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Partial regularity of minimizers of quasiconvex integrals with subquadratic growth: the general case

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Abstract. We prove partial regularity for minimizers of the functional $\int_\Omega f(x,u(x),Du(x))\,dx$ where the integrand $f(x,u,\xi)$ is quasiconvex with subquadratic growth: $|f(x,u,\xi)|\le L(1+|\xi|^p)$, $p<2$. We also obtain the same results for ω-minimizers.

1. Introduction. In this paper we study the partial regularity of minimizers of the functional

$$I(u)=\int_\Omega f(x,u(x),Du(x))\,dx \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, u is a $W^{1,p}(\Omega,\mathbb{R}^N)$ function with $p>1$ and $f(x,u,\xi):\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}$ is a uniformly strictly quasiconvex function, i.e.

$$\begin{aligned}\int_\Omega & f(x,u,\xi+D\phi(x))\,dx \\ &\ge \int_\Omega [f(x,u,\xi)+\nu(1+|\xi|^2+|D\phi(x)|^2)^{(p-2)/2}|D\phi(x)|^2]\,dx\end{aligned} \tag{1.2}$$

for any $(u,\xi)\in\mathbb{R}^N\times\mathbb{R}^{nN}$ and $\phi\in C^1_0(\Omega,\mathbb{R}^N)$.

The condition (1.2), in a slightly different form, was introduced in [E] in order to obtain partial regularity results for minimizers of the functional (1.1) in the case $p>2$. Evans assumed the integrand to depend on the gradient of u only, and a control condition on the second derivatives of the energy density f.

Next, in [FH], [GM] and [AF4], the same regularity results in the case of f depending on x, u, Du and without any control condition on the second derivatives were obtained.

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In 1991 Šverák gave examples of genuine quasiconvex and not polyconvex functions with subquadratic growth. Only then the problem of partial regularity in the subquadratic case became to be seriously considered.

We notice that a first regularity result in this direction was obtained in [CP] under the more restrictive assumption $2n/(n+2) < p < 2$. The case $1 < p < 2$ was treated in [CFM] with the function f only depending on the gradient Du.

In this paper we consider the regularity problem in its full generality, namely f is supposed also to depend on x and u, i.e. $f = f(x, u, Du)$. To face the problem we have to adapt a certain number of techniques used in the superquadratic case, $p > 2$ (see [AF3], [FH]), and combine them a suitable way with the new tools developed in [CFM]. The proof of the regularity of u is based as usual on a blow-up argument aiming to establish a decay estimate for the excess function

$$E(x_0, R) = R^\delta + \fint_{B_R(x_0)} |V(Du(x)) - V((Du)_{x_0,R})|^2\, dx$$

where $\delta > 0$ and

$$V(\xi) = (1 + |\xi|^2)^{(p-2)/4}\xi$$

where the structure of E reflects the quasiconvexity condition (1.2) and the term R^δ is due to a technical complication arising in the use of a sort of "freezing argument" based on Ekeland's variational principle (see Lemma 3.7). We mention that in order to get a crucial higher integrability result we use a new Poincaré type inequality on increasing spheres:

$$\text{(1.3)}\quad \Big(\fint_{B_R} \Big|V\Big(\frac{u-u_R}{R}\Big)\Big|^{2(1+\sigma)}\, dx\Big)^{1/(2(1+\sigma))} \le c\Big(\fint_{B_{3R}} |V(Du)|^\alpha\, dx\Big)^{1/\alpha}$$

provided $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $2/p < \alpha < 2$ and $\sigma > 0$ where the function V is used as a quasinorm (see Lemma 2.1 for the basic properties of V).

Finally, the same regularity results are given for an ω-minimizer of the functional, i.e. a function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that

$$I(u, B_r) \le [1 + \omega(r)]I(u + \phi, B_r)$$

for any function $\phi \in W_0^{1,p}(B_r, \mathbb{R}^N)$, $B_r \subset\subset \Omega$, with $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous nondecreasing concave function such that $\omega(0) = 0$ and $\omega(r) \le cr^{\overline{\sigma}}$, $\overline{\sigma} > 0$, completing the regularity result for ω-minimizers given in [Gi], Ch. 9.

2. Preliminary results. In the following Ω will denote a bounded open set in $\mathbb{R}^n$, $B_R(x_0)$ the ball $\{x \in \mathbb{R}^n : |x - x_0| < R\}$, and if h is an integrable function we define

$$(h)_{x_0,R} := \fint_{B_R(x_0)} h(x)\,dx = \frac{1}{\omega_n R^n} \int_{B_R(x_0)} h(x)\,dx\,,$$

where ω_n is the Lebesgue measure of the n-dimensional unit ball. When no confusion may arise we write simply $(h)_R$ in place of $(h)_{x_0,R}$ and B_R in place of $B_R(x_0)$. Throughout the paper p will be a number between 1 and 2 and for $\xi \in \mathbb{R}^k$ we define

$$V(\xi) = (1+|\xi|^2)^{(p-2)/4}\xi. \tag{2.1}$$

The following statement contains some useful properties of the function V.

LEMMA 2.1. *Let* $1 < p < 2$, *and let* $V : \mathbb{R}^k \to \mathbb{R}^k$ *be defined by* (2.1). *Then for any* $\xi, \eta \in \mathbb{R}^k$, $t > 0$,

(i) $$2^{(p-2)/4} \min\{|\xi|, |\xi|^{p/2}\} \le |V(\xi)| \le \min\{|\xi|, |\xi|^{p/2}\},$$

(ii) $$|V(t\xi)| \le \max\{t, t^{p/2}\}|V(\xi)|,$$

(iii) $$|V(\xi+\eta)| \le c(p)[|V(\xi)| + |V(\eta)|],$$

(iv) $$\frac{p}{2}|\xi-\eta| \le \frac{|V(\xi)-V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{(p-2)/4}} \le c(k,p)|\xi-\eta|,$$

(v) $$|V(\xi)-V(\eta)| \le c(k,p)|V(\xi-\eta)|,$$

(vi) $$|V(\xi-\eta)| \le c(p,M)|V(\xi)-V(\eta)| \quad \text{if } |\eta| \le M \text{ and } \xi \in \mathbb{R}^k,$$

(vii) $$\forall \varepsilon > 0\ \exists c_\varepsilon > 0 \quad |\xi| \le \varepsilon |V(\xi)|^2 + c_\varepsilon.$$

Proof. See Lemma 2.1 of [CFM] for (i)–(vi) while (vii) trivially follows from the definition of V. ∎

THEOREM 2.2. *If* $1 < p < 2$, *there exist* $2/p < \alpha < 2$ *and* $\sigma > 0$ *such that if* $u \in W^{1,p}(B_{3R}(x_0), \mathbb{R}^N)$, *then*

$$\Bigg(\fint_{B_R(x_0)} \left|V\left(\frac{u-(u)_{x_0,R}}{R}\right)\right|^{2(1+\sigma)} dx\Bigg)^{1/(2(1+\sigma))} \le c\Big(\fint_{B_{3R}(x_0)} |V(Du)|^\alpha\,dx\Big)^{1/\alpha}, \tag{2.2}$$

where $c \equiv c(n,p,N)$ *is independent of* R *and* u.

Proof. See [CFM]. ∎

REMARK 2.3. The Sobolev–Poincaré type inequality stated above has been proven in [CFM] and it is essential in order to get our regularity result (see Theorem 3.2, Step 3). The proof is essentially based on some estimates for the maximal function combined with the properties of $V(t)$ stated in Lemma 2.1.

The following is a technical result used in the proof of Lemma 2.5 and a straightforward generalization of a classical interpolation lemma (see [Gi], Lemma 6.1).

LEMMA 2.4. *Let* $f : [r/2, r] \to [0, \infty[$ *be a bounded function such that for all* $r/2 < t < s < r$,

$$f(t) \le \theta f(s) + A \int_{B_r} \left| V\left(\frac{h(x)}{s-t}\right)\right|^2 dx,$$

where $h \in L^p(B_r)$, $A > 0$, *and* $0 < \theta < 1$. *Then there exists* $c \equiv c(\theta)$ *such that*

$$f\left(\frac{r}{2}\right) \le c(\theta) A \int_{B_r} \left| V\left(\frac{h(x)}{r}\right)\right|^2 dx.$$

Proof. See [CFM]. ∎

We are now in a position to prove the following higher integrability result (see [AF3] for the case $p \ge 2$).

LEMMA 2.5. *Let* $g : \mathbb{R}^{nN} \to \mathbb{R}$ *be a function of class* C^1 *such that*

$$|g(\xi)| \le c(1 + \lambda^2|\xi|^2)^{(p-2)/2}|\xi|^2,$$
$$\int_{\Omega} g(D\phi(x))\, dx \ge \nu \int_{\Omega} \frac{1}{\lambda^2}|V(\lambda D\phi(x))|^2\, dx$$

for any $\phi \in C_0^1(\mathbb{R}^n, \mathbb{R}^N)$, *for suitable positive constants* c, ν, $0 < \lambda < 1$. *Fix* $0 \le \mu < 1$ *and let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *satisfy*

$$\int_{\Omega} g(Du(x))\, dx \le \int_{\Omega} [g(Du(x) + D\phi(x)) + \mu|D\phi(x)|]\, dx$$

for all $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$. *Then there exist* c_0, δ, *depending only on* p, n, N, L, ν *but independent of* λ, u *and* μ, *such that for any* $B_r(x_0) \subset \Omega$,

$$\fint_{B_{r/2}} |V(\lambda Du)|^{2(1+\delta)}\, dx \le c_0 \Big(\fint_{B_{3r}} (\lambda^2 \mu + |V(\lambda Du)|^2)\, dx\Big)^{1+\delta}. \tag{2.3}$$

Proof. Fix $B_r \subset \Omega$, let $\frac{1}{2}r < t < s < r$ and take a cut-off function $\zeta \in C_0^1(B_s)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ on B_t and $|D\zeta| \le 2/(s-t)$. Set

$$\phi_1 = [u - (u)_r]\zeta, \qquad \phi_2 = [u - (u)_r](1 - \zeta).$$

Then $Du = D\phi_1 + D\phi_2$. Using the growth conditions, the minimality of u and Lemma 2.1(vii) we easily get

$$\frac{\nu}{\lambda^2}\int_{B_s}|V(\lambda D\phi_1)|^2\,dx \le \int_{B_s} g(D\phi_1)\,dx = \int_{B_s} g(Du - D\phi_2)\,dx$$
$$= \int_{B_s} g(Du)\,dx + \int_{B_s}[g(Du-D\phi_2) - g(Du)]\,dx$$
$$\le \int_{B_s\setminus B_t} g(D\phi_2)\,dx + \int_{B_s\setminus B_t}[g(Du-D\phi_2)-g(Du)]\,dx + \int_{B_s}\mu|D\phi_1|\,dx$$
$$\le c\int_{B_s\setminus B_t}\frac{1}{\lambda^2}(|V(\lambda D\phi_2)|^2 + |V(\lambda(Du-D\phi_2))|^2 + |V(\lambda Du)|^2)\,dx$$
$$+ \frac{\mu\nu}{2}\int_{B_s}|V(D\phi_1)|^2\,dx + c(\nu)\mu r^n$$
$$\le c\int_{B_s\setminus B_t}\frac{1}{\lambda^2}(|V(\lambda D\phi_2)|^2 + |V(\lambda(Du-D\phi_2))|^2 + |V(\lambda Du)|^2)\,dx$$
$$+ \frac{\mu\nu}{2\lambda^2}\int_{B_s}|V(\lambda D\phi_1)|^2\,dx + c(\nu)\mu r^n.$$

By absorbing the last integral above in the left hand side and using Lemma 2.1, we have

$$\frac{1}{\lambda^2}\int_{B_t}|V(\lambda Du)|^2\,dx \le \frac{1}{\lambda^2}\int_{B_s}|V(\lambda D\phi_1)|^2\,dx$$
$$\le \widetilde{c}(\nu)\left[\mu r^n + \int_{B_s\setminus B_t}\frac{1}{\lambda^2}\left(|V(\lambda Du)|^2 + \left|V\left(\lambda\frac{u-(u)_r}{s-t}\right)\right|^2\right)dx\right].$$

We "fill the hole" by adding to both sides the term

$$\widetilde{c}\,\frac{1}{\lambda^2}\int_{B_t}|V(\lambda Du)|^2\,dx,$$

then we divide by $\widetilde{c}+1$, thus obtaining

$$\frac{1}{\lambda^2}\int_{B_t}|V(\lambda Du)|^2\,dx$$
$$\le \frac{\widetilde{c}}{\widetilde{c}+1}\left[\frac{1}{\lambda^2}\int_{B_s}|V(\lambda Du)|^2\,dx + c\int_{B_r}\left(\mu + \frac{1}{\lambda^2}\left|V\left(\lambda\frac{u-(u)_r}{s-t}\right)\right|^2\right)dx\right].$$

Now, by Lemma 2.4 above, we get

$$\int_{B_{r/2}}|V(\lambda Du)|^2\,dx \le c\int_{B_r}\left|V\left(\lambda\frac{u-(u)_r}{r}\right)\right|^2 dx + c(\nu)\int_{B_r}\lambda^2\mu\,dx$$

and so, by (2.2), we get

$$\fint_{B_{r/2}} |V(\lambda Du)|^2\,dx \le c\Big(\fint_{B_r}\Big|V\Big(\lambda\frac{u-(u)_r}{r}\Big)\Big|^{2(1+\sigma)}dx\Big)^{1/(1+\sigma)} + c\int_{B_r}\lambda^2\mu\,dx$$
$$\le c\Big(\fint_{B_{3r}}|V(\lambda Du)|^\alpha\,dx\Big)^{2/\alpha} + c\int_{B_r}\lambda^2\mu\,dx$$

with $2/p<\alpha<2$. From this inequality the result follows immediately just by applying the version of the Gehring Lemma due to Giaquinta and Modica (see [G], Theorem 1.1, Chapter 5). ∎

Now we give a list of useful lemmas. The following lemma is a slightly modified version of the approximation result proved in [AF2].

LEMMA 2.6. *Let $u\in W^{1,q}(\mathbb{R}^n,\mathbb{R}^N)$ with $q\ge 1$. For every $K>0$, set*

$$H_K=\{x\in\mathbb{R}^n : M(Du)\le K\}.$$

Then there exists a Lipschitz function $w:\mathbb{R}^n\to\mathbb{R}^N$ such that $\|Dw\|_\infty\le cK$, $w=u$ on H_K and

$$\operatorname{meas}(\mathbb{R}^n\setminus H_K)\le c\|Du\|_q^q/K^q,$$

where c depends only on n,N,q.

The next result is a simple consequence of a priori estimates for solutions of linear elliptic systems with constant coefficients.

PROPOSITION 2.7. *Let $u\in W^{1,1}(\Omega,\mathbb{R}^N)$ be such that*

$$\int_\Omega A^{ij}_{\alpha\beta}D_\alpha u^i D_\beta\phi^j\,dx=0 \tag{2.4}$$

for any $\phi\in C^1_0(\Omega,\mathbb{R}^N)$, where $(A^{ij}_{\alpha\beta})$ is a constant matrix satisfying the strong Legendre–Hadamard condition:

$$A^{ij}_{\alpha\beta}\lambda^i\lambda^j\mu_\alpha\mu_\beta\ge\nu|\lambda|^2|\mu|^2\quad\text{for any } \lambda\in\mathbb{R}^N,\ \mu\in\mathbb{R}^n.$$

Then u is C^∞ and for any $B_R(x_0)\subset\Omega$,

$$\sup_{B_{R/2}}|Du|\le\frac{c}{R^n}\int_{B_R}|Du|\,dx, \tag{2.5}$$

where c depends only on n, N, p, ν and $\max A^{ij}_{\alpha\beta}$.

Proof. See [CFM], Prop. 2.10. ∎

The following selection theorem due to Eisen [Ei] is also useful.

LEMMA 2.8. *Let G be a measurable subset of $\mathbb{R}^k$ with $\operatorname{meas}(G)<\infty$. Assume (M_h) is a sequence of measurable subsets of G such that, for some $\varepsilon>0$,*

$$\operatorname{meas}(M_h)\ge\varepsilon\quad\text{for all } h\in\mathbb{N}.$$

Then a subsequence (M_{h_k}) can be selected such that $\bigcap_k M_{h_k}\neq\emptyset$.

We conclude this section by recalling a well known variational lemma due to Ekeland. It will be one of the main technical tools in the next section.

THEOREM 2.9 (Ekeland). *Let (X,d) a complete metric space and $F : X \to (-\infty,\infty)$ a lower semicontinuous functional such that*

$$-\infty < \inf_X F < \infty.$$

Let $\varepsilon > 0$ and $x \in X$ be such that

$$F(x) \le \inf_X F + \varepsilon.$$

Then there exists $y \in X$ such that

$$d(x,y) < 1, \quad F(y) \le F(x), \quad F(x) \le F(z) + \varepsilon d(y,z) \quad \forall z \in X.$$

Proof. See [Ek] and [Gi], Chap. 5. ∎

3. Proof of the main result. In this section we will prove the partial regularity of minimizers of the functional

$$I(v) := \int_\Omega f(x, v(x), Dv(x))\,dx,$$

where $v \in W^{1,p}(\Omega,\mathbb{R}^N)$, $1 < p < 2$, and $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to \mathbb{R}$ is a function satisfying the following assumptions:

$$(H_1) \qquad |f(x,v,\xi)| \le L(1+|\xi|^p);$$

$$(H_2) \qquad \int_\Omega f(x_0,v_0,\xi+D\phi(x))\,dx \ge \int_\Omega [f(x_0,v_0,\xi) + \nu(1+|\xi|^2+|D\phi(x)|^2)^{(p-2)/2}|D\phi(x)|^2]\,dx \quad \forall (x_0,v_0,\xi) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN},\ \forall \phi \in C^1_0(\Omega,\mathbb{R}^N);$$

$$(H_3) \qquad |f(x,u,\xi) - f(y,v,\xi)| \le C(1+|\xi|^p)\gamma(|x-y|^p + |u-v|^p)$$

where $\gamma(t) \le t^\sigma$, $0 < \sigma < 1/p$ and γ is bounded, concave, nonnegative and increasing;

$$(H_4) \qquad f_{\xi\xi}(x,u,\xi) \quad \text{is continuous;}$$

there is a continuous function $\psi : \mathbb{R}^{nN} \to \mathbb{R}$ satisfying

$$(H_5) \qquad f(x,u,\xi) \ge \psi(\xi)$$

and

$$\int_\Omega \psi(D\phi(x))\,dx \ge \int_\Omega [\psi(0) + \nu|D\phi(x)|^p]\,dx$$

for every $\phi \in C^1_0(\Omega,\mathbb{R}^N)$ with $C,\nu > 0$.

REMARK 3.1. Condition (H_2), introduced in [E] in the case $p \geq 2$, is called *uniform strict quasiconvexity* and implies that for any $(x, v, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$, $\lambda \in \mathbb{R}^N$, $\mu \in \mathbb{R}^n$,

$$\frac{\partial^2 f}{\partial \xi_\alpha^i \partial \xi_\beta^j}(x, v, \xi)\lambda^i \lambda^j \mu_\alpha \mu_\beta \geq c\nu(1 + |\xi|^2)^{(p-2)/2}|\lambda|^2|\mu|^2,$$

with c independent of ξ, λ, μ.

Notice that we do not assume any control on second derivatives. However, if a function f is *quasiconvex*, i.e. satisfies (H_2) with $\nu = 0$, and has the growth control (H_1), then it is well known (see [AF1], [M]) that

$$|Df(x, v, \xi)| \leq c(n, N, p)L(1 + |\xi|^{p-1}). \tag{3.1}$$

We also recall that a function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ is a *minimizer* of $I(v)$ if for any function $\phi \in W_0^{1,p}(\Omega, \mathbb{R}^N)$,

$$I(u, \Omega) \leq I(u + \phi, \Omega);$$

while an ω-*minimizer* is a function $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that

$$I(u, B_r) \leq [1 + \omega(r)]I(u + \phi, B_r)$$

for any function $\phi \in W_0^{1,p}(B_r, \mathbb{R}^N)$, $B_r \subset\subset \Omega$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing concave function such that $\omega(0) = 0$. It is easy to check that a minimizer is also an ω-minimizer if we take $\omega = 0$.

We can now state the main result of this section.

THEOREM 3.2. *Let f be a C^2 function satisfying (H_i) for $i = 1, \ldots, 5$ and let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be an ω-minimizer of a functional $I(v)$ with $\omega(r) \leq cr^{\overline{\delta}}$, $\overline{\delta} > 0$. Then there exists an open subset Ω_0 of Ω such that* $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$ *and u is in $C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for some $\alpha < 1$.*

A standard technique in order to prove such results is to look at the decay in small balls around a point x_0 of the so-called *excess* of the gradient of the solution u. Roughly speaking the excess $E(x_0, R)$ measures how far the gradient is from being constant in the ball $B_R(x_0)$. In our case, following the techniques introduced in [FH], in view of some estimates provided by applications of Ekeland's variational pinciple, it will be convenient to define

$$E(x_0, R) = \fint_{B_R(x_0)} |V(Du(x)) - V((Du)_{x_0,R})|^2\, dx + R^\delta,$$

where $\delta > 0$ is a suitable positive constant and V is given by (2.1).

We recall here a semicontinuity result:

LEMMA 3.3. *Let $p \geq 1$ and let $f : \mathbb{R}^{nN} \to \mathbb{R}$ be a quasiconvex function of class C^1 satisfying*

$$|f(\xi)| \leq L(1 + |\xi|^p).$$

Then for every $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ the functional $\int_\Omega f(Dw)\,dx$ is sequentially lower semicontinuous on the Dirichlet class $u+W_0^{1,p}(\Omega,\mathbb{R}^N)$ endowed with the weak topology of $W^{1,p}$.

Proof. The proof is a simple consequence of the semicontinuity theorem of [AF1] (see also [AF3]). ∎

The following higher integrability result can be found for example in [Gi]; see also [AF3] and [M].

THEOREM 3.4. *Let f satisfy (H_1)–(H_5) and*

$$|f(x,u,\xi+\eta)-f(x,u,\xi)| \le c(1+|\xi|^{p-1}+|\eta|^{p-1})|\eta|,$$

and let $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ be a minimizer of I. Then there are $q_0 > p$ and $C_0 > 0$, independent of u, such that $u \in W^{1,q_0}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and for every $B_r \subset \Omega$,

$$\Big(\fint_{B_{r/2}} |Du|^{q_0}\,dx\Big)^{1/q_0} \le C_0\Big(\fint_{B_r}(1+|Du|)^p\,dx\Big)^{1/p}.$$

Proof. See Lemma IV.3 of [AF3]. ∎

REMARK 3.5. We remark that it is possible to get higher integrability up to the boundary. In fact for $\Omega = B_r$, following [Gi], page 112 (see also [AF3], Remark (IV.4)), if there is a function $u_0 \in W^{1,q}(\mathbb{R}^n,\mathbb{R}^N)$ such that

$$u-u_0 \in W_0^{1,p}(B,\mathbb{R}^N), \qquad q \ge p,$$

then there exist q_0, C_0, $p<q_0<q$, such that $u \in W^{1,q_0}(B,\mathbb{R}^N)$ and

$$\Big(\fint_B |Du|^{q_0}\,dx\Big)^{1/q_0} \le c_0\Big(\fint_B(1+|Du|^p)\,dx\Big)^{1/p} + \Big(\fint_B |Du_0|^{q_0}\,dx\Big)^{1/q_0}.$$

LEMMA 3.6. *Let f satisfy (H_1), (H_2) and fix $x_0 \in \Omega$ and $u_0 \in \mathbb{R}^N$. If B_r is any ball in $\mathbb{R}^n$ and $u \in W^{1,p}(B_r,\mathbb{R}^N)$, then the functional $\int_{B_r} f(x_0,u_0,Dw(x))\,dx$ is sequentially weakly lower semicontinuous on $u+W_0^{1,p}(B_r,\mathbb{R}^N)$ and satisfies*

$$\int_{B_r} f(x_0,u_0,Dw(x))\,dx \ge \nu\int_{B_r}|Dw|^p\,dx - c\int_{B_r}(1+|Du|^p)\,dx.$$

Proof. The semicontinuity follows from Lemma 3.3, since (H_2) implies quasiconvexity.

Now, let $\widetilde{u} \in (u)_r + W_0^{1,p}(B_{2r},\mathbb{R}^N)$ be an extension of u such that $\int_{B_{2r}}|D\widetilde{u}|^p\,dx \le c\int_{B_r}|Du|^p\,dx$; if we set, for every $w \in u+W_0^{1,p}(B_r,\mathbb{R}^N)$,

$$\widetilde{w} = \begin{cases} w & \text{in } B_r,\\ \widetilde{u} & \text{in } B_{2r}\setminus B_r,\end{cases}$$

then by (H_2) and Lemma 2.1(vii),

$$\begin{aligned}
&\int_{B_r} f(x_0,u_0,Dw)\,dx + \int_{B_{2r}\setminus B_r} f(x_0,u_0,D\widetilde{u})\,dx \\
&\quad = \int_{B_{2r}} f(x_0,u_0,D\widetilde{w})\,dx \ge \int_{B_r} \nu|V(Dw)|^2\,dx + \int_{B_{2r}} f(x_0,u_0,0)\,dx \\
&\quad = \int_{B_r\cap\{|Dw|>1\}} \nu|V(Dw)|^2\,dx + \int_{B_r\cap\{|Dw|\le 1\}} \nu|V(Dw)|^2\,dx \\
&\qquad + \int_{B_{2r}} f(x_0,u_0,0)\,dx \\
&\quad \ge \int_{B_r\cap\{|Dw|>1\}} \nu|Dw|^p\,dx + \int_{B_r\cap\{|Dw|\le 1\}} \nu|Dw|\,dx - c + \int_{B_{2r}} f(x_0,u_0,0)\,dx \\
&\quad \ge \int_{B_r} \nu|Dw|^p\,dx - c + \int_{B_{2r}} f(x_0,u_0,0)\,dx.
\end{aligned}$$

The assertion follows easily by (H_1). ■

LEMMA 3.7. *There exist constants* $0<\beta_1<\beta_2<1$ *and* $c_k>0$, *for every* $k>0$, *such that if* u *is a minimizer of* I, $r<1$, $B_{2r}(x_0)\subset\Omega$ *and* $(|Du|^p)_{x_0,2r}\le K$, *then there is a* $v\in u+W_0^{1,p}(B_r(x_0),\mathbb{R}^N)$ *such that*

$$(k) \qquad \Big(\int_{B_{r/2}} |Dv-Du|^p\,dx\Big)^{1/p} \le c_K r^{\beta_1}$$

and

$$\begin{aligned}
(kk) \qquad &\int_{B_r} f(x_0,(u)_{x_0,r},Dv(x))\,dx \\
&\qquad \le \int_{B_r} f(x_0,(u)_{x_0,r},Dv(x)+D\phi(x))\,dx + r^{\beta_2}\int_{B_r}|D\phi|\,dx
\end{aligned}$$

for every $\phi\in C_0^1(B_r(x_0),\mathbb{R}^N)$.

Proof. By Theorem 3.4, there exist $q_0>p$ and c_0 such that $u\in W^{1,q_0}_{\rm loc}(\Omega)$ and

$$(3.2) \qquad \Big(\fint_{B_{r/2}} |Du|^{q_0}\,dx\Big)^{1/q_0} \le c_0\Big(\fint_{B_r}(1+|Du|^p)\,dx\Big)^{1/p}$$

for any $B_r\subset\subset\Omega$. By Lemma 3.6 there exists $\overline{u}\in u+W_0^{1,p}(B_r,\mathbb{R}^N)$, solution of the problem

$$\min\{I_r^0(w) : w\in W_0^{1,p}(B_r,\mathbb{R}^N)\}$$

where

$$I_r^0(w) = \fint_{B_r} f(x_0, (u)_r, Dw)\,dx.$$

By Remark 3.5 we can pick $q_1 > 0$ such that $p < q_1 \leq q_0$ with

$$\begin{aligned}(3.3)\quad &\Big(\fint_{B_r} |D\overline{u}|^{q_1}\,dx\Big)^{1/q_1} \\ &\leq c_0\Big[\Big(\fint_{B_r}(1+|D\overline{u}|^p)\,dx\Big)^{1/p} + \Big(\fint_{B_r}(1+|Du|^{q_1})\,dx\Big)^{1/q_1}\Big].\end{aligned}$$

From Lemma 3.6, the minimality of $\overline{u}$ and the growth assumption we have

$$\begin{aligned}(3.4)\qquad \fint_{B_r} |D\overline{u}|^p\,dx &\leq \fint_{B_r} f(x_0,(u)_r,Du)\,dx + c\fint_{B_r}(1+|Du|^p)\,dx \\ &\leq c\fint_{B_r}(1+|Du|^p)\,dx.\end{aligned}$$

Finally, by (3.2)–(3.4) and Hölder's inequality we have

$$\begin{aligned}(3.5)\qquad \Big(\fint_{B_r}|D\overline{u}|^{q_1}\,dx\Big)^{1/q_1} &\leq c\Big(\fint_{B_r}(1+|Du|^p)\,dx\Big)^{1/p} \\ &\leq c\Big(\fint_{B_r}(1+|Du|^{q_1})\,dx\Big)^{1/q_1}.\end{aligned}$$

Now we estimate the difference $I_r^0(u,B_r) - I_r^0(\overline{u},B_r)$. We have

$$\begin{aligned}I_r^0(u,B_r) - I_r^0(\overline{u},B_r) &= \fint_{B_r}[f(x_0,(u)_r,Du) - f(x_0,(\overline{u})_r,D\overline{u})]\,dx \\ &= \fint_{B_r}[f(x_0,(u)_r,Du) - f(x,u,Du)]\,dx \\ &\quad + \fint_{B_r}[f(x,\overline{u},D\overline{u}) - f(x_0,(\overline{u})_r,D\overline{u})]\,dx \\ &\quad + \fint_{B_r}[f(x,u,Du) - f(x,\overline{u},D\overline{u})]\,dx = \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

By ω-minimality we get

$$\mathrm{III} \leq c\omega(r)\fint_{B_r} f(x,\overline{u},D\overline{u})\,dx.$$

Moreover, by the growth condition, (3.4) and (3.2) we obtain

$$\begin{aligned}\mathrm{III} &\leq c\omega(r)\fint_{B_r}(1+|D\overline{u}|^p) \leq c\,\omega(r)\fint_{B_r}(1+|Du|^p)\,dx \\ &\leq c(k)\omega(r) \leq c(k)r^{\overline{\sigma}}.\end{aligned}$$

Now let us estimate I using the hypotheses on γ and the Hölder and Sobolev–Poincaré inequalities:

$$
\begin{aligned}
I &\le \fint_{B_r} (1+|Du|^p)\gamma(|x-x_0|^p + |u-(u)_r|^p)\,dx \\
&\le c\Big(\fint_{B_r} (1+|Du|^{q_0})\,dx\Big)^{p/q_0} \\
&\quad \times \Big(\fint_{B_r} \gamma(|x-x_0|^p + |u-(u)_r|^p)^{q_0/(q_0-p)}\,dx\Big)^{1-p/q_0} \\
&\le c(k)\gamma\Big(r^p\Big(1+\fint_{B_r}|Du|^p\,dx\Big)\Big)^{1-p/q_0} \le c(k)r^{p\sigma(1-p/q_0)} = c(k)r^{\delta_1}.
\end{aligned}
$$

As for I and III, we obtain

$$
\begin{aligned}
\mathrm{II} &\le c\Big(\fint_{B_r}(1+|D\overline{u}|^{q_1})\,dx\Big)^{p/q_1}\Big(\fint_{B_r}\gamma(|x-x_0|^p + |u-u_r|^p)^{q_1/(q_1-p)}\,dx\Big)^{1-p/q_1} \\
&\le c(k)\gamma\Big(r^p\Big(1+\fint_{B_r}|Du|^p\,dx\Big)\Big)^{1-p/q_1} \le c(k)r^{p\sigma(1-p/q_1)} = c(k)r^{\delta_2}.
\end{aligned}
$$

Then we get

$$I_r^0(u) - I_r^0(\overline{u}) \le c(k)r^{\delta_3}$$

where $\delta_3 = \min\{\delta_1,\, \delta_2,\, \overline{\sigma}\}$.

Now we consider the complete metric space $u + W_0^{1,1}(B_r)$ endowed with the metric

$$d(v,w) = (c(k)r^{\delta_3/2})^{-1}\fint_{B_r}|Dv - Dw|\,dx$$

and we set

$$J(w) = \begin{cases} I_r^0(w) & \text{if } w \in u + W_0^{1,p}(B_r), \\ \infty & \text{otherwise.}\end{cases}$$

This functional is lower semicontinuous in $u + W_0^{1,1}(B_r)$ by Lemma 3.6 and by definition of $\overline{u}$ it turns out that

$$\inf J = I_r^0(\overline{u}).$$

Now we apply Ekeland's theorem to find a function $v \in u + W_0^{1,1}(B_r)$ with

$$\fint_{B_r}|Dv - Dw|\,dx < c(k)r^{\delta_3/2}$$

which minimizes the functional

$$\widetilde{J}(w) = J(w) + r^{\delta_3/2}\fint_{B_r}|Dv - Dw|\,dx,$$

that is, satisfies (kk) with $\beta_2 = \delta_3/2$ if we pick $w = u$ and we put $\phi = v - u$.

Applying Theorem 3.4 to the functional $\widetilde{J}(w)$ we see that there are constants q and c independent of k, r and satisfying $p < q < q_0$ such that $v \in W^{1,q}_{\mathrm{loc}}(B_r)$ and

$$\Big(\fint_{B_{r/2}} |Dv|^q \, dx \Big)^{1/q} \le c \Big(\fint_{B_r} (1 + |Dv|^p) \, dx \Big)^{1/p}.$$

Now, if $\tau = (q-p)/((q-1)p)$ so that $1/p = \tau + (1-\tau)/q$, interpolating we have

$$\Big(\fint_{B_{r/2}} |Dv - Du|^p \, dx \Big)^{1/p} \le c \Big(\fint_{B_{r/2}} |Dv - Du| \, dx \Big)^{\tau} \Big(\fint_{B_{r/2}} |Dv - Du|^q \, dx \Big)^{(1-\tau)/q}.$$

Using the previous inequalities and the assumptions on $(Du)_r$, we get

$$\Big(\fint_{B_{r/2}} |Dv - Du|^p \, dx \Big)^{1/p} \le c(k) r^{\delta_3 \tau/2}$$

and the assertion follows with $\beta_1 = \delta_3 \tau/2 < \beta_2$. ∎

To complete the proof of Theorem 3.2, we need the following technical lemma (see [AF4] for the proof):

LEMMA 3.8. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a function of class C^2 satisfying, for any $\xi \in \mathbb{R}^k$,*

$$|Df(\xi)| \le L(1 + |\xi|^2)^{(p-1)/2}$$

with $1 < p < 2$. Then for any $M > 0$ there exists a constant c, depending only on M, p, L, such that if we set, for any $\lambda > 0$ and $A \in \mathbb{R}^k$ with $|A| \le M$,

$$f_{A,\lambda}(\xi) = \lambda^{-2}[f(A + \lambda\xi) - f(A) - \lambda Df(A)\xi],$$

then

$$|f_{A,\lambda}(\xi)| \le c(p, L, M)(1 + |\lambda\xi|^2)^{(p-2)/2} |\xi|^2.$$

Due to the fact that we will use comparison functions provided by Theorem 2.9, we have to modify our excess function for the gradient of the minimizer u. Namely we define

$$E(x_0, R) = R^\delta + \fint_{B_R(x_0)} |V(Du) - V((Du)_{x_0,R})|^2 \, dx$$

with $0 < \delta < \beta_1$, β_1 given by Lemma 3.7.

We can now establish the decay estimate of $E(x_0, R)$. The proof we give is based on an idea contained in [EG], later modified in [AF4] in order to deal with functionals with no control on the second derivatives (see also [CP]). We will follow closely the various steps of the proof as presented in [CFM].

PROPOSITION 3.9 (Decay estimate). *Fix $M > 0$. There exists a constant C_M such that for every $0 < \tau < 1/8$ there is an $\varepsilon \equiv \varepsilon(\tau, M)$ such that if*

$|(u)_{x_0,R}| \le M$, $|(Du)_{x_0,R}| \le M$ *and* $E(x_0,R) < \varepsilon$ *then*

$$E(x_0,\tau R) \le C_M \tau^{\delta} E(x_0,R).$$

Proof. Fix M and τ. We shall determine C_M at the end of the proof.

Step 1: blow-up. We argue by contradiction, assuming that there is a sequence $B_{4R_h}(x_h)$ of balls contained in Ω such that $|(u)_{x_h,4R_h}| \le M$ $|(Du)_{x_h,4R_h}| \le M$, $\lim_h E(x_h,4R_h) = 0$ and

$$E(x_h,4\tau R_h) > C_M \tau^{\delta} E(x_h,4R_h). \tag{3.6}$$

Set

$$\lambda_h^2 = E(x_h,4R_h).$$

Using Lemma 2.1(i)&(iv) and the previous assumptions, we have

$$\begin{aligned}\lambda_h^2 &\ge c \fint_{B_{4R_h}(x_h)} |V(Du) - V((Du)_{x_h,4R_h})|^2\,dx \\ &\ge c \fint_{B_{4R_h}(x_h)} |Du - (Du)_{x_h,4R_h}|^2 (1 + M^2 + |Du|^2)^{(p-2)/2}\,dx \\ &\ge c \fint_{B_{4R_h}(x_h)} |Du - (Du)_{x_h,4R_h}|^2 (1 + 2M^2 + |Du - (Du)_{x_h,4R_h}|^2)^{(p-2)/2}\,dx \\ &\ge c \fint_{B_{4R_h}(x_h)} (1 + 2M^2 + |Du - (Du)_{x_h,4R_h}|^2)^{p/2}\,dx - K_M.\end{aligned}$$

Hence

$$\fint_{B_{4R_h}(x_h)} |Du - (Du)_{x_h,4R_h}|^p \le c$$

for a suitable constant $c > 0$. From the assumptions at the beginning, we get

$$\fint_{B_{4R_h}(x_h)} |Du|^p\,dx \le c' < \infty.$$

Now, by Lemma 3.7, we choose $u_h \in u + W_0^{1,p}(B_{2R_h}(x_h),\mathbb{R}^N)$ such that

$$\left(\fint_{B_{2R_h}(x_h)} |Du - Du_h|^p\,dx\right)^{1/p} \le cR_h^{\beta_1}$$

and, for every $\phi \in C_0^1(B_{2R_h},\mathbb{R}^N)$,

$$\begin{aligned}&\int_{B_{2R_h}(x_h)} f(x_h,(u)_{2R_h},Du_h(x))\,dx \\ &\quad\le \int_{B_{2R_h}(x_h)} f(x_h,(u)_{2R_h},Du_h(x) + D\phi(x))\,dx + (2R_h)^{\beta_2} \int_{B_{2R_h}} |D\phi|\,dx.\end{aligned} \tag{3.7}$$

Hence

$$|(Du_h)_{x_h,R_h}| \le \fint_{B_{R_h}(x_h)} |Du - Du_h|\,dx + |(Du)_{x_h,R_h}|$$
$$\le \Big(\fint_{B_{R_h}(x_h)} |Du - Du_h|^p\,dx \Big)^{1/p} + |(Du)_{x_h,R_h}| \le c(M).$$

Now we put

$$A_h = (Du_h)_{x_h,R_h}, \quad a_h = (u_h)_{x_h,R_h},$$
$$\mu_h^2 = (4R_h)^\delta + \fint_{B_{4R_h}(x_h)} |V(Du_h) - V(A_h)|^2\,dx;$$

and rescale the functions u_h in each ball $B_{R_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$:

$$v_h(y) = \frac{1}{\mu_h R_h}[u_h(x_h + R_h y) - (u_h)_{x_h,R_h} - R_h A_h y].$$

Clearly, we have

$$Dv_h(y) = \frac{1}{\mu_h}[Du_h(x_h + R_h y) - A_h], \quad (v_h)_{0,1} = 0, \quad (Dv_h)_{0,1} = 0.$$

Now, let us prove that

$$\lambda_h^2 \ge c\mu_h^2, \quad c > 0,$$

a relation useful later on. To this end observe that by Lemma 2.1(vi),

$$\fint_{B_{4R_h}(x_h)} |V(Du) - V((Du)_{x_h,4R_h})|^2\,dx$$
$$\ge c \fint_{B_{4R_h}(x_h)} |V(Du - (Du)_{x_h,4R_h})|^2\,dx$$
$$= c \fint_{B_{4R_h}(x_h)} |V[(Du_h - A_h) - (Du_h - Du) - ((Du)_{x_h,4R_h} - A_h)]|^2\,dx$$
$$\ge c \fint_{B_{4R_h}(x_h)} |V(Du_h - A_h)|^2\,dx - c \fint_{B_{4R_h}(x_h)} |V(Du_h - Du)|^2\,dx$$
$$- c \fint_{B_{4R_h}(x_h)} |V((Du)_{x_h,4R_h} - A_h)|^2\,dx$$
$$\ge c\Big[\fint_{B_{4R_h}(x_h)} |V(Du_h) - V(A_h)|^2\,dx - I_{1,h} - I_{2,h}\Big].$$

From this estimate we deduce

$$\mu_h^2 \le c[\lambda_h^2 + I_{1,h} + I_{2,h}].$$

We observe that

$$I_{1,h} + I_{2,h} \le c \fint_{B_{4R_h}(x_h)} |Du_h - Du|^p \, dx \le c R_h^{\beta_1 p} \le c R_h^{\delta} \le c\lambda_h^2$$

and so $\lambda_h^2 \ge c\mu_h^2$, $c > 0$.

Now, we prove that

$$\fint_{B_1(0)} |Dv_h|^p \, dx \le c < \infty.$$

In fact, by Lemma 2.1 we have

$$\begin{aligned}
&\fint_{B_1(0)} \left| \frac{V(Du_h(x_h + R_h y)) - V(A_h)}{\mu_h} \right|^2 dy \\
&\ge c \frac{1}{\mu_h^2} \fint_{B_1(0)} |Du_h(x_h + R_h y) - A_h|^2 (1 + |Du_h(x_h + R_h y)|^2 + |A_h|^2)^{(p-2)/2} \, dy \\
&= c \fint_{B_1(0)} |Dv_h|^2 \left(1 + \left| \frac{Du_h(x_h + R_h y) - A_h + A_h}{\mu_h} \right|^2 \mu_h^2 + |A_h|^2 \right)^{(p-2)/2} dy \\
&\ge c \fint_{B_1(0)} |Dv_h|^2 (1 + \mu_h^2 |Dv_h|^2 + |A_h|^2)^{(p-2)/2} \, dy \\
&\ge c \fint_{B_1(0)} |Dv_h|^2 (k_M + |Dv_h|^2)^{(p-2)/2} \, dy \ge c \fint_{B_1(0)} (k_M + |Dv_h|^2)^{p/2} \, dy - k'_M
\end{aligned}$$

and we note that the first integral in the above estimate is dominated by 1. Passing possibly to a subsequence we may conclude that (Dv_h) is bounded in $L^p(B_1, \mathbb{R}^{nN})$:

$$\|Dv_h\|_p \le c \quad \text{for any } h, \tag{3.8}$$

and assume, without loss of generality, that $v_h \to v$ weakly in $W^{1,p}(B_1, \mathbb{R}^N)$, and, since $|A_h| \le M$ for all h, that $A_h \to A$.

Step 2: v solves a linear system. From the Euler–Lagrange system for u, rescaled in each $B_{R_h}(x_h)$, we deduce that for every $\phi \in C_0^1(B_1, \mathbb{R}^N)$ and $x \in \Omega$, $a \in \mathbb{R}^N$ and $A \in \mathbb{R}^{nN}$,

$$\int_{B_1} \frac{\partial^2 f}{\partial \xi_\alpha^i \partial \xi_\beta^j}(x, a, A) D_\alpha v^i D_\beta \phi^j \, dy = 0. \tag{3.9}$$

Consider the functional

$$G(\phi) = \int_{B_{R_h}} f(x_h, a_h, Du_h + D\phi)\, dx + cR_h^{\beta_2} \int_{B_{R_h}} |D\phi|\, dx.$$

From Step 1 writing the Euler–Lagrange equation of G and rescaling we get

$$\frac{1}{\mu_h} \int_{B_1} \left\langle \frac{\partial f}{\partial \xi_\alpha^i}(x_h, a_h, A_h + \mu_h Dv_h) - \frac{\partial f}{\partial \xi_\alpha^i}(x_h, a_h, A_h), D_\alpha \phi^i \right\rangle dy + c\frac{R_h^{\beta_2}}{\mu_h} \int_{B_1} |D\phi|\, dy = 0.$$

By compactness we may suppose that $x_h \to x \in \Omega$ and $a_h \to a \in \mathbb{R}^N$. Now we observe that

$$R_h^{\beta_2}/\mu_h \to 0. \tag{3.10}$$

In fact $\delta < \beta_1 < \beta_2$ and

$$\mu_h^2 \geq (4R_h)^\delta, \qquad R_h \to 0$$

and so we get (3.10).

Performing the same computations of Step 2 of Proposition 3.4 in [CFM], we see that

$$\lim_h \frac{1}{\mu_h} \int_{B_1} \left[\frac{\partial f}{\partial \xi_\alpha^i}(A_h + \mu_h Dv_h) - \frac{\partial f}{\partial \xi_\alpha^i}(A_h)\right] D_\alpha \phi^i\, dy = \int_{B_1} \frac{\partial^2 f}{\partial \xi_\alpha^i \partial \xi_\beta^j}(A) D_\alpha v^i D_\beta \phi^j\, dy = 0$$

by (3.4). By Remark 3.1 the coefficients of this linear system satisfy the inequality

$$c(\nu, M)|\lambda|^2|\mu|^2 \leq \frac{\partial^2 f}{\partial \xi_\alpha^i \partial \xi_\beta^j}(x, a, A)\lambda^i \lambda^j \mu_\alpha \mu_\beta \leq c(M)|\lambda|^2|\mu|^2,$$

hence from Proposition 2.7 we deduce that v is C^∞ in B_1. Moreover from the theory of linear systems (see [G], Th. 2.1, Ch. 3) and by (2.5) and (3.3) we see that if $0 < \tau < 1/2$ then

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2\, dy \leq c(M)\tau^2 \fint_{B_{1/2}} |Dv - (Dv)_{1/2}|^2\, dy \leq c(M)\tau^2 \sup_{B_{1/2}} |Dv|^2 \leq c(M)\tau^2 \Big(\fint_{B_1} |Dv|^p\, dx\Big)^{2/p} \leq C^*(M)\tau^\delta.$$

Step 3: higher integrability of v_h. Set

$$f_h(\xi) := \mu_h^{-2}[f(x_h, a_h, A_h + \mu_h \xi) - f(x_h, a_h, A_h) - \mu_h Df(x_h, a_h, A_h)\xi].$$

Then thanks to Lemma 3.8 and applying Lemma 2.5 to the functions $f_h(\xi)$, we get, by (2.3) and Lemma 2.1(vi),

$$\tag{3.11} \begin{aligned} &\fint_{B_{1/2}} |V(\mu_h Dv_h)|^{2(1+\delta)}\,dy \\ &\quad \le c\Big(\fint_{B_{4R_h}(x_h)} |V(Du_h(x)-A_h)|^2\,dx + \mu_h^2 R_h^{\beta_2}\Big)^{1+\delta} \\ &\quad \le c(M)\Big(\fint_{B_{4R_h}(x_h)} |V(Du_h(x))-V(A_h)|^2\,dx + \mu_h^2 R_h^{\beta_2}\Big)^{1+\delta} \\ &\quad \le c[\mu_h^{2(1+\delta)} + (\mu_h^2 R_h^{\beta_2})^{(1+\delta)}]. \end{aligned}$$

Arguing as in Step 1 we conclude that the sequence (Dv_h) is bounded in $L^{p(1+\delta)}(B_{1/2},\mathbb{R}^{nN})$.

Step 4: upper bound. Fix $r < 1/3$ and set

$$I_r^h(w) = \int_{B_r} f_h(Dw(y))\,dy.$$

Passing to a (not relabelled) subsequence, we may always assume that $\lim_h[I_r^h(v_h) - I_r^h(v)]$ exists. We claim that

$$\lim_h [I_r^h(v_h) - I_r^h(v)] \le 0.$$

Choose $s < r$ and take $\zeta \in C_0^\infty(B_r)$ such that $0 \le \zeta \le 1$, $\zeta = 1$ on B_s and $|D\zeta| \le 2/(r-s)$. If we set $\phi_h = (v - v_h)\zeta$ we can go on as in Step 4 of Proposition 3.4 in [CFM]. Namely we have, rescaling inequality (3.7) above,

$$\begin{aligned} &I_r^h(v_h) - I_r^h(v) \\ &\quad \le I_r^h(v_h+\phi_h) - I_r^h(v) + c\frac{R_h^{\beta_2}}{\mu_h^2}\int_{B_r} |D\phi_h|\,dy \\ &\quad \le \int_{B_r\setminus B_s} \Big(1 + \frac{1}{\mu_h^2}|V(\mu_h(v-v_h)D\xi + \mu_h\zeta Dv + \mu_h(1-\zeta)Dv_h)|^2\Big)\,dy \\ &\qquad + c\frac{R_h^{\beta_2}}{\mu_h^2}\int_{B_r} |D\phi_h|\,dy \\ &\quad \le \frac{c}{\mu_h^2}\int_{B_r\setminus B_s} (\mu_h^2 + |V(\mu_h Dv)|^2 + |V(\mu_h Dv_h)|^2 \\ &\qquad + \max\{|D\zeta|^2, |D\zeta|^p\}|V(\mu_h(v-v_h)|^2)\,dy + c\frac{R_h^{\beta_2}}{\mu_h^2}\int_{B_r} |D\phi_h|\,dy \end{aligned}$$

$$\le \frac{c}{\mu_h^2} \int_{B_r\setminus B_s} \left(\mu_h^2 + |V(\mu_h Dv_h)|^2 + \frac{1}{(r-s)^2} |V(\mu_h(v-v_h))|^2 \right) dy$$

$$+ c\frac{R_h^{\beta_2}}{\mu_h^2} \int_{B_r} |D\phi_h| \, dy.$$

By the Hölder inequality and (3.11) of Step 3, we get

$$\int_E \frac{1}{\mu_h^2} |V(\mu_h Dv_h)|^2 \, dy \le c|E|^{\delta/(1+\delta)}.$$

From the above estimate we get

$$I_r^h(v_h) - I_r^h(v) \le c(r-s)^{\delta/(1+\delta)} + \frac{c}{\mu_h^2 (r-s)^2} \int_{B_r\setminus B_s} |V(\mu_h(v-v_h))|^2 \, dy$$

$$+ c\frac{R_h^{\beta_2}}{\mu_h^2} \int_{B_r} |D\phi_h| \, dy.$$

By (2.2) and taking θ such that $1/2 = \theta + (1-\theta)/(2(1+\sigma))$, we obtain, using Lemma 2.1(ii)&(iii),

$$\int_{B_r\setminus B_s} |V(\mu_h(v-v_h))|^2 \, dy$$

$$\le \Big(\int_{B_r\setminus B_s} |V(\mu_h(v-v_h))| \, dy \Big)^{2\theta} \Big(\int_{B_r\setminus B_s} |V(\mu_h(v-v_h))|^{2(1+\sigma)} \, dy \Big)^{(1-\theta)/(1+\sigma)}$$

$$\le c\mu_h^{2\theta} \Big(\int_{B_1} |v-v_h| \, dy \Big)^{2\theta} \Big(\int_{B_{1/3}} |V(\mu_h(v-v_h) - \mu_h(v-v_h)_{0,1/3})|^{2(1+\sigma)} dy$$

$$+ |V(\mu_h(v-v_h)_{0,1/3})|^{2(1+\sigma)} \Big)^{(1-\theta)/(1+\sigma)}$$

$$\le c\mu_h^{2\theta} \Big(\int_{B_1} |v-v_h| \, dy \Big)^{2\theta} \Big[\Big(\int_{B_1} |V(\mu_h Dv_h)|^2 \, dy \Big)^{1-\theta} + \mu_h^{2(1-\theta)} \Big]$$

$$\le c\mu_h^2 \Big(\int_{B_1} |v-v_h| \, dy \Big)^{2\theta},$$

where we have used the estimate (see (3.11))

$$\fint_{B_1} |V(\mu_h Dv_h)|^2 \, dy \le c\mu_h^2. \tag{3.12}$$

Therefore we obtain

$$\int_{B_r\setminus B_s} |V(\mu_h(v-v_h))|^2 \, dy \le c\mu_h^2 \Big(\int_{B_1} |v-v_h| \, dy \Big)^{2\theta}.$$

Moreover

$$I_r^h(v_h) - I_r^h(v) \le c\Big[(\sup_{B_r} |Dv|^2)(r-s) + (r-s)^{\delta/(1+\delta)} + \frac{1}{(r-s)^2}\Big(\int_{B_1} |v - v_h|\,dy\Big)^{2\theta} + c\frac{R_h^{\beta_2}}{\mu_h^2}\int_{B_r} |D\phi_h|\,dy\Big].$$

Since $v_h \to v$ in $L^p(B_1, \mathbb{R}^N)$, letting first $h \to \infty$ and then $s \to r$ we prove the claim.

Step 5: lower bound. We claim that for $t < r < 1/6$,

$$\limsup_h \int_{B_t} |V(\mu_h(Dv - Dv_h))|^2\,dy \le c \lim_h [I_r^h(v_h) - I_r^h(v)].$$

Let $\phi \in C_0^1(B_{1/6})$ be such that $0 \le \phi \le 1$, $\phi = 1$ on $B_{1/8}$ and $|D\phi| \le c$. Set

$$\widetilde{v}_h = v_h\phi, \qquad \widetilde{v} = v\phi.$$

We may always assume that the exponent δ given by the higher integrability estimate (3.11) is less than or equal to the exponent σ provided by the Sobolev–Poincaré inequality (2.2). Therefore by (3.11) and (3.12) we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} |V(\mu_h D\widetilde{v}_h)|^{2(1+\delta)}\,dy \\
&\quad\le c\int_{B_{1/6}} |V(\mu_h Dv_h)|^{2(1+\delta)}\,dy + c\int_{B_{1/6}} |V(\mu_h v_h)|^{2(1+\delta)}\,dy \\
&\quad\le c\int_{B_{1/6}} |V(\mu_h Dv_h)|^{2(1+\delta)}\,dy + c\int_{B_{1/6}} |V(\mu_h v_h - \mu_h (v_h)_{0,1/6})|^{2(1+\delta)}\,dy \\
&\qquad + c|V(\mu_h(v_h)_{0,1/6})|^{2(1+\delta)} \\
&\quad\le c\mu_h^{2(1+\delta)} + c\Big(\int_{B_1} |V(\mu_h Dv_h)|^2\,dy\Big)^{1+\delta} \le c\mu_h^{2(1+\delta)}.
\end{aligned}$$

From this estimate and Proposition 2.3 of [CFM], it then follows that

$$\mu_h^{-1}[\|V(\mu_h D\widetilde{v}_h)\|_{L^{2(1+\delta)}(\mathbb{R}^n)} + \|V(\mu_h M(D\widetilde{v}_h))\|_{L^{2(1+\delta)}(\mathbb{R}^n)}] \le c \tag{3.13}$$

for all h. Fix $\varepsilon > 0$. From the estimate above it is clear that there exists $\eta > 0$ such that if $G \subset \mathbb{R}^n$ is a measurable set with $\operatorname{meas}(G) < \eta$, then

$$\frac{1}{\mu_h^2}\Big[\int_G |V(\mu_h D\widetilde{v}_h)|^2\,dy + \int_G |V(\mu_h M(D\widetilde{v}_h))|^2\,dy\Big] < \varepsilon. \tag{3.14}$$

Notice that (3.13) also implies that $(\widetilde{v}_h)$ is bounded in $W^{1,p(1+\delta)}(\mathbb{R}^n, \mathbb{R}^N)$, therefore by the continuity of the maximal function in L^q spaces we deduce that there exists $K > 1$ such that, setting $S_h = \{y \in \mathbb{R}^n : M(D\widetilde{v}_h)(y) > K\}$,

$$\text{meas}(S_h) < \eta \quad \text{for all } h. \tag{3.15}$$

Having chosen K, we now apply Lemma 2.9 to find a sequence of functions $w_h \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$w_h = \widetilde{v}_h \quad \text{on } \mathbb{R}^n \setminus S_h, \qquad \|Dw_h\|_\infty \le cK. \tag{3.16}$$

Therefore, passing to a (not relabelled) subsequence we may also suppose that

$$w_h \rightharpoonup w \quad \text{weak}^* \text{ in } W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^N).$$

Notice that by (3.14), (3.15) and the definition of S_h we have the estimate

$$\text{meas}(S_h)(1 + \mu_h^2 K^2)^{(p-2)/2} K^2 \le \frac{1}{\mu_h^2} \int_{S_h} |V(\mu_h M(D\widetilde{v}_h))|^2 \, dy \le \varepsilon,$$

which gives

$$\text{meas}(S_h) \le \varepsilon \frac{(1 + \mu_h^2 K^2)^{(2-p)/2}}{K^2} < \frac{2\varepsilon}{K^2} \tag{3.17}$$

for h large enough. We now turn to estimate the difference

$$\begin{aligned} I_r^h(v_h) - I_r^h(v) &= [I_r^h(\widetilde{v}_h) - I_r^h(w_h)] + [I_r^h(w_h) - I_r^h(w)] \\ &\quad + [I_r^h(w) - I_r^h(v)] = R_1^h + R_2^h + R_3^h. \end{aligned} \tag{3.18}$$

By Lemma 3.8 and (3.14)–(3.16) we get

$$\begin{aligned} |R_1^h| &\le \int_{S_h \cap B_r} |f_h(D\widetilde{v}_h) - f_h(Dw_h)| \, dy \\ &\le \frac{c}{\mu_h^2} \int_{S_h} [V(\mu_h D\widetilde{v}_h)|^2 + V(\mu_h M(D\widetilde{v}_h))|^2] \, dy < c\varepsilon. \end{aligned} \tag{3.19}$$

Now choose $t < s < r$ and take a cut-off function ζ as in Step 4. Setting $\psi_h = (w_h - w)\zeta$ we split R_2^h as follows:

$$\begin{aligned} R_2^h &= [I_r^h(w_h) - I_r^h(w + \psi_h)]r \\ &\quad + [I_r^h(w + \psi_h) - I_r^h(w) - I_r^h(\psi_h)] + I_r^h(\psi_h) \\ &= R_4^h + R_5^h + R_6^h. \end{aligned} \tag{3.20}$$

Again by Lemma 3.8, (3.16) and Lemma 2.1(i)&(ii) we have

$$\begin{aligned} |R_4^h| &\le \int_{B_r \setminus B_s} |f_h(Dw_h) - f_h(Dw + D\psi_h)| \, dy \\ &\le \frac{c}{\mu_h^2} \int_{B_r \setminus B_s} \left[|V(\mu_h Dw_h)|^2 + |V(\mu_h Dw)|^2 + \frac{1}{(r-s)^2} |V(\mu_h(w_h - w))|^2 \right] dy \\ &\le c(K)(r-s) + \frac{c}{(r-s)^2} \int_{B_r \setminus B_s} |w_h - w|^2 \, dy. \end{aligned}$$

Since $w_h \to w$ uniformly we conclude that

$$\limsup_h |R_4^h| \le c(K)(r-s). \tag{3.21}$$

To bound R_5^h we observe that for any $A, B \in \mathbb{R}^{nN}$,

$$f_h(A+B) - f_h(A) - f_h(B) = \int_0^1\int_0^1 D^2 f_h(sA+tB)\,ds\,dt$$

and therefore

$$R_5^h = \int_{B_r} dx \int_0^1\int_0^1 D^2 f(A_h + s\mu_h Dw_h + t\mu_h D\psi_h) Dw D\psi_h\,ds\,dt.$$

Since $D^2 f(A_h + s\mu_h Dw_h + t\mu_h D\psi_h)$ converges to $D^2 f(A)$ uniformly, and $w_h \rightharpoonup w$ weak* in $W^{1,\infty}$, we easily get

$$\lim_h R_5^h = 0. \tag{3.22}$$

Moreover (H_2) implies that

$$\begin{aligned} R_6^h = \int_{B_r} f_h(D\psi_h)\,dy &\ge \frac{\nu}{\mu_h^2}\int_{B_r} |V(\mu_h D\psi_h)|^2\,dy \\ &\ge \frac{\nu}{\mu_h^2}\int_{B_s} |V(\mu_h(Dw_h - Dw))|^2\,dy. \end{aligned}$$

Passing possibly to a subsequence we may suppose that $\lim_h R_2^h$ also exists. Therefore by (3.20)–(3.22) we deduce

$$\lim_h R_2^h \ge \limsup_h \frac{\nu}{\mu_h^2}\int_{B_s} |V(\mu_h(Dw_h - Dw))|^2\,dy - c(K)(r-s). \tag{3.23}$$

To deal with R_3^h we use a technique introduced in [AF1]. First we prove that

$$\text{meas}\{y \in B_r : v(y) \ne w(y)\} \le 3\varepsilon/K^2. \tag{3.24}$$

Set $S = \{y \in B_r : v(y) \ne w(y)\}$ and

$$\widetilde{S} = S \cap \{y \in B_r : v(y) = \lim_h v_h(y)\}.$$

Then $\text{meas}(S) = \text{meas}(\widetilde{S})$. We argue by contradiction. If $\text{meas}(S) > 3\varepsilon/K^2$ then by (3.17), $\text{meas}(\widetilde{S} \setminus S_h) > \varepsilon/K^2$ for h large enough and by Lemma 2.8 there exists $\overline{y} \in B_r$ such that $\overline{y} \in \widetilde{S} \setminus S_h$ for infinitely many h. Passing to this subsequence, we have

$$v(\overline{y}) = \lim_h v_h(\overline{y}) = \lim_h w_h(\overline{y}) = w(\overline{y});$$

hence $\overline{y} \notin S$, which is a contradiction. This proves (3.24). Now, since $Dv = Dw$ a.e. in $B_r \setminus S$, by (3.7) and (3.24) we get

$$
\begin{aligned}
(3.25)\qquad |R_3^h| &\le \int_{B_r \cap S} |f_h(Dw) - f_h(Dv)|\,dy \\
&\le \frac{c}{\mu_h^2} \int_{B_r \cap S} [|V(\mu_h Dv)|^2 + |V(\mu_h Dw)|^2]\,dy \\
&\le c(1+K^2)\mathrm{meas}(S) \le \frac{c(1+K^2)\varepsilon}{K^2} \le c\varepsilon,
\end{aligned}
$$

since $K > 1$. By this inequality, (3.18), (3.19) and (3.23) we conclude that

$$
\begin{aligned}
(3.26)\qquad \lim_h [I_r^h(v_h) - I_r^h(v)] \ge\ & \limsup_h \frac{\nu}{\mu_h^2} \int_{B_s} |V(\mu_h(Dw_h - Dw))|^2\,dy \\
& - c(K)(r-s) - c\varepsilon.
\end{aligned}
$$

By Lemma 2.1(iii) we then have

$$
\begin{aligned}
(3.27)\qquad \frac{1}{\mu_h^2} \int_{B_t} & |V(\mu_h(Dv - Dv_h))|^2\,dy \\
&\le \frac{c}{\mu_h^2} \int_{B_s} |V(\mu_h(Dw - Dw_h))|^2\,dy \\
&\quad + \frac{c}{\mu_h^2} \int_{B_s \cap S_h} |V(\mu_h(Dw_h - Dv_h))|^2\,dy \\
&\quad + \frac{c}{\mu_h^2} \int_{B_s \cap S_h} |V(\mu_h(Dv - Dw))|^2\,dy.
\end{aligned}
$$

With the same argument used to prove (3.24) we also get

$$
\begin{aligned}
(3.28)\qquad \frac{c}{\mu_h^2} \int_{B_s \cap S} & |V(\mu_h(Dv - Dw))|^2\,dy \\
&\le \frac{c}{\mu_h^2} \int_{B_r \cap S} [|V(\mu_h Dv)|^2 + |V(\mu_h Dw)|^2]\,dy \le c\varepsilon
\end{aligned}
$$

and as in (3.19) we get

$$
\begin{aligned}
\frac{c}{\mu_h^2} \int_{B_s \cap S_h} & |V(\mu_h(Dv_h - Dw_h))|^2\,dy \\
&\le \frac{c}{\mu_h^2} \int_{S_h} [|V(\mu_h D\widetilde{v}_h)|^2 + |V(\mu_h M(D\widetilde{v}_h))|^2]\,dy \le c\varepsilon.
\end{aligned}
$$

From this estimate and (3.26)–(3.28) we finally conclude that

$$\limsup_h \frac{1}{\mu_h^2} \int_{B_t} |V(\mu_h(Dv - Dv_h))|^2 \, dy \le c \lim_h [I_r^h(v_h) - I_r^h(v)] + c\varepsilon + c(K)(r-s).$$

The proof of the claim then follows by letting first $s \to r$ and then $\varepsilon \to 0^+$.

Step 6: conclusion of the proof. From the previous two steps we see that for any $0 < \tau < 1/8$,

$$\lim_h \frac{1}{\mu_h^2} \int_{B_\tau} |V(\mu_h(Dv - Dv_h))|^2 \, dy = 0.$$

Now,

$$\begin{aligned}
\lim_h \frac{E(x_h, 4\tau R_h)}{\lambda_h^2} &\le c \lim_h \frac{E(x_h, 4\tau R_h)}{\mu_h^2} \\
&= c \lim_h \frac{1}{\mu_h^2} \fint_{B_{4\tau R_h}(x_h)} |V(Du) - V((Du)_{x_h, 4\tau R_h})|^2 \, dx + c \lim_h \frac{\tau^\delta R_h^\delta}{\mu_h^2} \\
&\le \lim_h \frac{c}{\mu_h^2} \fint_{B_{4\tau R_h}(x_h)} (|V(Du - Du_h)|^2 + |V(Du_h - (Du_h)_{x_h, 4\tau R_h})|^2 \\
&\quad + |V((Du)_{x_h, 4\tau R_h} - (Du_h)_{x_h, 4\tau R_h})|^2) \, dx + c \lim_h \frac{\tau^\delta R_h^\delta}{\mu_h^2} \\
&\le \lim_h \frac{c}{\mu_h^2} \Big(\fint_{B_\tau} |V(\mu_h(Dv_h - (Dv_h)_\tau))|^2 \, dy \\
&\quad + \fint_{B_{\tau R_h}(x_h)} |Du - Du_h|^p \, dx \Big) + c \lim_h \frac{\tau^\delta R_h^\delta}{\mu_h^2} \\
&\le \lim_h \frac{c}{\mu_h^2} \fint_{B_\tau} (|V(\mu_h(Dv_h - Dv))|^2 + |V(\mu_h(Dv - (Dv)_\tau))|^2 \\
&\quad + |V(\mu_h((Dv)_\tau - (Dv_h)_\tau))|^2) \, dy + c \lim_h \frac{\tau^\delta R_h^\delta}{\mu_h^2} \\
&\le C^*(M) \tau^\delta,
\end{aligned}$$

and since $Dv_h \rightharpoonup Dv$ weakly in $L^p(B_1, \mathbb{R}^{nN})$ we deduce that

$$\lim_h \frac{E(x_h, 4\tau R_h)}{\lambda_h^2} \le C^*(M) \tau^\delta,$$

which contradicts (3.6) if we choose $C_M = 2C^*(M)$. ■

Proof of Theorem 3.2. With the same techniques used in [CFM] (see also [FH]) we obtain the assertion. ■

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