat{x^{j_k}},\dots,x^{j_p})$$
$$= (-1)^{k-1} \sum_{j,j_1,\dots,j_p=1}^{n} \tfrac{\alpha_l^{j_k}\alpha_1^{j_1}\dots\alpha_{k-1}^{j_{k-1}}\alpha_k^{j}\alpha_{k+1}^{j_{k+1}}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-e_{j_p}}, x^{j_1},\dots,x^{j_p})$$

for every $l \in \{1,\dots,p\}$. Hence if $l \neq k$, then transposing x^{j_k} and x^{j_l}, using the skew-symmetry of f in the last p variables, and exchanging the positions of j_k and j_l in the right hand side of (3.11), we see that

$$\text{(3.12)} \qquad \sum_{j,j_1,\dots,j_p=1}^{n} \tfrac{\alpha_l^j\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p}-e_j}, x^j, x^{j_1},\dots,\widehat{x^{j_k}},\dots,x^{j_p}) = 0.$$

Substituting (3.11) with $l = k$ and (3.12) into (3.10) gives

$$\text{(3.13)} \qquad (h_{p-1,U}\partial_{p,U}f)(x^\alpha, x^{\alpha_1},\dots,x^{\alpha_p})$$
$$= \sum_{k=1}^{p}(-1)^{k-1} \sum_{j,j_1,\dots,j_p=1}^{n} \tfrac{\alpha^j\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-\widehat{e_{j_k}}-\dots-e_{j_p}-e_j}, x^j, x^{j_1},\dots,\widehat{x^{j_k}},\dots,x^{j_p})$$
$$+ \sum_{j_1,\dots,j_p=1}^{n} \tfrac{|\alpha_1+\dots+\alpha_p|\alpha_1^{j_1}\dots\alpha_p^{j_p}}{|\alpha+\alpha_1+\dots+\alpha_p|} f(x^{\alpha+\alpha_1+\dots+\alpha_p-e_{j_1}-\dots-e_{j_p}}, x^{j_1},\dots,x^{j_p}).$$

From (2.3) and (3.5) it follows that

$$(3.14)\qquad (\partial_{p+1,U}h_{p,U}f)(x^{\alpha},x^{\alpha_1},\ldots,x^{\alpha_p}) = (h_{p,U}f)(1,x^{\alpha},x^{\alpha_1},\ldots,x^{\alpha_p})$$
$$= \sum_{j,j_1,\ldots,j_p=1}^{n} \tfrac{\alpha^j\alpha_1^{j_1}\ldots\alpha_p^{j_p}}{|\alpha+\alpha_1+\cdots+\alpha_p|} f(x^{\alpha+\alpha_1+\cdots+\alpha_p-e_{j_1}-\cdots-e_{j_p}},x^{j_1},\ldots,x^{j_p})$$
$$+\sum_{k=1}^{p}(-1)^k \sum_{j,j_1,\ldots,j_p=1}^{n} \tfrac{\alpha^j\alpha_1^{j_1}\ldots\alpha_p^{j_p}}{|\alpha+\alpha_1+\cdots+\alpha_p|}$$
$$\times f(x^{\alpha+\alpha_1+\cdots+\alpha_p-e_j-e_{j_1}-\cdots-\widehat{e_{j_k}}-\cdots-e_{j_p}},x^{j},x^{j_1},\ldots,\widehat{x^{j_k}},\ldots,x^{j_p}).$$

Adding (3.13) to (3.14), we have

$$(3.15)\qquad (h_{p-1,U}\partial_{p,U}f+\partial_{p+1,U}h_{p,U}f)(x^{\alpha},x^{\alpha_1},\ldots,x^{\alpha_p})$$
$$= \sum_{j_1,\ldots,j_p=1}^{n} \tfrac{(|\alpha_1+\cdots+\alpha_p|+|\alpha|)\alpha_1^{j_1}\ldots\alpha_p^{j_p}}{|\alpha+\alpha_1+\cdots+\alpha_p|} f(x^{\alpha+\alpha_1+\cdots+\alpha_p-e_{j_1}-\cdots-e_{j_p}},x^{j_1},\ldots,x^{j_p}).$$

Comparing (3.9) and (3.15) we get (3.8), which completes the proof of the lemma. ∎

THEOREM 3.2. *The sequence* (3.3) *is a fine resolution of the constant sheaf* $\mathcal{R} = M\times\mathbb{R}$.

Proof. From Lemma 3.1 we deduce that if (U,ξ) is a chart on M such that $\xi(U)$ is star-shaped, then the sequence

$$0 \xrightarrow{\partial^*_{-1,U}} \mathbb{R} \xrightarrow{\partial^*_{0,U}} C_0(\mathcal{C}^\infty(U,\mathbb{R}))^* \xrightarrow{\partial^*_{1,U}} C_1(\mathcal{C}^\infty(U,\mathbb{R}))^* \xrightarrow{\partial^*_{2,U}} C_2(\mathcal{C}^\infty(U,\mathbb{R}))^* \xrightarrow{\partial^*_{3,U}} \cdots$$

is exact. It follows that the sequence (3.3) is exact.

Fix $p\ge 0$ and a locally finite cover $\{U_i\}_{i\in I}$ of M by open sets. Let $\{\psi_i\}_{i\in I}$ be a smooth partition of unity on M subordinate to $\{U_i\}_{i\in I}$. For every open subset U of M we define $L_{i,p,U} : C_p(\mathcal{C}^\infty(U,\mathbb{R}))\to C_p(\mathcal{C}^\infty(U,\mathbb{R}))$ by setting

$$(L_{i,p,U}f)(\varphi,\varphi_1,\ldots,\varphi_p) = f(\psi_i|_U\varphi,\varphi_1,\ldots,\varphi_p)$$

for every $f\in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$ and all $\varphi,\varphi_1,\ldots,\varphi_p\in\mathcal{C}^\infty(U,\mathbb{R})$. Obviously, $L_{i,p,V}j_{p,U,V} = j_{p,U,V}L_{i,p,U}$ for all open subsets U and V of M with $U\subset V$. Hence for each $i\in I$ the maps $L^*_{i,p,U}$, where U is an open subset of M, form a presheaf endomorphism. We will denote by L_i^p the sheaf endomorphism $\mathcal{C}^p(M)\to\mathcal{C}^p(M)$ associated with this presheaf endomorphism. It is easily seen that $\operatorname{supp} L_i^p\subset U_i$ for every $i\in I$, because $L_{i,p,U}=0$ if $U\cap\operatorname{supp}\psi_i=\emptyset$, and that $\sum_{i\in I}L_i^p = \mathrm{id}_{\mathcal{C}^p}$, because $\sum_{i\in I}L_{i,p,U} = \mathrm{id}_{C_p(\mathcal{C}^\infty(U,\mathbb{R}))}$ for U such that $U\cap U_i$ for only finitely many i. This means that the sheaf $\mathcal{C}^p(M)$ is fine and the theorem is proved. ∎

As usual, the fine resolution (3.3) of the constant sheaf $\mathcal{R} = M \times \mathbb{R}$ canonically determines a cohomology theory for M with coefficients in sheaves of real vector spaces over M. For such a sheaf $\mathcal{S}$ we denote by $\Gamma(\mathcal{S})$ the real vector space of all global continuous sections of $\mathcal{S}$ and for a homomorphism $\nu : \mathcal{S} \to \mathcal{T}$ of such sheaves we write $\Gamma(\nu)$ for the induced linear map $\Gamma(\mathcal{S}) \to \Gamma(\mathcal{T})$ with $\sigma \mapsto \nu \circ \sigma$. With the fine resolution (3.3) and each sheaf $\mathcal{S}$ of real vector spaces over M we associate the chain complex

$$0 \to \Gamma(\mathcal{C}^0(M) \otimes \mathcal{S}) \xrightarrow{\Gamma(\partial^1 \otimes \mathrm{id}_{\mathcal{S}})} \Gamma(\mathcal{C}^1(M) \otimes \mathcal{S}) \xrightarrow{\Gamma(\partial^2 \otimes \mathrm{id}_{\mathcal{S}})} \Gamma(\mathcal{C}^2(M) \otimes \mathcal{S}) \xrightarrow{\Gamma(\partial^3 \otimes \mathrm{id}_{\mathcal{S}})} \cdots$$

and get $H^p(M, \mathcal{S}) = \ker(\Gamma(\partial^{p+1} \otimes \mathrm{id}_{\mathcal{S}}))/\mathrm{im}(\Gamma(\partial^p \otimes \mathrm{id}_{\mathcal{S}}))$ for every $p \geq 1$ and $H^0(M, \mathcal{S}) = \ker(\Gamma(\partial^1 \otimes \mathrm{id}_{\mathcal{S}}))$. It is well known that for every $p \geq 0$ the vector space $H^p(M, \mathcal{R})$ is canonically isomorphic to the pth de Rham cohomology group of M as well as to the pth singular cohomology group of M with real coefficients in both the continuous and differentiable versions.

Our next goal is to show that for each $p \geq 0$ the presheaf consisting of $C_p(\mathcal{C}^\infty(U, \mathbb{R}))^*$, where U is an open subset of M, and $j^*_{p,U,V}$, where U and V are open subsets of M such that $U \subset V$, is complete. This, of course, will give us a canonical isomorphism between $C_p(\mathcal{C}^\infty(M, \mathbb{R}))^*$ and $\Gamma(\mathcal{C}^p(M)) = \Gamma(\mathcal{C}^p(M) \otimes \mathcal{R})$, and consequently between $H_p(\mathcal{C}^\infty(M, \mathbb{R}))^*$ and $H^p(M, \mathcal{R})$ for every $p \geq 0$ as desired.

LEMMA 3.3. *Let U be an open subset of M, $p \geq 0$ and $f \in C_p(\mathcal{C}^\infty(U, \mathbb{R}))$. Then there is a compact subset K of U such that $f(\varphi, \varphi_1, \ldots, \varphi_p) = 0$ for all $\varphi, \varphi_1, \ldots, \varphi_p \in \mathcal{C}^\infty(U, \mathbb{R})$ such that $K \cap \operatorname{supp} \varphi = \emptyset$.*

Proof. Fix an imbedding $U \to \mathbb{R}^m$ of class $\mathcal{C}^\infty$ for an $m \in \mathbb{N}$ (it exists due to the Whitney imbedding theorem). Thus U will be thought of as a submanifold of $\mathbb{R}^m$ with the relative topology. Choose a locally finite open cover $\{V_i\}_{i \in \mathbb{N}}$ of U such that $\overline{V_i} \subset U$ and $\overline{V_i}$ is compact for each $i \in \mathbb{N}$ (here $\overline{V_i}$ denotes the closure of V_i in $\mathbb{R}^m$).

Let $k_1, \ldots, k_p \in \{1, \ldots, m\}$. Write $g(\varphi) = f(\varphi, x^{k_1}|_U, \ldots, x^{k_p}|_U)$ for every $\varphi \in \mathcal{C}^\infty(U, \mathbb{R})$, where $x^1, \ldots, x^m$ stand for the standard coordinates on $\mathbb{R}^m$ and each x^j is treated as the function $\mathbb{R}^m \to \mathbb{R}$ with $x \mapsto x^j$. We claim that there are only finitely many $i \in \mathbb{N}$ for which there is a $\varphi \in \mathcal{C}^\infty(U, \mathbb{R})$ such that $\operatorname{supp} \varphi \subset V_i$ and $g(\varphi) \neq 0$. Suppose, on the contrary, that we have $i_0 < i_1 < i_2 < \cdots$ and $\varphi_0, \varphi_1, \varphi_2, \ldots \in \mathcal{C}^\infty(U, \mathbb{R})$ such that $\operatorname{supp} \varphi_j \subset V_{i_j}$ and $c_j = g(\varphi_j) \neq 0$ for every $j \in \mathbb{N}$. Let

$$\varphi(x) = \sum_{j=0}^{\infty} \frac{1}{c_j} \varphi_j(x)$$

for every $x \in U$. Since $\{V_i\}_{i \in \mathbb{N}}$ is locally finite, the definition of φ makes sense and $\varphi \in \mathcal{C}^\infty(U,\mathbb{R})$. By the local finiteness of $\{V_i\}_{i\in\mathbb{N}}$ and the continuity of $g : \mathcal{C}^\infty(U,\mathbb{R}) \to \mathbb{R}$,

$$g(\varphi) = g\Big(\lim_{l\to\infty} \sum_{j=0}^{l} \frac{1}{c_j}\varphi_j\Big) = \lim_{l\to\infty} \sum_{j=0}^{l} \frac{1}{c_j} g(\varphi_j) = \lim_{l\to\infty} l = \infty,$$

which is impossible.

From (2.2) it follows that for every $\varphi \in \mathcal{C}^\infty(U,\mathbb{R})$ and all $\eta_1,\dots,\eta_p \in \mathbb{R}[x^1,\dots,x^m]$ we have

$$f(\varphi, \eta_1|_U,\dots,\eta_p|_U) = \sum_{k_1,\dots,k_p=1}^{m} f\Big(\varphi \frac{\partial\eta_1}{\partial x^{k_1}}\Big|_U \cdots \frac{\partial\eta_p}{\partial x^{k_p}}\Big|_U, x^{k_1}|_U,\dots,x^{k_p}|_U\Big).$$

Hence from what has already been proved, we deduce that there are only finitely many $i \in \mathbb{N}$ for which there are $\varphi \in \mathcal{C}^\infty(U,\mathbb{R})$ and $\eta_1,\dots,\eta_p \in \mathbb{R}[x^1,\dots,x^m]$ such that $\operatorname{supp}\varphi \subset V_i$ and $f(\varphi,\eta_1|_U,\dots,\eta_p|_U) \neq 0$. But the set of polynomials $\mathbb{R}[x^1,\dots,x^m]$ (treated as functions $U \to \mathbb{R}$) is dense in $\mathcal{C}^\infty(U,\mathbb{R})$ (to prove this, one can observe that for each $\varphi \in \mathcal{C}^\infty(U,\mathbb{R})$ and each compact subset L of U there is $\widetilde{\varphi} \in \mathcal{C}^\infty(\mathbb{R}^m,\mathbb{R})$ such that $\varphi(x) = \widetilde{\varphi}(x)$ for every member x of an open subset of U containing L, and then use the fact that the set of polynomials $\mathbb{R}[x^1,\dots,x^m]$ is dense in $\mathcal{C}^\infty(\mathbb{R}^m,\mathbb{R})$ with the $\mathcal{C}^\infty$-compact-open topology). Therefore there are only finitely many $i \in \mathbb{N}$ for which there are $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $\operatorname{supp}\varphi \subset V_i$ and $f(\varphi,\varphi_1,\dots,\varphi_p) \neq 0$.

Summing up, we have $l \in \mathbb{N}$ and $i_1,\dots,i_l \in \mathbb{N}$ with the property that $f(\varphi,\varphi_1,\dots,\varphi_p) = 0$ for every $i \in \mathbb{N} \setminus \{i_1,\dots,i_l\}$ and all $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $\operatorname{supp}\varphi \subset V_i$. Set

$$K = \bigcup_{j=1}^{l} \overline{V_{i_j}}.$$

Obviously, K is compact. Assume that $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ and $K \cap \operatorname{supp}\varphi = \emptyset$. Taking a smooth partition of unity $\{\psi_i\}_{i\in\mathbb{N}}$ on U subordinate to $\{V_i\}_{i\in\mathbb{N}}$, we get

$$\begin{aligned} f(\varphi,\varphi_1,\dots,\varphi_p) &= f\Big(\sum_{i=1}^{\infty} \psi_i\varphi, \varphi_1,\dots,\varphi_p\Big) \\ &= \sum_{i=1}^{\infty} f(\psi_i\varphi,\varphi_1,\dots,\varphi_p) = \sum_{j=1}^{l} f(\psi_{i_j}\varphi,\varphi_1,\dots,\varphi_p) = 0, \end{aligned}$$

because f is continuous, $\operatorname{supp}(\psi_i\varphi) \subset V_i$ for every $i \in \mathbb{N}$ and $\psi_{i_j}\varphi = 0$ for every $j \in \{1,\dots,l\}$. ∎

LEMMA 3.4. *Let U be an open subset of M, $p \geq 0$, $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$, and B a closed subset of U with the property that $f(\varphi,\varphi_1,\ldots,\varphi_p) = 0$ for all $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $B \cap \operatorname{supp}\varphi = \emptyset$. Then we have $f(\varphi,\varphi_1,\ldots,\varphi_p)=0$ for every $q \in \{1,\ldots,p\}$ and all $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $B \cap \operatorname{supp}\varphi_q = \emptyset$.*

Proof. We can assume that $q = 1$. Take $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $B \cap \operatorname{supp}\varphi_1 = \emptyset$, and a $\xi \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $B \cap \operatorname{supp}\xi = \emptyset$ and $\xi(x) = 1$ for every $x \in \operatorname{supp}\varphi_1$ (it is easily seen that such a ξ exists). From (2.2) we have

$$\begin{aligned} f(\varphi,\varphi_1,\ldots,\varphi_p) &= f(\varphi,\xi\varphi_1,\varphi_2,\ldots,\varphi_p) \\ &= f(\varphi\xi,\varphi_1,\ldots,\varphi_p) + f(\varphi_1\varphi,\xi,\varphi_2,\ldots,\varphi_p) = 0, \end{aligned}$$

because $\varphi_1 = \xi\varphi_1$, $B \cap \operatorname{supp}(\varphi\xi) = \emptyset$ and $B \cap \operatorname{supp}(\varphi_1\varphi) = \emptyset$, and the lemma follows. ∎

LEMMA 3.5. *Let U be an open subset of M, $p \geq 0$, $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$, and let B be a closed subset of U with the property that $f(\varphi,\varphi_1,\ldots,\varphi_p) = 0$ for all $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $B \cap \operatorname{supp}\varphi = \emptyset$. Then for each open subset V of M such that $B \subset V \subset U$ there is a $g \in C_p(\mathcal{C}^\infty(V,\mathbb{R}))$ such that $j_{p,V,U}\, g = f$.*

Proof. Fix an open subset V of M such that $B \subset V \subset U$. Observe first that for each $\varphi \in \mathcal{C}^\infty(V,\mathbb{R})$ there is $\widetilde{\varphi} \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for every point x of an open subset of V containing B. Indeed, one can take any $\xi \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $\operatorname{supp}\xi \subset V$ and $\xi(x) = 1$ for every point x of an open subset of U containing B, and set

$$\widetilde{\varphi}(x) = \begin{cases} \xi(x)\varphi(x) & \text{if } x \in V, \\ 0 & \text{if } x \in U \setminus \operatorname{supp}\xi. \end{cases} \tag{3.16}$$

If $\varphi,\varphi_1,\ldots,\varphi_p \in \mathcal{C}^\infty(V,\mathbb{R})$, then we define

$$g(\varphi,\varphi_1,\ldots,\varphi_p) = f(\widetilde{\varphi},\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_p),$$

where $\widetilde{\varphi},\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_p$ are any functions from $\mathcal{C}^\infty(U,\mathbb{R})$ such that $\widetilde{\varphi}(x)=\varphi(x)$, $\widetilde{\varphi}_1(x) = \varphi_1(x),\ldots,\widetilde{\varphi}_p(x) = \varphi_p(x)$ for every point x of an open subset of V containing B. This definition is independent of the choice of $\widetilde{\varphi},\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_p$. Indeed, if $\widetilde{\varphi}',\widetilde{\varphi}_1',\ldots,\widetilde{\varphi}_p' \in \mathcal{C}^\infty(U,\mathbb{R})$ are also such that $\widetilde{\varphi}'(x) = \varphi(x)$, $\widetilde{\varphi}_1'(x) = \varphi_1(x),\ldots,\widetilde{\varphi}_p'(x) = \varphi_p(x)$ for every point x of an open subset of V containing B, then $B \cap \operatorname{supp}(\widetilde{\varphi}' - \widetilde{\varphi}) = \emptyset$, $B \cap \operatorname{supp}(\widetilde{\varphi}_1' - \widetilde{\varphi}_1) = \emptyset$, $\ldots$, $B \cap \operatorname{supp}(\widetilde{\varphi}_p' - \widetilde{\varphi}_p) = \emptyset$ and it is a simple matter to use the $(p+1)$-linearity of f, the assumed property of B and Lemma 3.4 to show that $f(\widetilde{\varphi}',\widetilde{\varphi}_1',\ldots,\widetilde{\varphi}_p') = f(\widetilde{\varphi},\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_p)$.

Of course, g is $(p+1)$-linear and skew symmetric in the last p variables. It is continuous, because so is the map $\mathcal{C}^\infty(V,\mathbb{R}) \to \mathcal{C}^\infty(U,\mathbb{R})$ which sends φ to $\widetilde{\varphi}$ given by formula (3.16). It is also easy to see that g satisfies (2.2).

Summing up, $g \in C_p(\mathcal{C}^\infty(V,\mathbb{R}))$. The fact that $j_{p,V,U}g = f$ is an immediate consequence of the definition of g. ∎

Lemma 3.6. *Let U be an open subset of M, A any set, $\{U_\alpha\}_{\alpha\in A}$ a family of open subsets of M such that $U = \bigcup_{\alpha\in A} U_\alpha$, $p \ge 0$ and $f \in C_p(\mathcal{C}^\infty(U,\mathbb{R}))$. Then there are $s \in \mathbb{N}$, $\alpha_1,\dots,\alpha_s \in A$ and $f_1 \in C_p(\mathcal{C}^\infty(U_{\alpha_1},\mathbb{R})),\dots,f_s \in C_p(\mathcal{C}^\infty(U_{\alpha_s},\mathbb{R}))$ such that $f = j_{p,U_{\alpha_1},U}f_1 + \dots + j_{p,U_{\alpha_s},U}f_s$.*

Proof. From Lemma 3.3 we have a compact subset K of U with the property that $f(\varphi,\varphi_1,\dots,\varphi_p) = 0$ for all $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(U,\mathbb{R})$ such that $K \cap \operatorname{supp}\varphi = \emptyset$. Since K is compact, there are $\alpha_1,\dots,\alpha_s \in A$ such that $K \subset U_{\alpha_1} \cup \dots \cup U_{\alpha_s} =: V$. From Lemma 3.5 we have a $g \in C_p(\mathcal{C}^\infty(V,\mathbb{R}))$ such that $j_{p,V,U}g = f$. Take a smooth partition of unity $\{\xi_1,\dots,\xi_s\}$ on V subordinate to the cover $\{U_{\alpha_1},\dots,U_{\alpha_s}\}$. For every $i \in \{1,\dots,s\}$ we define $g_i \in C_p(\mathcal{C}^\infty(V,\mathbb{R}))$ by

$$g_i(\varphi,\varphi_1,\dots,\varphi_p) = g(\xi_i\varphi,\varphi_1,\dots,\varphi_p)$$

for all $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(V,\mathbb{R})$. We see at once that, for each $i \in \{1,\dots,s\}$, $\operatorname{supp}\xi_i$ is a closed subset of V with the property that $g_i(\varphi,\varphi_1,\dots,\varphi_p) = 0$ for all $\varphi,\varphi_1,\dots,\varphi_p \in \mathcal{C}^\infty(V,\mathbb{R})$ such that $\operatorname{supp}\xi_i \cap \operatorname{supp}\varphi = \emptyset$, and in addition $\operatorname{supp}\xi_i \subset U_{\alpha_i}$. Therefore from Lemma 3.5 for each $i \in \{1,\dots,s\}$ we have an $f_i \in C_p(\mathcal{C}^\infty(U_{\alpha_j},\mathbb{R}))$ such that $g_i = j_{p,U_{\alpha_i},V}f_i$. The maps $f_1,\dots,f_s$ are as needed, because $g = \sum_{i=1}^s g_i$ and so

$$f = j_{p,V,U}g = \sum_{i=1}^{s} j_{p,V,U}g_i = \sum_{i=1}^{s} j_{p,V,U}j_{p,U_{\alpha_i},V}f_i = \sum_{i=1}^{s} j_{p,U_{\alpha_i},U}f_i. \ ∎$$

Lemma 3.7. *Let U be an open subset of M, A any set, $\{U_\alpha\}_{\alpha\in A}$ a family of open subsets of M such that $U = \bigcup_{\alpha\in A} U_\alpha$, $p \ge 0$, $s \in \mathbb{N}$, $\alpha_1,\dots,\alpha_s \in A$, and $f_1 \in C_p(\mathcal{C}^\infty(U_{\alpha_1},\mathbb{R})),\dots,f_s \in C_p(\mathcal{C}^\infty(U_{\alpha_s},\mathbb{R}))$ such that*

$$j_{p,U_{\alpha_1},U}f_1 + \dots + j_{p,U_{\alpha_s},U}f_s = 0. \tag{3.17}$$

Then for all $k,l \in \{1,\dots,s\}$ with $k < l$ there are $f_{kl} \in C_p(\mathcal{C}^\infty(U_{\alpha_k}\cap U_{\alpha_l},\mathbb{R}))$ such that

$$f_i = \sum_{l=i+1}^{s} j_{p,U_{\alpha_i}\cap U_{\alpha_l},U_{\alpha_i}}f_{il} - \sum_{k=1}^{i-1} j_{p,U_{\alpha_k}\cap U_{\alpha_i},U_{\alpha_i}}f_{ki}$$

for every $i \in \{1,\dots,s\}$.

Proof. Note first that for any open subsets V and W of M such that $V \subset W$ the map $j_{p,V,W} : C_p(\mathcal{C}^\infty(V,\mathbb{R})) \to C_p(\mathcal{C}^\infty(W,\mathbb{R}))$ is injective, because the image of the map $\mathcal{C}^\infty(W,\mathbb{R}) \to \mathcal{C}^\infty(V,\mathbb{R})$ with $\varphi \mapsto \varphi|_V$ is dense in $\mathcal{C}^\infty(V,\mathbb{R})$. In particular, the map $j_{p,\bigcup_{i=1}^s U_{\alpha_i},U} : C_p(\mathcal{C}^\infty(\bigcup_{i=1}^s U_{\alpha_i},\mathbb{R})) \to C_p(\mathcal{C}^\infty(U,\mathbb{R}))$ is injective, and from (3.17) it follows that $j_{p,U_{\alpha_1},\bigcup_{i=1}^s U_{\alpha_i}}f_1 + \dots + j_{p,U_{\alpha_s},\bigcup_{i=1}^s U_{\alpha_i}}f_s = 0$. Therefore without restriction of generality we can assume $U = U_{\alpha_1} \cup \cdots \cup U_{\alpha_s}$. To shorten the notation, we will write U_i instead of U_{α_i} for $i \in \{1, \ldots, s\}$.

We begin by proving the lemma in the special case when $s = 2$. Hence $U = U_1 \cup U_2$, $f_1 \in C_p(\mathcal{C}^\infty(U_1, \mathbb{R}))$, $f_2 \in C_p(\mathcal{C}^\infty(U_2, \mathbb{R}))$, and

$$j_{p,U_1,U} f_1 + j_{p,U_2,U} f_2 = 0.$$

From Lemma 3.3 for each $i \in \{1, 2\}$ we have a compact subset K_i of U_i with the property that $f_i(\varphi, \varphi_1, \ldots, \varphi_p) = 0$ for all $\varphi, \varphi_1, \ldots, \varphi_p \in \mathcal{C}^\infty(U_i, \mathbb{R})$ such that $K_i \cap \operatorname{supp} \varphi = \emptyset$. Take an open subset W of U such that

$$K_1 \cap K_2 \subset W \subset \overline{W} \subset U_1 \cap U_2,$$

where $\overline{W}$ denotes the closure of W in U.

We prove that if $\psi, \psi_1, \ldots, \psi_p \in \mathcal{C}^\infty(U_1, \mathbb{R})$ are such that $\overline{W} \cap \operatorname{supp} \psi = \emptyset$, then $f_1(\psi, \psi_1, \ldots, \psi_p) = 0$. It is easy to see that there are $\widetilde{\psi}, \widetilde{\psi}_1, \ldots, \widetilde{\psi}_p \in \mathcal{C}^\infty(U, \mathbb{R})$ which coincide with $\psi, \psi_1, \ldots, \psi_p$ on an open subset of U_1 containing K_1, and satisfy the condition $\overline{W} \cap \operatorname{supp} \widetilde{\psi} = \emptyset$. Taking a smooth partition of unity $\{\chi_1, \chi_2\}$ on U subordinate to the cover $\{U \setminus L_2, U \setminus L_1\}$, where $L_2 = K_2 \setminus W$ and $L_1 = K_1 \setminus W$, we have

$$\begin{aligned}
f_1(\psi, \psi_1, \ldots, \psi_p) &= f_1(\widetilde{\psi}|_{U_1}, \widetilde{\psi}_1|_{U_1}, \ldots, \widetilde{\psi}_p|_{U_1}) \\
&= f_1((\chi_1\widetilde{\psi})|_{U_1}, \widetilde{\psi}_1|_{U_1}, \ldots, \widetilde{\psi}_p|_{U_1}) + f_1((\chi_2\widetilde{\psi})|_{U_1}, \widetilde{\psi}_1|_{U_1}, \ldots, \widetilde{\psi}_p|_{U_1}) \\
&= f_1((\chi_1\widetilde{\psi})|_{U_1}, \widetilde{\psi}_1|_{U_1}, \ldots, \widetilde{\psi}_p|_{U_1}) = (j_{p,U_1,U} f_1)(\chi_1\widetilde{\psi}, \widetilde{\psi}_1, \ldots, \widetilde{\psi}_p) \\
&= -(j_{p,U_2,U} f_2)(\chi_1\widetilde{\psi}, \widetilde{\psi}_1, \ldots, \widetilde{\psi}_p) = -f_2((\chi_1\widetilde{\psi})|_{U_2}, \widetilde{\psi}_1|_{U_2}, \ldots, \widetilde{\psi}_p|_{U_2}) = 0,
\end{aligned}$$

which is due to the fact that $K_1 \cap \operatorname{supp}(\chi_2\widetilde{\psi}) = \emptyset$ (as $L_1 \cap \operatorname{supp}(\chi_2\widetilde{\psi}) = \emptyset$, $\overline{W} \cap \operatorname{supp}(\chi_2\widetilde{\psi}) = \emptyset$, $K_1 \subset L_1 \cup \overline{W}$) and similarly $K_2 \cap \operatorname{supp}(\chi_1\widetilde{\psi}) = \emptyset$.

Since $\overline{W}$ is a closed subset of U_1 contained in $U_1 \cap U_2$, on account of what has just been proved, from Lemma 3.5 we have an $f_{12} \in C_p(\mathcal{C}^\infty(U_1 \cap U_2, \mathbb{R}))$ such that

$$j_{p,U_1 \cap U_2,U_1} f_{12} = f_1.$$

It remains to show that $-j_{p,U_1 \cap U_2,U_2} f_{12} = f_2$. Let $\varphi, \varphi_1, \ldots, \varphi_p \in \mathcal{C}^\infty(U_2, \mathbb{R})$. For any $\widetilde{\varphi}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_p \in \mathcal{C}^\infty(U, \mathbb{R})$ which coincide with $\varphi, \varphi_1, \ldots, \varphi_p$ on an open subset of U_2 containing $\overline{W}$, we get

$$\begin{aligned}
-(j_{p,U_1 \cap U_2,U_2} f_{12})(\varphi, \varphi_1, \ldots, \varphi_p) &= -f_{12}(\varphi|_{U_1 \cap U_2}, \varphi_1|_{U_1 \cap U_2}, \ldots, \varphi_p|_{U_1 \cap U_2}) \\
&= -f_1(\widetilde{\varphi}|_{U_1}, \widetilde{\varphi}_1|_{U_1}, \ldots, \widetilde{\varphi}_p|_{U_1}) \\
&= -(j_{p,U_1,U} f_1)(\widetilde{\varphi}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_p) \\
&= (j_{p,U_2,U} f_2)(\widetilde{\varphi}, \widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_p) \\
&= f_2(\varphi, \varphi_1, \ldots, \varphi_p),
\end{aligned}$$

the last equality being a consequence of the fact that if $\psi, \psi_1, \ldots, \psi_p \in C_p(\mathcal{C}^\infty(U_2, \mathbb{R}))$ are such that $\overline{W} \cap \operatorname{supp} \psi = \emptyset$, then $f_2(\psi, \psi_1, \ldots, \psi_p) = 0$, which can clearly be proved in the same manner as above for f_1. This establishes the lemma in the case $s = 2$.

Now we proceed by induction on s. Suppose $s \geq 3$ and the assertion of the lemma is true for $s-1$. Put $V = \bigcup_{i=1}^{s-1} U_i$ and

$$g = \sum_{i=1}^{s-1} j_{p,U_i,V} f_i.$$

Of course $U = V \cup U_s$ and, by (3.17), $j_{p,V,U} g + j_{p,U_s,U} f_s = 0$. Therefore, from what has already been proved, we have an $h \in C_p(\mathcal{C}^\infty(V \cap U_s))$ such that $g = j_{p,V\cap U_s,V} h$ and $f_s = -j_{p,V\cap U_s,U_s} h$.

Obviously we have $V \cap U_s = \bigcup_{i=1}^{s-1}(U_i \cap U_s)$. From Lemma 3.6 we obtain $e_1 \in C_p(\mathcal{C}^\infty(U_1 \cap U_s, \mathbb{R}))$, …, $e_{s-1} \in C_p(\mathcal{C}^\infty(U_{s-1} \cap U_s, \mathbb{R}))$ such that

$$h = \sum_{i=1}^{s-1} j_{p,U_i\cap U_s,V\cap U_s} e_i.$$

Write $d_1 = j_{p,U_1\cap U_s,U_1} e_1$, …, $d_{s-1} = j_{p,U_{s-1}\cap U_s,U_{s-1}} e_{s-1}$. We see that

$$\sum_{i=1}^{s-1} j_{U_i,V}(f_i - d_i) = g - \sum_{i=1}^{s-1} j_{p,U_i\cap U_s,V} e_i = g - j_{p,V\cap U_s,V} h = g - g = 0.$$

Thus from the assertion of the lemma with $s-1$, for $k, l \in \{1, \ldots, s-1\}$ such that $k < l$, there are maps $f_{kl} \in C_p(\mathcal{C}^\infty(U_k \cap U_l, \mathbb{R}))$ satisfying

$$f_i - d_i = \sum_{l=i+1}^{s-1} j_{p,U_i\cap U_l,U_i} f_{il} - \sum_{k=1}^{i-1} j_{p,U_k\cap U_i,U_i} f_{ki}$$

for $i \in \{1, \ldots, s-1\}$. Moreover, we put

$$f_{is} = e_i$$

for $i \in \{1, \ldots, s-1\}$. We get

$$\begin{aligned}\sum_{l=i+1}^{s} j_{p,U_i\cap U_l,U_i} f_{il} - \sum_{k=1}^{i-1} j_{p,U_k\cap U_i,U_i} f_{ki} &= (f_i - d_i) + j_{p,U_i\cap U_s,U_i} e_i \\ &= (f_i - d_i) + d_i = f_i\end{aligned}$$

for $i \in \{1, \ldots, s-1\}$, and

$$-\sum_{k=1}^{s-1} j_{p,U_k\cap U_s,U_s} f_{ks} = -\sum_{k=1}^{s-1} j_{p,U_k\cap U_s,U_s} e_k = -j_{p,V\cap U_s,U_s} h = -(-f_s) = f_s. \blacksquare$$

THEOREM 3.8. *For each $p \geq 0$ the presheaf consisting of $C_p(\mathcal{C}^\infty(U, \mathbb{R}))^*$, where U is an open subset of M, and $j^*_{p,U,V}$, where U and V are open subsets of M such that $U \subset V$, is complete, that is, for every open subset U of M*

and every family $\{U_\alpha\}_{\alpha\in A}$ *of open subsets of* M *such that* $U=\bigcup_{\alpha\in A}U_\alpha$ *the following conditions hold:*

(a) *whenever* $\sigma,\tau\in C_p(\mathcal{C}^\infty(U,\mathbb{R}))^*$ *are such that* $j^*_{p,U_\alpha,U}\sigma=j^*_{p,U_\alpha,U}\tau$ *for all* $\alpha\in A$, *then* $\sigma=\tau$,

(b) *whenever* $\sigma_\alpha\in C_p(\mathcal{C}^\infty(U_\alpha,\mathbb{R}))^*$, *where* $\alpha\in A$, *satisfy* $j^*_{p,U_\alpha\cap U_\beta,U_\alpha}\sigma_\alpha = j^*_{p,U_\alpha\cap U_\beta,U_\beta}\sigma_\beta$ *for all* $\alpha,\beta\in A$, *then there exists* $\sigma\in C_p(\mathcal{C}^\infty(U,\mathbb{R}))^*$ *such that* $j^*_{p,U_\alpha,U}\sigma=\sigma_\alpha$ *for every* α.

Proof. Choose any order on A. Lemmas 3.6 and 3.7 say that the sequence of vector spaces and linear maps

$$\sum_{(\alpha,\beta)\in A^2,\,\alpha<\beta} C_p(\mathcal{C}^\infty(U_\alpha\cap U_\beta,\mathbb{R}))\xrightarrow{\Phi}\sum_{\alpha\in A}C_p(\mathcal{C}^\infty(U_\alpha,\mathbb{R}))$$
$$\xrightarrow{\Psi} C_p(\mathcal{C}^\infty(U,\mathbb{R}))\to 0,$$

where $\Phi=\sum_{(\alpha,\beta)\in A^2,\,\alpha<\beta}(j_{p,U_\alpha\cap U_\beta,U_\alpha}-j_{p,U_\alpha\cap U_\beta,U_\beta})$ and $\Psi=\sum_{\alpha\in A}j_{p,U_\alpha,U}$, is exact. Thus so is the sequence of the dual vector spaces and the dual linear maps

$$\prod_{(\alpha,\beta)\in A^2,\,\alpha<\beta} C_p(\mathcal{C}^\infty(U_\alpha\cap U_\beta,\mathbb{R}))^*\xleftarrow{\Phi^*}\prod_{\alpha\in A}C_p(\mathcal{C}^\infty(U_\alpha,\mathbb{R}))^*$$
$$\xleftarrow{\Psi^*} C_p(\mathcal{C}^\infty(U,\mathbb{R}))^*\leftarrow 0,$$

where $\Phi^*=\prod_{(\alpha,\beta)\in A^2,\,\alpha<\beta}(j^*_{p,U_\alpha\cap U_\beta,U_\alpha}-j^*_{p,U_\alpha\cap U_\beta,U_\beta})$ and $\Psi^*=\prod_{\alpha\in A}j^*_{p,U_\alpha,U}$. But this means that (a) and (b) are satisfied, and the proof is complete. ∎

Let $S_p(M,\mathbb{R})$ for each $p\ge 0$ denote the real vector space of differentiable singular p-chains on M with real coefficients. For each $c\in S_p(M,\mathbb{R})$ we define $f_{c,M}:\mathcal{C}^\infty(M,\mathbb{R})\times\cdots\times\mathcal{C}^\infty(M,\mathbb{R})\to\mathbb{R}$ by the formula

$$f_{c,M}(\varphi,\varphi_1,\ldots,\varphi_p)=\int_c\varphi\,d\varphi_1\wedge\cdots\wedge d\varphi_p.$$

It is clear that $f_{c,M}$ is $(p+1)$-linear, skew-symmetric in the last p variables and satisfies (2.2). A standard verification shows that it is also continuous. Thus $f_{c,M}\in C_p(\mathcal{C}^\infty(M,\mathbb{R}))$.

From Stokes' theorem we have

$$\begin{aligned}f_{\partial c,M}(\varphi,\varphi_1,\ldots,\varphi_{p-1})&=\int_{\partial c}\varphi\,d\varphi_1\wedge\cdots\wedge d\varphi_{p-1}\\&=\int_c d(\varphi\,d\varphi_1\wedge\cdots\wedge d\varphi_{p-1})=\int_c 1\,d\varphi\wedge d\varphi_1\wedge\cdots\wedge d\varphi_{p-1}\\&=f_{c,M}(1,\varphi,\varphi_1,\ldots,\varphi_{p-1})=(\partial_{p,M}f_{c,M})(\varphi,\varphi_1,\ldots,\varphi_{p-1})\end{aligned}$$

for every $p \geq 1$, every $c \in S_p(M,\mathbb{R})$ and all $\varphi, \varphi_1, \ldots, \varphi_{p-1} \in \mathcal{C}^\infty(M,\mathbb{R})$. This means that the linear maps

$$S_p(M,\mathbb{R}) \to C_p(\mathcal{C}^\infty(M,\mathbb{R})), \quad c \mapsto f_{c,M}, \tag{3.18}$$

where $p \geq 0$, form a chain map between the complex of differentiable singular chains on M with real coefficients and the chain complex (3.1).

We can now formulate and prove our main result.

THEOREM 3.9. *For each $p \geq 0$ the linear map* (3.18) *induces an isomorphism between the pth differentiable singular homology group of M with real coefficients and $H_p(\mathcal{C}^\infty(M,\mathbb{R}))$.*

Proof. The cohomology vector spaces of the cochain complex dual to a chain complex of vector spaces can be naturally identified with the vector spaces dual to the homology vector spaces of this chain complex. Consequently, the maps between the cohomology vector spaces induced by the cochain map dual to a chain map between chain complexes of vector spaces can be naturally identified with the maps dual to the maps between the homology vector spaces induced by this chain map. Moreover, a linear map between vector spaces is an isomorphism if and only if the dual map is an isomorphism. Therefore the theorem will be proved once we show that the cochain map dual to the chain map consisting of the maps (3.18) induces isomorphisms between the cohomology vector spaces.

Fix $p \geq 0$. For every open subset U of M we define the map

$$S_p(U,\mathbb{R}) \to C_p(\mathcal{C}^\infty(U,\mathbb{R})), \quad c \mapsto f_{c,U}, \tag{3.19}$$

where $S_p(U,\mathbb{R})$ denotes the real vector space of differentiable singular p-chains on U with real coefficients, and $f_{c,U} : \mathcal{C}^\infty(U,\mathbb{R}) \times \cdots \times \mathcal{C}^\infty(U,\mathbb{R}) \to \mathbb{R}$ is given by

$$f_{c,U}(\varphi, \varphi_1, \ldots, \varphi_p) = \int_c \varphi \, d\varphi_1 \wedge \cdots \wedge d\varphi_p.$$

The maps dual to (3.19) form a homomorphism between our presheaf consisting of the vector spaces $C_p(\mathcal{C}^\infty(U,\mathbb{R}))^*$, where U is an open subset of M, and the maps $j^*_{p,U,V}$, where U and V are open subsets of M such that $U \subset V$, and the presheaf of differentiable singular p-chains on M with real coefficients. Hence we have the homomorphism of the associated sheaves

$$\mathcal{C}^p(M) \to \mathcal{S}^p(M,\mathbb{R}), \tag{3.20}$$

induced by this presheaf homomorphism.

We have the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{R} & \xrightarrow{\partial^0} & \mathcal{C}^0(M) & \xrightarrow{\partial^1} & \mathcal{C}^1(M) & \xrightarrow{\partial^2} & \mathcal{C}^2(M) & \xrightarrow{\partial^3} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{S}^0(M,\mathbb{R}) & \longrightarrow & \mathcal{S}^1(M,\mathbb{R}) & \longrightarrow & \mathcal{S}^2(M,\mathbb{R}) & \longrightarrow & \cdots
\end{array} \tag{3.21}$$

where the first vertical arrow is the identity map on $\mathcal{R}$, the other vertical arrows are the maps (3.20), the first row is the fine resolution (3.3) of $\mathcal{R}$ from Theorem 3.2 and the second row is the well known fine resolution of $\mathcal{R}$ which is used in the proof that the classical differentiable singular cohomology is isomorphic with the sheaf cohomology with coefficients in $\mathcal{R}$ ([7, 5.31]). That the first square in (3.21) commutes may be easily seen directly (the sheaf homomorphism $\mathcal{R} \to \mathcal{S}^0(M, \mathbb{R})$ is induced by the presheaf homomorphism consisting of the maps dual to the linear maps $S_0(U, \mathbb{R}) \to \mathbb{R}$, where U is an open subset of M, which send every $x \in U$ to 1). The other squares in (3.21) commute, because the maps (3.19) commute, by Stokes' theorem, with the boundary operators.

Consider the following commutative diagram of cochain complexes of real vector spaces:

$$\begin{array}{ccc} C^*(\mathcal{C}^\infty(M,\mathbb{R})) & \longrightarrow & \Gamma(\mathcal{C}^*(M)) \\ \downarrow & & \downarrow \\ S^*(M,\mathbb{R}) & \longrightarrow & \Gamma(\mathcal{S}^*(M,\mathbb{R})) \end{array} \tag{3.22}$$

where $C^*(\mathcal{C}^\infty(M, \mathbb{R}))$ denotes the cochain complex dual to (3.1), $S^*(M, \mathbb{R})$ the complex of differentiable singular cochains on M with real coefficients, $\Gamma(\mathcal{C}^*(M))$ and $\Gamma(\mathcal{S}^*(M, \mathbb{R}))$ the complexes of global continuous sections for the sheaf complexes $\mathcal{C}^0(M) \to \mathcal{C}^1(M) \to \mathcal{C}^2(M) \to \cdots$ and $\mathcal{S}^0(M, \mathbb{R}) \to \mathcal{S}^1(M, \mathbb{R}) \to \mathcal{S}^2(M, \mathbb{R}) \to \cdots$, which are parts of the rows of (3.21); the cochain map $C^*(\mathcal{C}^\infty(M, \mathbb{R})) \to S^*(M, \mathbb{R})$ is dual to the chain map consisting of the maps (3.18), the cochain map $\Gamma(\mathcal{C}^*(M)) \to \Gamma(\mathcal{S}^*(M, \mathbb{R}))$ is induced by the maps (3.20), and finally $C^*(\mathcal{C}^\infty(M, \mathbb{R})) \to \Gamma(\mathcal{C}^*(M))$ and $S^*(M, \mathbb{R}) \to \Gamma(\mathcal{S}^*(M, \mathbb{R}))$ consist of the maps sending any element α of $C_p(\mathcal{C}^\infty(M, \mathbb{R}))^*$ or $S_p(M, \mathbb{R})^*$ to the section $M \to \mathcal{C}^p(M)$ or $M \to \mathcal{S}^p(M, \mathbb{R})$ with $x \mapsto \rho_x \alpha$ for $p \geq 0$, where $\rho_x \alpha$ denotes the germ of α at x.

The cochain map $C^*(\mathcal{C}^\infty(M, \mathbb{R})) \to \Gamma(\mathcal{C}^*(M))$ in (3.22) is an isomorphism, since for every $p \geq 0$ the presheaf consisting of $C_p(\mathcal{C}^\infty(U, \mathbb{R}))^*$, where U is an open subset of M, and $j^*_{p,U,V}$, where U and V are open subsets of M such that $U \subset V$, is complete on account of Theorem 3.8. The cochain map $\Gamma(\mathcal{C}^*(M)) \to \Gamma(\mathcal{S}^*(M, \mathbb{R}))$ in (3.22) induces isomorphisms on cohomology, because of the homomorphism (3.21) between the two fine resolutions of $\mathcal{R}$ ([7, 5.24]). Finally, it is well known that the cochain map $S^*(M, \mathbb{R}) \to \Gamma(\mathcal{S}^*(M, \mathbb{R}))$ in (3.22) induces isomorphisms on cohomology [7, 5.32]. Consequently, the cochain map $C^*(\mathcal{C}^\infty(M, \mathbb{R})) \to S^*(M, \mathbb{R})$ in (3.22) also induces isomorphisms on cohomology and the proof is complete. ■

4. The relation to the de Rham complex. Let us now suppose that A is an associative and commutative F-algebra with unit. In this special case the de Rham cochain complex of A can be defined (see for instance [6, Chapter XIX]). We recall briefly the construction of this complex. Note that it uses no topologies on A and F.

If M is an A-module, then an F-linear map $D : A \to M$ is called a *derivation* if $D(ab) = aDb + bDa$ for all $a, b \in A$. The *universal derivation* of A is, by definition, a derivation $d : A \to \Omega$, where Ω is an A-module, such that for every A-module M and every derivation $D : A \to M$ there is a unique A-linear map $f : \Omega \to M$ satisfying $f \circ d = D$. Such a universal derivation exists and is uniquely determined up to a unique isomorphism. The A-module Ω is called the *module of Kähler differentials*. For each $p \in \mathbb{N}$ the A-module $\Omega^p_{A/F}$ is defined to be the pth exterior power $\bigwedge^p \Omega$ of the A-module Ω. Each $\Omega^p_{A/F}$ is generated as a vector space over F by vectors of the form $a\,db_1 \wedge \cdots \wedge db_p$, where $a, b_1, \ldots, b_p \in A$. Furthermore, for every $p \geq 1$ there is a unique F-linear map $d^p : \Omega^{p-1}_{A/F} \to \Omega^p_{A/F}$ with the property that

$$d^p(a\,db_1 \wedge \cdots \wedge db_{p-1}) = da \wedge db_1 \wedge \cdots \wedge db_{p-1}$$

for all $a, b_1, \ldots, b_{p-1} \in A$. Then $d^p \circ d^{p-1} = 0$ for every $p \geq 2$. The cochain complex of vector spaces over F:

$$\Omega^0_{A/F} \xrightarrow{d^1} \Omega^1_{A/F} \xrightarrow{d^2} \Omega^2_{A/F} \xrightarrow{d^3} \cdots \tag{4.1}$$

is called the *de Rham complex* of A.

The construction of the chain complex (2.1) described in Section 2 can be also carried out without any topologies on A and F by dropping the assumption that the maps forming C_pA for $p \in \mathbb{N}$ are continuous. Of course, this is equivalent to taking the discrete topology on A. We now show that in the special case of an associative and commutative algebra A with the discrete topology the chain complex (2.1) is (up to an isomorphism) dual to the de Rham cochain complex (4.1).

For each $p \in \mathbb{N}$ we define the F-linear map

$$\varphi_p : (\Omega^p_{A/F})^* \to C_pA \tag{4.2}$$

by the formula

$$(\varphi_p \vartheta)(a, b_1, \ldots, b_p) = \vartheta(a\,db_1 \wedge \cdots \wedge db_p)$$

for every $\vartheta \in (\Omega^p_{A/F})^*$ and all $a, b_1, \ldots, b_p \in A$ (here, as usual, $(\Omega^p_{A/F})^*$ stands for the vector space over F consisting of all F-linear maps $\Omega^p_{A/F} \to F$).

Theorem 4.1. *If A is an associative and commutative F-algebra with unit endowed with the discrete topology, then the maps* (4.2) *form an isomorphism between the chain complex dual to the de Rham cochain complex* (4.1) *and the chain complex* (2.1).

Proof. A trivial verification shows that for every $p \geq 1$ we have

$$\varphi_{p-1} \circ d^{p*} = \partial_p \circ \varphi_p,$$

where $d^{p*} : (\Omega^p_{A/F})^* \to (\Omega^{p-1}_{A/F})^*$ is given by $d^{p*}(\vartheta) = \vartheta \circ d^p$, which means that the maps (4.2) form a homomorphism of chain complexes.

Fix a $p \geq 0$. The proof will be completed as soon as we can find the map $\psi_p : C_pA \to (\Omega^p_{A/F})^*$ inverse to φ_p.

We first observe that A^* is an A-module with multiplication given by $(a\vartheta)(b) = \vartheta(ab)$ for every $\vartheta \in A^*$ and all $a, b \in A$. Hence for every $q \geq 0$ the set S^qA consisting of all skew-symmetric A-q-linear maps $\Omega \times \cdots \times \Omega \to A^*$ is an A-module too.

Let $f \in C_pA$. We prove by induction that for every $q \in \{0, \ldots, p\}$ and all $c_1, \ldots, c_{p-q} \in A$ there is a uniquely determined $\alpha^f_{c_1,\ldots,c_{p-q}} \in S^qA$ such that

$$(\alpha^f_{c_1,\ldots,c_{p-q}}(db_1, \ldots, db_q))(a) = f(a, c_1, \ldots, c_{p-q}, b_1, \ldots, b_q) \tag{4.3}$$

for all $a, b_1, \ldots, b_q \in A$. This is obvious for $q = 0$. Assume $q \in \{1, \ldots, p\}$ and the assertion holds for $q - 1$. Let $c_1, \ldots, c_{p-q} \in A$. It is easy to see that

$$\beta^f_{c_1,\ldots,c_{p-q}} : A \to S^{q-1}A, \quad c \mapsto \alpha^f_{c_1,\ldots,c_{p-q},c}$$

is a derivation, because Ω is generated as an A-module by vectors of the form db, where $b \in A$. Since d is the universal derivation, we have the A-linear map $\gamma^f_{c_1,\ldots,c_{p-q}} : \Omega \to S^{q-1}A$ such that $\gamma^f_{c_1,\ldots,c_{p-q}} \circ d = \beta^f_{c_1,\ldots,c_{p-q}}$. Thus

$$\big((\gamma^f_{c_1,\ldots,c_{p-q}}(db_1))(db_2, \ldots, db_q)\big)(a) = f(a, c_1, \ldots, c_{p-q}, b_1, \ldots, b_q)$$

for all $a, b_1, \ldots, b_q \in A$ and it suffices to put

$$\alpha^f_{c_1,\ldots,c_{p-q}}(\omega_1, \ldots, \omega_q) = (\gamma^f_{c_1,\ldots,c_{p-q}}\omega_1)(\omega_2, \ldots, \omega_q)$$

for all $\omega_1, \ldots, \omega_q \in \Omega$. The skew-symmetry of $\alpha^f_{c_1,\ldots,c_{p-q}}$ is a consequence of (4.3), and the induction is complete.

In particular, for $q = p$ we have $\alpha^f \in S^pA$ such that

$$(\alpha^f(db_1, \ldots, db_p))(a) = f(a, b_1, \ldots, b_p)$$

for all $a, b_1, \ldots, b_p \in A$. We now define $\delta^f : \Omega^p_{A/F} \to A^*$ to be the A-linear map such that

$$\delta^f(\omega_1 \wedge \cdots \wedge \omega_p) = \alpha^f(\omega_1, \ldots, \omega_p)$$

for all $\omega_1, \ldots, \omega_p \in \Omega$, and finally

$$(\psi_p f)(\omega) = (\delta^f \omega)(1)$$

for every $\omega \in \Omega^p_{A/F}$. Therefore $\psi_p f \in (\Omega^p_{A/F})^*$ and

$$(\psi_p f)(a\, db_1 \wedge \cdots \wedge db_p) = f(a, b_1, \ldots, b_p)$$

for all $a, b_1, \ldots, b_p \in A$, which means that ψ_p is the map inverse to φ_p. ∎

Corollary 4.2. *If A is an associative and commutative F-algebra with unit endowed with the discrete topology, then for each $p \geq 0$ the homology vector space H_pA of the chain complex* (2.1) *is isomorphic to the vector space dual to the pth cohomology vector space of the de Rham cochain complex* (4.1).

It is worth pointing out that Theorem 4.1 not only gives a link between the chain complex (2.1) and the de Rham complex, but also shows clearly that the choice of a particular topology on the algebra A in the construction of (2.1) influences essentially its homology vector spaces. In fact, Theorem 3.9 would not be true if the $\mathcal{C}^\infty$-compact-open topology on $\mathcal{C}^\infty(M, \mathbb{R})$ was replaced by the discrete one. This follows immediately from Theorem 4.1, because on the one hand, there is no vector space such that its dual vector space has a countably infinite basis, and on the other hand, there exist second countable Hausdorff manifolds of class $\mathcal{C}^\infty$ such that some of their differentiable singular homology groups with real coefficients have countably infinite bases.

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