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Mixed type semicontinuous differential inclusions in Banach spaces

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Abstract. We consider a class of differential inclusions in (nonseparable) Banach spaces satisfying mixed type semicontinuity hypotheses and prove the existence of solutions for a problem with state constraints. The cases of dissipative type conditions and with time lag are also studied. These results are then applied to control systems.

1. Preliminaries. Let E be a Banach space and let $\emptyset \neq D \subset E$ be a closed set. Let moreover F be a nonempty compact-valued multifunction from $I \times D$ into E. We consider the following problem:

$$\dot{x}(t) \in F(t, x(t)), \quad x(t) \in D, \quad x(0) = x_0 \in D. \tag{1}$$

Here $t \in I = [0,1]$. When F satisfies compactness type conditions, the existence of (local) solutions can be obtained under various semicontinuity assumptions (see [8] for instance). When $E = D$ one can easily prove the existence under a growth condition combined with one of the following assumptions:

- F is almost lower semicontinuous (LSC);
- $F(\cdot, x)$ has a strongly measurable selection, $F(t, \cdot)$ has closed graph and convex values.

In the first case one uses Fryszkowski's continuous selection theorem (see Lemma 9.1 of [8]) or Bressan–Colombo's Γ^M-continuous selection theorem (Theorem 2 of [6]).

The second case is well known (cf. [5, 8]).

Problem (1) with constraints is more complicated and one has to use stronger hypotheses.

In case $E = \mathbb{R}^n$ problem (1) is also considered when:

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(H1) $F(\cdot, x)$ is measurable and $F(t, \cdot)$ is upper semicontinuous (USC) with compact values and continuous at the points where it is not convex-valued [14, 16].

(H2) F is $\mathcal{L} \otimes \mathcal{B}$ (Lebesgue–Borel) measurable and $F(t, \cdot)$ is USC with convex compact values or $F(t, \cdot)$ restricted to some neighbourhood is LSC at the points where it is not convex-valued [15, 17].

The tedious proofs of the existence of solutions in [14–16] are simplified in [17] with the help of Fryszkowski's selection theorem. The results of the last paper are generalized recently in [12], where a special selection theorem for mixed semicontinuous mappings with decomposable values is proved. In the case of autonomous F a very short proof is presented in [8] with the help of Γ^M-continuous selections. The most general results in this direction when $\mathbb{R}^n = D$ appear in the very recent papers [1, 2]. Thy deal with more general operator inclusions which embrace (1) and also boundary value problems of second order as well as integral inclusions. In the case of (1), however, the approach of [2] is applicable only when $E = D$ is reflexive and separable (if infinite-dimensional). Only finite-dimensional problems are considered also in [1].

In this paper we consider problem (1) mainly when $D \neq E$. We use natural tangential conditions when the right-hand side is almost semicontinuous and a stronger condition when it is not. We show how the assumptions of [2] can be reduced to our case when E is separable and $D = E$. Since the space is infinite-dimensional we use compactness type assumptions. For E^* uniformly convex we also consider the case of one-sided Lipschitz right-hand side.

Moreover, we consider differential inclusions with time lag:

$$\dot{x}(t) \in F(t, x_t), \quad x_0 = \phi; \quad \phi(s) \in D,\ x(t) \in D, \tag{2}$$

where $x_t \in X$, $F : I \times X \to P_{\mathrm{f}}(E)$ and $X = \{\alpha \in C([-\tau, 0], E) : \alpha(0) \in D$ and $\alpha(s) \in \overline{\mathrm{co}}\, D,\ \forall s \in [-\tau, 0]\}$ and $C([-\tau, 0], E)$ is the usual space of continuous functions.

In the last section the results are applied to control systems $F(t, x) = f(t, x, U)$. We extend a result of [7].

Now we recall the main definitions and notations. All the concepts not discussed in detail can be found in [8].

By $P_{\mathrm{f}}(E)$ (resp. $P_{\mathrm{c}}(E)$) we denote the set of all nonempty compact (resp. convex compact) sets in E. If $A \subset E$ then $\overline{A}$ (resp. $\mathrm{co}\, A$) is the closed (resp. convex) hull of A. For $A, B \subset E$ we write $\mathrm{dist}(A, B) = \inf_{a \in A;\, b \in B} |a - b|$ and if $e \in E$ then $\mathrm{dist}(e, A) = \mathrm{dist}(\{e\}, A)$. Moreover, $D^+(A, B) = \sup_{a \in A} \mathrm{dist}(a, B)$, and $D_{\mathrm{H}}(A, B) = \max\{D^+(A, B), D^+(B, A)\}$ is the Hausdorff distance. Notice that $P_{\mathrm{f}}(E)$ and $P_{\mathrm{c}}(E)$ equipped with the Hausdorff distance become complete metric spaces. By U we denote the open unit ball.

For $x \in E^*$ the *support function* of the set $A \in P_{\mathrm{f}}(E)$ is

$$\sigma(x, A) = \max_{a \in A} \langle x, a \rangle.$$

DEFINITION 1. The multifunction $F : E \to P_{\mathrm{f}}(E)$ is called *LSC* (resp. *USC*) at x if for every open V with $V \cap F(x) \neq \emptyset$ there exists a neighbourhood A of x such that $V \cap F(y) \neq \emptyset$ for every $y \in A$ (resp. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(y) \subset F(x) + \varepsilon U$ when $|x - y| < \delta$). The multifunction F is said to be *semicontinuous* on $A \subset E$ if for every $x \in A$, F is USC at x or F is LSC on some relatively open neigbourhood of x.

Denote by $\mathcal{L}$ the Lebesgue measurable subsets of I and by $\mathcal{B}$ the Borel measurable subsets of E. Let $A \subset I \times D$ be $\mathcal{L} \otimes \mathcal{B}$-measurable such that for every t with $(t, x) \in A$ (for some $x \in D$) the projection $\{z : (t, z) \in A\}$ is relatively open. The multifunction F is called A-*almost semicontinuous* (A-ASC) on $I \times D$ when there exist a null set N and a sequence $\{I_i\}_{i=1}^{\infty}$ of pairwise disjoint compact sets such that $I \backslash N = \bigcup_{i=1}^{\infty} I_i$ and for every $n \geq 1$, F is LSC on $(I_n \times D) \cap A$ and USC with convex values on $(I_n \times D) \setminus A$.

Define $\mathrm{graph}(F) = \{(z, u) : z \in I \times D \text{ and } u \in F(z)\}$. When $\mathrm{graph}(F)$ is closed we say that F *has a closed graph*. The (*Bouligand*) *tangent cone* to D at x is

$$T_D(x) = \{y \in E : \liminf_{\lambda \to 0^+} \lambda^{-1} \operatorname{dist}(x + \lambda y, D) = 0\}.$$

DEFINITION 2. The multifunction F is said to satisfy the *growth condition* when there exists an integrable $\lambda(\cdot)$ such that $|F(t, x)| \leq \lambda(t)\{1 + |x|\}$. F is said to satisfy the *tangential condition* if for every $x \in D : F(t, x) \cap T_D(x) \neq \emptyset$ when $F(t, \cdot)$ is USC and $F(t, x) \subset T_D(x)$ when $F(t, \cdot)$ is LSC.

Given $M > 0$ and $A \subset I \times D$ define the cone $\varGamma^M = \{(t, x) \in A \times D : |x| \leq Mt\}$. The (single-valued) function $f : A \to E$ is said to be $\varGamma^M$-*continuous* when $(t_n - t, x_n - x) \in \varGamma^M$ and $(t_n, x_n) \to (t, x)$ implies $f(t_n, x_n) \to f(t, x)$ as $n \to \infty$.

THEOREM 1 (Theorem 2 of [6]). *Let X, Z be Banach spaces and let $M > 0$. If $\varOmega \subset I \times X$ is nonempty then any closed-valued LSC multifunction $F : \varOmega \to Z$ admits a $\varGamma^M$-continuous selection.*

Consider the *Hausdorff measure of noncompactness*:

$$\beta(K) = \inf\{r : K \text{ can be covered by finitely many balls with radius } \leq r\}.$$

DEFINITION 3. The multifunction F is said to satisfy the *compactness condition* if there exists a Kamke function $w(\cdot, \cdot)$ such that

$$\lim_{h \to 0} \beta(F([t, t + h], C)) \leq w(t, \beta(C))$$

for every bounded $C \subset D$ (or $C \subset X$ in the case of (2)).

When E is separable one can replace this inequality by $\beta(F(t,C)) \le w(t,\beta(C))$.

Recall that a Carathéodory function $w : I \times \mathbb{R} \to \mathbb{R}^+$ is called a *Kamke function* when $w(t,0) = 0$, $w(\cdot,\cdot)$ is integrably bounded on bounded sets ($\lambda(t) = \sup_{x\in C} |w(t,x)|$ is integrable when C is bounded) and the unique solution of $\dot{r} = w(t,r)$, $r(0) = 0$ is $r(t) \equiv 0$.

Let F satisfy the compactness condition and be A-ASC. We will use the following lemma.

Lemma 1. *Suppose* $|F(t,x)| \le K$. *Define*

$$G(t,x) = \begin{cases} F(t,x), & (t,x) \in (I_n \times D) \setminus \Omega_n, \\ G_n(t,x), & (t,x) \in \Omega_n,\ n = 1,2,\ldots, \\ 0, & t \in N, \end{cases}$$

where $\Omega_n = (I_n \times D) \cap A$ *and* I_n, N *are from Definition* 1. *Here* $G_n(t,x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, f_n(A_\varepsilon)$, $A_\varepsilon = ([t-\varepsilon, t+\varepsilon] \times (x + \varepsilon U)) \cap \Omega_n$ *and* $f_n(\cdot,\cdot)$ *are* Γ^M-*continuous selections of* $F(\cdot,\cdot)$ *on* Ω_n. *Then* $G(\cdot,\cdot)$ *is almost USC. Furthermore if* $M > K$ *then every solution of*

$$\dot{x}(t) \in G(t,x(t)), \qquad x(0) = x_0, \tag{3}$$

is also a solution of (1).

Proof. We first prove that $G(\cdot,\cdot)$ is almost USC. If $(t,x) \in (I_n \times D) \setminus \Omega_n$ then $G(\cdot,\cdot)$ is USC at (t,x) since $G(t,x) = F(t,x)$. If $(t,x) \in \Omega_n \cap (I_n \times D)$ then there exists $\delta > 0$ such that $[(t-\delta, t+\delta) \cap I_n] \times [(x+\delta U) \cap D] \subset \Omega_n$. By the definition of $G(\cdot,\cdot)$, taking into account the compactness condition one finds that $G(\cdot,\cdot)$ is USC at (t,x).

We follow with modifications the proof of Lemma 4.1 of [4] and use some arguments of the proof of Theorem 2.1 of [2]. Let $x(\cdot)$ be a solution of (3). Since $x(\cdot)$ is continuous, the set $\{z \in I \times E : z = (t,x(t))\}$ is $\mathcal{L} \otimes \mathcal{B}$-measurable. Consider the set $\mathcal{K} = \{t \in I : (t,x(t)) \in A\}$. Obviously $\mathcal{K}$ is measurable. If $\mathrm{meas}(\mathcal{K}) = 0$ then $x(\cdot)$ is a solution of (1). Suppose $\mathrm{meas}(\mathcal{K}) > 0$. Then for almost every $t \in I_n \cap \mathcal{K}$ one has $(t,x(t)) \in \Omega_n$ and moreover t is a point of density of $I_n \cap \mathcal{K}$. By Lusin's theorem one can suppose that $\dot{x}(\cdot)$ is continuous on I_n. Let $t_n \to t^+$ be such that $(t_n, x(t_n)) \in \Omega_n$. Then $\dot{x}(t_n) \to \dot{x}(t)$ and for sufficiently small $\delta = \delta(k)$ one has $[t_k - \delta, t_k + \delta] \times (x(t_k) + \delta U) \subset (t,x(t)) + \Gamma^M$. Thus $\dot{x}(t) = f_n(t,x(t))$ for almost every $t \in \mathcal{K}$. ∎

2. Main results. In this section we present our main results concerning the existence of solutions. Some particular cases in which the assumptions can be relaxed are also described. First we consider the case with state constraints.

THEOREM 2. *Let $F : I \times D \to P_{\mathrm{f}}(E)$ be A-ASC and satisfy the compactness, growth and tangential conditions. Define $\Omega_n = (I_n \times D) \cap A$. If the Ω_n are relatively open in $I_n \times D$, then problem* (1) *has a solution.*

Proof. One can assume $|F(t,x)| \le 1$ (see [8], p. 52). There exists a Γ^2-continuous selection (Definition 2) $f_n(t,x) \in F(t,x)$ for $(t,x) \in \Omega_n$. Define a new orientor field as follows:

$$G(t,x) = \begin{cases} F(t,x), & (t,x) \in (I_n \times D) \setminus \Omega_n, \\ G_n(t,x), & (t,x) \in \Omega_n,\ n = 1,2,\ldots, \\ 0, & t \in N. \end{cases}$$

Here $G_n(t,x)$ and A_ε are as in Lemma 1. By Lemma 1, $G(\cdot,\cdot)$ is almost USC. From Theorem 9.1 of [8] we know that the problem

$$\dot{y}(t) \in G(t, y(t)), \quad y(0) = y_0,$$

has a solution $y(\cdot)$. Obviously $y(\cdot)$ is Lipschitz with constant 1. The proof is complete thanks to Lemma 1. ∎

Theorem 2 gives an (affirmative) answer to Problem 5.4 of [8] (see also Theorem 5 below).

REMARK 1. The conditions of Theorem 2 seem to be more restrictive than those in [12, 15, 17]. However when E is separable one can easily reduce the cases of [12, 14–17] to the conditions of Theorem 2—see Theorem 3 below.

Consider the case when E is separable. We will replace the A-ASC of F by existence of a set A such that:

A1. $A \subset I \times D$, $A \in \mathcal{L} \otimes \mathcal{B}$ and for every t the projection A_t is relatively open.

A2. F is almost LSC on A, $F(\cdot,x)$ is measurable for every $x \in D$, $F(t,\cdot)$ is USC and convex-valued on $\big(I \times D\big) \setminus A$.

THEOREM 3. *Suppose $F(\cdot,\cdot)$ satisfies the compactness, growth and tangential conditions. Under assumptions* A1, A2 *there exists $F_0(t,x) \subset F(t,x)$ satisfying the assumptions of Theorem* 2 *except the assumption that the Ω_n are relatively open in $I_n \times D$.*

Proof. Define

$$G(t,x) = \begin{cases} F(t,x), & (t,x) \in (I_n \times D) \setminus \Omega_n, \\ G_n(t,x), & (t,x) \in \Omega_n,\ n = 1,2,\ldots, \\ 0, & t \in N. \end{cases}$$

Here N is a null set and $G_n(t,x) = \bigcap_{\varepsilon > 0} \overline{\mathrm{co}}\, f_n((t, x + \varepsilon U) \cap \Omega_n)$, where Ω_n and f_n are as in Lemma 1. By the compactness condition, $G(t,\cdot)$ is USC. Furthermore the definition of $G(\cdot,\cdot)$ yields that $T_D(x) \cap G(\cdot,x)$ has a measurable selection. Now, from Proposition 5.1 of [8] it follows that there exists an almost USC $G_0(t,x) \subset G(t,x)$ with convex and compact values satisfying the compactness, tangential and growth conditions and such that $v(t) \in G_0(t,u(t))$ for every measurable $u(\cdot), v(\cdot)$ with $v(t) \in G(t,u(t))$. We only have to see that $G_0(\cdot,\cdot)$ is nonempty-valued.

Let $A = \{x_i\}_{i=1}^{\infty}$ be a dense subset of D. Let $\varepsilon > 0$ be given. Let $f_i(t) \in G(t,x_i) \cap D$ be measurable. Then there exists a compact $I_\varepsilon \subset I$ with $\operatorname{meas}(I_\varepsilon) > 1-\varepsilon$ such that $f_i(\cdot)$ is continuous on I_ε for every i. By the compactness condition for every $t \in I_\varepsilon$ the sequence $\{f_i(t)\}_{i=1}^{\infty}$ has a density point, say $f(t)$. Let $x_i \to x$. Then $f(t) \in G_0(t,x)$, by the upper semicontinuity of $G_0(t,\cdot)$. Hence $G_0(\cdot,\cdot)$ has nonempty values and therefore the multimap G_0 satisfies all the conditions of Theorem 2. ∎

COROLLARY 1. *Under the assumptions of Theorem* 3, *problem* (1) *has a solution.*

REMARK 2. It is easy to see that the conditions of Corollary 1 are weaker than (H1) from Section 1. Namely as shown in [14] if $F(t,\cdot)$ is continuous at x then it is continuous on a neighbourhood U of x. That is, $F(t,\cdot)$ is continuous on an open set A. Using the Scorza Dragoni theorem it is easy to see that $F(\cdot,\cdot)$ is LSC on $A \in \mathcal{L} \otimes \mathcal{B}$ and for every t the projection A_t is relatively open. The conditions of [15, 17] are obviously stronger than those of Corollary 1. So are the conditions of [12] under which the existence of solutions is proved, although in case $D \equiv \mathbb{R}^n$ Corollary 1 can be proved with the help of Theorem 2.2 of [12]. Notice that Theorem 2.5 of [2] is more general than Corollary 1 in case $D \equiv \mathbb{R}^n$.

Now we consider problem (1) under dissipative conditions. Suppose E^* is uniformly convex. Let $J(\cdot)$ be the duality map and let $D = E$. The multifunction F is said to be *one-sided Lipschitz* (OSL) if for every $x, y \in E$,

$$\sigma(J(x-y), F(t,x)) - \sigma(J(x-y), F(t,y)) \le L(t)|x-y|^2$$

where $L(\cdot)$ is Lebesgue integrable (see [9] for more details).

Define

$$H(t,x) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\, F(t, x+\varepsilon U) \in P_{\mathrm{c}}(E).$$

THEOREM 4. *Let F be OSL and let $H(\cdot,\cdot)$ be compact-valued and almost USC. If F is bounded on bounded sets*, *then under* A1, A2 *problem* (1) *has a solution. Furthermore the solution set of* (1) *is dense in the solution set of*

$$\dot{x}(t) \in H(t, x(t)), \qquad x(0) = x_0. \tag{4}$$

Proof. Notice first that one can replace $L(\cdot)$ by 1 if needed (see [8, 9] for instance). Furthermore one can suppose that F is integrably bounded. Indeed, for every AC $x(\cdot)$ with $\operatorname{dist}(\dot{x}(t), F(t, x(t)+\overline{U})) \le 1$ one has $\langle J(x(t)), \dot{x}(t)\rangle \le |x(t)|\cdot|F(t,U)| + |x(t)|^2$. Thus $\frac{d}{dt}|x(t)|^2 \le |x(t)|\cdot|F(t,U)| +$

$|x(t)|^2$. Since $|x(\cdot)|$ is AC one has $\frac{d}{dt}|x(t)| \le |x(t)| \cdot |F(t,U)| + |x(t)|$. Thus there exists a positive constant K such that $|F(t,x(t))| \le K$ whenever $t \in I$ and $|x| \le C$. So we suppose that $F(\cdot,\cdot)$ is bounded.

From Theorem 1 of [9] we know that (4) has a nonempty R_δ solution set. Furthermore, as pointed out in Remark 2.2 of [10], if $R(t,x) \subset F(t,x)$ is strongly measurable in t and with closed graph in x, then the differential inclusion (1) with $F(\cdot,\cdot)$ replaced by $R(\cdot,\cdot)$ has a nonempty R_δ solution set.

It remains to prove that the solution set of (1) is dense in the solution set of (4). To this end consider the differential inclusion $\dot{x}(t) \in F(t, x(t) + \varepsilon U)$, $x(0) = x_0$. Let $y(\cdot)$ be its solution. Fix $\delta > 0$ and consider the multimap

$$F_\delta(t,z) = \mathrm{cl}\{v \in F(t,z) : \langle J(y(t)-z), \dot{y}(t) - v\rangle < |y(t)-z|^2 \\ + \mathrm{Mo}_J(\varepsilon) + l^2(t) + 2l(t)|y(t)-z| + \delta\}.$$

Here $\mathrm{Mo}_J(\delta) = \sup\{|J(x)-J(y)| : |x-y| \le \delta;\ x,y \in CU\}$ is the modulus of continuity of the duality map J on the bounded set CU and $l(\cdot)$ is a positive continuous function with $|l(t)| \le \varepsilon$.

It is not difficult to see that $F_\delta(\cdot,\cdot)$ is LSC at (t,x) when $F(\cdot,\cdot)$ is LSC at (t,x). Moreover if $F(t,\cdot)$ is USC then $F_\delta(t,\cdot)$ is USC and $F_\delta(\cdot,x)$ admits a strongly measurable selection. Consequently, the differential inclusion

$$\dot{x}(t) \in F_\delta(t,x(t)), \qquad x(0) = x_0,$$

has a solution $z_{\delta,\varepsilon}(\cdot)$. It is standard to prove that

$$\lim_{\delta \to 0^+;\, \varepsilon \to 0^+} |z_{\delta,\varepsilon}(t) - y(t)| = 0$$

uniformly on I. It is also easy to see that if $y(\cdot)$ is a solution of (4) then there exists $\varepsilon_i \to 0$ such that $x_i(t) \to y(t)$ uniformly on I for some $\dot{x}_i(t) \in F(t, x_i(t) + \varepsilon_i U)$, $x_i(0) = x_0$. ■

REMARK 3. Theorem 4 deals with "dissipative type conditions". Here we essentially use the results of [9]. However, in that paper the author considers the case when F is either almost LSC, or almost USC, or a sum of an almost LSC and an almost USC multifunction.

Obviously when F satisfies the growth condition one can suppose that $F(t,\cdot)$ is locally OSL. Namely for every $x \in E$ there exist a neighbourhood V of x and L such that

$$\sigma(J(x-y), F(t,x)) - \sigma(J(x-y), F(t,y)) \le L|x-y|^2$$

when $y \in V$.

Furthermore one can suppose that $F(\cdot,\cdot)$ is bounded on bounded sets. Indeed, as in the previous proof there exists an integrable $\lambda(\cdot)$ such that $|F(t,x(t))| \le \lambda(t)$ whenever $t \in I$ and $|x| \le C$. If we set $y(t) = x(c(t))$, where $c(\cdot)$ is the inverse of $\int_0^t \lambda(s)\,ds$, then

$$\dot{y}(t) \in \frac{1}{\lambda(t)} F(t, y(t)).$$

The right-hand side of the last differential inclusion is bounded (cf. [8]).

When $D = E$ and E is nonseparable we can prove the following theorem.

THEOREM 5. *Assume that $F(\cdot,\cdot)$ satisfies the growth and compactness conditions. Let A satisfy* A1 *and let $F(\cdot,\cdot)$ be almost LSC on A. Assume also that $F(\cdot,x)$ has a strongly measurable selection for every $x \in E$. If $F(t,\cdot)$ is convex-valued and USC on $((I \setminus N) \times E) \setminus A$, then problem* (1) *has a solution.*

Proof. Consider the multifunction $G(t,x)$ as in the proof of Theorem 3. Obviously $G(\cdot,x)$ admits a strongly measurable selection since $f_n(\cdot,x)$ is strongly mesurable for every n. Furthermore $G(t,\cdot)$ is USC for almost every t in I by the compactness condition. Let $\{x_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of approximate solutions, i.e. $\{\dot{x}_n\}_{n=1}^{\infty}$ is integrably bounded, $x_n(\cdot)$ is AC for every n and $\lim_{n\to\infty} \operatorname{dist}((x_n,\dot{x}_n), \operatorname{graph}(G)) = 0$. Since $\dot{x}_n(\cdot)$ is strongly measurable for every n, there exists a null set N such that the space

$$E_0 = \overline{\operatorname{span} \bigcup_{t\in I\setminus N} \bigcup_{n=1}^{\infty} \{\dot{x}_n(t)\}}$$

is separable. Therefore $x_n(t) \in E_0$ for almost every $t \in I$ and every n. Hence applying Proposition 9.3 of [8] for E_0 one can prove that $\{x_n\}_{n=1}^{\infty}$ is relatively compact in $C(I,E_0)$, while $\{\dot{x}_n\}_{n=1}^{\infty}$ is relatively weakly compact in $L_1(I,E_0)$. It is standard to show, by passing to subsequences if necessary, that $x(t) = \lim_{n\to\infty} x_n(t)$ exists uniformly and that $\dot{x}(t) = \lim_{n\to\infty} \dot{x}_n(t)$ L_1-weakly. Taking into account Lemma 1 we conclude that $x(\cdot)$ is a solution of (1). ■

Consider problem (2). Define $C = (I \times X) \setminus A$. We need the following assumptions:

B1. There exists $A \in \mathcal{L} \otimes \mathcal{B}(X)$ which is relatively open in X for every $t \in I$ such that F is almost LSC on A and $F(t,\alpha) \subset T_D(\alpha(0))$ for $(t,\alpha) \in A$.

B2. There exists a null set N such that for every $(t,\alpha) \in C \cap [(I \setminus N) \times X]$ the set $F(t,\alpha)$ is convex, $F(t,\cdot)$ has closed graph and $F(\cdot,\alpha)$ admits a strongly measurable selection.

B3. F satisfies the growth and compactness conditions and

$$\liminf_{h\to 0} h^{-1} \operatorname{dist}\left(\alpha(0) + \int_t^{t+h} F(s,\alpha)\,ds, D\right) = 0$$

for every $t \in I$ and every $\alpha \in X$. Here the integral is in the sense of Aumann.

THEOREM 6. *Under assumptions* B1–B3 *problem* (2) *has a solution.*

Proof. Without loss of generality we may assume $|F(t,\alpha)| \le 1$. Define

$$G(t,\alpha)=\begin{cases} F(t,\alpha), & (t,\alpha)\in((I\setminus N)\times X)\cap C,\\ G_n(t,\alpha), & (t,\alpha)\in\Omega_n,\ n=1,2,\ldots,\\ 0, & t\in N.\end{cases}$$

Here $\Omega_n=(I_n\times X)\cap A$, $F(\cdot,\cdot)$ is LSC on Ω_n and

$$G_n(t,x)=\bigcap_{\varepsilon>0}\overline{\operatorname{co}}\, f_n((t,\alpha+\varepsilon U)\cap\Omega_n).$$

Furthermore $f_n(t,\alpha)\in F(t,\alpha)$ are Γ^2-continuous selections. By the compactness condition, $G(t,\cdot)$ is USC. Obviously $G(\cdot,\alpha)$ admits a strongly measurable selection. Furthermore, B3 holds.

We follow the proof of Theorem 3.3 of [4]. Denote by V the open unit ball in X. Given $\varepsilon>0$ we look for $x^\varepsilon(\cdot)$ satisfying $x^\varepsilon(0)=\phi$ and $\dot{x}^\varepsilon(t)\in G(t,(x_t^\varepsilon+\varepsilon V)\cap D)+\varepsilon\overline{U}$ almost everywhere in I. Given $\alpha\in X$ there exist sequences $h_n\to 0^+$, $y^n\to 0$ and strongly measurable $w_n(\cdot)\in G(\cdot,\alpha)$ on $[0,a)$ (for some $a>0$) such that $\alpha(0)+\int_t^{t+h_n} w_n(s)\,ds+h_n y^n\in D$ for all $n\ge 1$. For $t=0$ and $\alpha=\psi$ we set

$$x(t)=\alpha(0)+\int_0^t (w_n(s)+y^n)\,ds$$

on $[0,\delta]$, where $\delta=h_n$ for sufficiently large n such that $h_n\le\varepsilon$, $|y^n|\le\varepsilon$ and get $\dot{x}(t)\in G(t,\psi)+\varepsilon U\subset G(t,,x_t+\varepsilon\overline{V})+\varepsilon\overline{V}$ a.e. on $[0,\delta]$. The usual procedure, using Zorn's lemma, yields $x^\varepsilon(\cdot)$ defined on the whole I.

Let $\varepsilon_i\to 0^+$ be decreasing. Set $x^i(\cdot)=x^{\varepsilon_i}(\cdot)$. Since $\dot{x}^i(\cdot)$ are strongly measurable, there exists a null set N such that

$$E_0=\overline{\operatorname{span}\bigcup_{t\in I\setminus N}\bigcup_{n=1}^{\infty}\{\dot{x}^n(t)\}}$$

is separable. Hence $x^n(t)\in E_0$ a.e. on I. Applying the lemma of Kisielewicz in the space E_0 (Proposition 9.3 of [8]) we obtain

$$r(t)=\beta\Big(\bigcup_{n=1}^{\infty}x_n(t)\Big)\le\int_0^t\beta\Big(\bigcup_{n=1}^{\infty}\dot{x}_n(s)\Big)\,ds\le\int_0^t w(s,r(s))\,ds.$$

Thus $\beta(\bigcup_{n=1}^\infty x_n(t))=0$ and hence $\{\bigcup_{n=1}^\infty x_n(\cdot)\}$ is relatively compact in $C(I,E)$. Let $x(\cdot)$ be its density point. Then $\dot{x}(t)\in G(t,x_t)$, $x_0=\phi$ and $x(t)\in D$. In the same fashion as in the proof of Lemma 1 one can prove that $x(\cdot)$ is a solution of (2). ∎

The following theorem extends Theorem 5.1 of [7] and gives affirmative answers to Problems 9.8 and 9.9 of [8] in the case of Hilbert spaces. Suppose

E is a Hilbert space. Define $H(t,x,p) = \max\{\langle p, v\rangle : v \in F(t,x)\}$. The *proximal normal cone* to D at $d \in D$ is

$$N_D^{\mathrm{P}}(d) = \{\lambda(x-d) : \lambda > 0;\ x \notin D \text{ is such that } |x-d| = \operatorname{dist}(x,D)\}.$$

We let $\operatorname{proj}_D^\delta(x) := \{s \in D : |x-s|^2 < \operatorname{dist}^2(x,D) + \delta^2\}$.

THEOREM 7. *Assume that condition* A1, *the growth condition and compactness condition hold*, $F(\cdot,\cdot)$ *is almost LSC with compact values on* A *and* $F(t,\cdot)$ *is USC with convex and compact values on* $D \setminus A$. *Let* E *be a Hilbert space and let there exist a null set* N *such that for* $t \notin N$ *either* $H(t,x,\xi) \le 0$ *for all* $\xi \in N_D^{\mathrm{P}}(x)$ *and all* $(t,x) \in A$, *or for every* $\xi \in N_D^{\mathrm{P}}(x)$ *and all* $(t,x) \in ((I\setminus N)\times D)\setminus A$ *there exists a strongly measurable selection* $f(t) \in F(t,x)$ *such that* $\langle f(t),\xi\rangle \le 0$ *for all* $(t,x) \in ((I\setminus N)\times D)\setminus A$. *Then problem* (1) *has a solution.*

Proof. As in [8], p. 52, one can reduce the problem to the case $|F(t,x)| \le 1$. Furthermore let $f(t,x)$ be a Γ^2-continuous selection of $F(t,x)$ on A. Define $G(t,x)$ as in the proof of Lemma 1. Hence $G(t,\cdot)$ is USC with convex and compact values. Furthermore $|G(t,x)| \le 1$ and for every $\xi \in N_D^{\mathrm{P}}(x)$ and all $(t,x) \in (I\setminus N)\times D$ there exists a strongly measurable selection $f(t) \in F(t,x)$ such that $\langle f(t),\xi\rangle \le 0$ for all $(t,x) \in I \times D$.

Suppose $x \in E$ is such that $\operatorname{dist}(x,D) < 1/2$. For any such x and any $\delta \in (0,1/2)$ we choose $s_\delta(x) \in \operatorname{proj}_D^\delta(x)$. It is easy to see, as in the proof of Theorem 3.1, p. 504 of [7], that there exists a strongly measurable $f_\delta(t,x) \in G(t, s_\delta(x) + \delta U)$ such that $|f_\delta(t,x)| \le 1$ and $\langle f_\delta(t,x), x - s_\delta(x)\rangle \le 4\delta$. Indeed, from Proposition 2.2 of [7] we know that for $x \in E\setminus D$, $\delta > 0$ and $s_\delta \in \operatorname{proj}_D^\delta(x)$ there exist $y_\delta \in E\setminus D$ and $\overline{s}_\delta \in D$ such that

$$y_\delta - \overline{s}_\delta \in N_D^P(\overline{s}_\delta), \qquad |(y_\delta - \overline{s}_\delta) - (x - s_\delta)| \le 2\delta, \qquad |s_\delta - \overline{s}_\delta| \le \delta.$$

Thus there exists a strongly measurable $f_\delta(t,x) \in G(t,\overline{s}_\delta)$ such that $\langle f_\delta(t,x), y_\delta - \overline{s}_\delta\rangle \le 0$. Hence $f_\delta(t,x) \in G(t, s_\delta(x) + \delta U)$, $|f_\delta(t,x)| \le 1$ and $\langle f_\delta(t,x), x - s_\delta(x)\rangle \le 4\delta$.

Let $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = 1\}$ be the subdivision of I with $t_i = i/N$. Set $x_i = x(t_i)$, $s_i = s_\delta(x_i)$ and $f_i(t) = f_\delta(t,x_i)$. Furthermore, let $x(t) = x_i + \int_{t_i}^t f_i(\tau)\,d\tau$. Then $|x(t) - x_i| \le t - t_i$ for $t \in [t_i, t_{i+1}]$. Therefore

$$\begin{aligned}\operatorname{dist}^2(x_{i+1},D) &\le |x_{i+1} - s_i|^2 \\ &= |x_{i+1} - x_i|^2 + |x_i - s_i|^2 + 2\langle x_{i+i} - x_i, x_i - s_i\rangle \\ &\le (t_{i+1} - t_i)^2 + \operatorname{dist}^2(x_i,D) + \delta^2 + 2\int\limits_{t_i}^{t_{i+1}} \langle f_i(t), x_i - s_i\rangle\,dt.\end{aligned}$$

Hence

$$\operatorname{dist}^2(x_{i+1},D) - \operatorname{dist}^2(x_i,D) \le (t_{i+1} - t_i)^2 + \delta^2 + 8\delta.$$

Thus

$$\operatorname{dist}^2(x_{i+1},D) \le i(\delta^2+8\delta)+\sum_{j=0}^{i}(t_{j+1}-t_j)^2$$
$$\le N(\delta^2+8\delta)+\sum_{j=0}^{i}N(t_{j+1}-t_j)^2=\frac{1}{N}+N(\delta^2+8\delta).$$

Given $\varepsilon>0$ one can choose $\delta>0$ so small and N so large that $\operatorname{dist}(x(t),D) \le \varepsilon$. Furthermore one can construct $x(t)$ such that $\dot{x}(t)\in G(t,x(t)+\varepsilon U)$. Consider the sequence $x_n(\cdot)$ of AC functions with $\dot{x}_n(t)\in G(t,x_n(t)+(1/n)U)$, $x(0)=x_0$ and $\operatorname{dist}^2(x_n(t),D)\le 1/n$. Using standard arguments one can conclude that $\{x_n(\cdot)\}_{n=1}^{\infty}$ is relatively compact in $C(I,E)$ (see the proof of Theorem 6). Passing to subsequences if necessary we have $x_n(t)\to x(t)$ uniformly on I. Moreover $\{\dot{x}_n(\cdot)\}_{n=1}^{\infty}$ is weakly precompact in $L_1(I,E)$ because $|\dot{x}_n(t)|\le 1$ for a.a. $t\in I$. Passing to a subsequence we have $\dot{x}_n(t)\to\dot{x}(t)$ weakly in $L_1(I,E)$. Taking into account Lemma 1 and Mazur's lemma one can conclude that

$$\dot{x}(t)\in F(t,x(t)),\qquad x(0)=x_0\quad\text{and}\quad x(t)\in D$$

for a.a. $t\in I$. ■

3. Applications to optimal control. Concluding remarks. In this section we present some applications of the previous results. We also show briefly some possible extensions of our results and compare them with the recent papers [2, 12]. Consider the system

$$\dot{x}(t)\in F(t,x(t)),\qquad x(0)=x_0. \tag{5}$$

Here $F(t,x)=\overline{\bigcup_{u\in V}G(t,x,u)}$, where $u\in V$, a closed set in a metric space, and $G: I\times D\times V\to P_{\mathrm{f}}(E)$. Suppose $G(\cdot,\cdot,\cdot)$ satisfies the assumptions of Theorem 2, Theorem 3 or Theorem 5. As a corollary of these theorems we obtain

PROPOSITION 1. *Problem* (5) *has a solution.*

However to obtain more convenient results we need additional assumptions. So let E^* be uniformly convex and let G be defined on the whole E.

THEOREM 8. *Suppose* $G(\cdot,\cdot,u)$ *satisfies the conditions of Theorem* 4 *uniformly in* u. *Then the solution set of* (5) *is dense in the solution set of*

$$\dot{x}(t)\in H(t,x),\qquad x(0)=x_0, \tag{6}$$

with respect to the $C(I,E)$ *topology. Here* $H(t,x)=\bigcap_{\varepsilon>0}\overline{\mathrm{co}}\,F(t,x+\varepsilon U)$.

The proof is the same as that of Theorem 1 of [9].

The last theorem implies that the control system (5) is "correct in the sense of relaxation". However, the "relaxed system" (6) is no more a control system in general.

Using the approach of the recent paper [7] one can consider the question of strong invariance of the solution set of (5).

DEFINITION 4. Let D be a closed set. Problem (5) is said to be D-*strongly invariant* if every solution of (5) remains in D when $x_0 \in D$.

The system (D, F) is said to be *approximately strongly invariant* if for any $\lambda > 0$, $T > 0$, and any $x_0 \in D$ there exists $\varepsilon = \varepsilon(x_0, \lambda, T) > 0$ such that every ε-solution $x(\cdot)$ of (5) satisfies $\operatorname{dist}(x(t), D) \le \lambda$ for all $t \in [0, T]$. We recall that the AC function $x(\cdot)$ is called an ε-*solution* when $\dot{x}(t) \in F(t, x(t) + \varepsilon U)$.

The following theorem partially generalizes Theorem 5.2 of [7].

THEOREM 9. *Suppose the conditions of Theorem* 8 *hold. Problem* (5) *is* D*-strongly invariant if the following Hamiltonian condition holds*:

$$H(t, x, \xi) \le 0 \quad \forall \xi \in N_D^{\mathrm{P}}(x)\ \forall x \in D \textit{ for a.a. } t \in I.$$

Proof. We will follow the proof of Theorem 5.2 of [7]. Set $d_D(x) = \operatorname{dist}(x, D)$. We let $\operatorname{proj}_D^{\delta}(x) := \{s \in D : |x - s|^2 < d_D^2(x) + \delta^2\}$. First we show that under the conditions of Theorem 9, problem (5) is approximately strongly invariant.

Indeed, define $\widehat{t} = \sup\{t' \in I : d_D(x(t)) < 1\ \forall t \in [0, t']\}$. Notice that $\widehat{t} > 0$. We let $w(t) = d_D^2(x(t))$. Let $x(\cdot)$ and $w(\cdot)$ be differentiable at $t \in [0, t')$. Choose $s_\delta \in \operatorname{proj}_D^{\delta}(x(t))$. Then

$$\begin{aligned} \dot{w}(t) &= \lim_{\delta \to 0^+} \frac{d_D^2(x(t+\delta)) - d_D^2(x(t))}{\delta} \\ &\le \limsup_{\delta \to 0^+} \frac{|x(t+\delta) - s_\delta|^2 - |x(t) - s_\delta|^2 + \delta^2}{\delta} \\ &= \limsup_{\delta \to 0^+} \langle x(t) - s_\delta, \dot{x}(t) \rangle. \end{aligned}$$

By Proposition 2.2 of [7] there exists a pair $(y_\delta, \overline{s}_\delta)$ with $\overline{s}_\delta \in D$ such that

$$y_\delta - \overline{s}_\delta \in N_D^{\mathrm{P}}(\overline{s}_\delta), \qquad |(y_\delta - \overline{s}_\delta) - (x - s_\delta)| \le 2\delta, \qquad |s_\delta - \overline{s}_\delta| \le \delta.$$

Consequently, $\langle x(t) - s_\delta, \dot{x}(t) \rangle \le \langle y_\delta - \overline{s}_\delta, \dot{x}(t) \rangle + 2K\delta$ because $|\dot{x}(t)| \le K$. Since $\dot{x}(t) \in F(t, x(t) + \varepsilon U)$, there exists a (strongly) measurable $v(t) \in F(t, \overline{s}_\delta)$ such that

$$\langle x(t) + h(t) - s_\delta, \dot{x}(t) - v(t) \rangle \le L|x(t) + h(t) - s_\delta|^2.$$

Using standard calculations one can show that there exists a positive constant C such that

$$\langle y_\delta - \overline{s}_\delta, \dot{x}(t) - v(t) \rangle \le C\{|y_\delta - \overline{s}_\delta| \cdot |\dot{x}(t) - v(t)| + \varepsilon\}.$$

Therefore

$$\begin{aligned}\langle y_\delta - \overline{s}_\delta, \dot{x}(t)\rangle &\le \langle y_\delta - \overline{s}_\delta, v(t)\rangle + C\{|y_\delta - \overline{s}_\delta| \cdot |\dot{x}(t) - v(t)| + \varepsilon\}\\ &\le C\{|y_\delta - \overline{s}_\delta| \cdot |\dot{x}(t) - v(t)| + \varepsilon\}.\end{aligned}$$

Thus

$$\limsup_{\delta\to 0^+}\langle x(t) - s_\delta, \dot{x}(t)\rangle \le L d_D^2(x(t)) + C\varepsilon.$$

Therefore $\dot{w}(t) \le 2Lw(t) + C\varepsilon$ and hence $w(t) \le e^{Lt}\int_0^t e^{-Ls}C\varepsilon\,ds$, i.e. $w(t) \le C\varepsilon^{1/2}$. Choosing ε sufficiently small one can assure $d_D(x(t)) < 1$ for all $t \in I$. Hence problem (5) is approximately strongly invariant.

Let $y(\cdot)$ be a solution of (5). Choosing sufficiently small $\varepsilon > 0$ one can show that $d_D(y(t)) \le \lambda$ for all $t \in I$ for fixed $\lambda > 0$. Since $\lambda > 0$ is arbitrary, $y(t) \in D$. ∎

When $F(t,\cdot)$ is locally Lipschitz the Hamiltonian condition is also necessary (cf. Theorem 5.2 of [7]).

Example 1. Let $E = \mathbb{R}^1$ and let $D = [-1, 0]$. Set $A = [-1, 1]$. Define

$$F(x) = \begin{cases} -1, & x > 0,\\ A, & x = 0,\\ 1, & x < 0.\end{cases}$$

Obviously in case $x = 0$ the Hamiltonian condition does not hold. However, the system (5) is strongly invariant. Therefore in our case the Hamiltonian condition is not necessary.

Remark 4. Some of our results can be extended to the situation when $D(\cdot)$ depends on t. We refer to [3, 4, 11, 13] and the references therein for the theory of such differential inclusions. We will formulate one typical result following [3, 4].

Let the graph $R = \{(t,x) : t \in I,\ x \in D(t)\}$ be left-closed, i.e.

$$(t_n, x_n) \in R \text{ with } t_n \nearrow t \text{ and } x_n \to x \quad \text{implies} \quad (t,x) \in R.$$

Theorem 10. *Let R be left-closed and $F : R \to P_{\mathrm{f}}(E)$ be A-ASC and satisfy the growth and compactness conditions. Set $[(I_n \times E) \cap R] \cap A = \Omega_n$. Suppose Ω_n is relatively open in R. Assume that for $t \in I \setminus N$ and $(t,x) \in R$ one has either $\{1 \times F(t,x)\} \cap T_R(t,x) \ne \emptyset$ when $F(t,\cdot)$ is USC or $1 \times F(t,x) \subset T_R(t,x)$ when $F(t,\cdot)$ is LSC. Suppose that $\{1 \times E\} \cap T_R(t,x) \ne \emptyset$ for $t \in N$ and $(t,x) \in R$ where N is a null set. Then the Cauchy problem*

$$\dot{x}(t) \in F(t,x), \quad x(t) \in D(t), \quad x(0) = x_0 \in D(0), \tag{7}$$

has a solution.

Since the theory of differential inclusions under time depending constraints is not considered in this paper, we will only sketch the proof.

Proof. Define $G(t,x)$ as in Lemma 1, obtained as in the proof of Theorem 4.1 of [4]. Evidently $G(\cdot,\cdot)$ is an almost USC map from R into E satisfying: $\{1\times G(t,x)\}\cap T_R(t,x)\neq\emptyset$ for $t\in I\setminus N$ with $(t,x)\in R$, and $\{1\times E\}\cap T_R(t,x)\neq\emptyset$ for $t\in N$.

In the same fashion as in the proof of Theorem 3.1 of [4] one can reduce the problem to the case $|G(t,x)|\le 1$. Afterwards Lemma 3.1 of [4] applies. Therefore problem (7) with F replaced by G has a solution $x(\cdot)$. By Lemma 1, $x(\cdot)$ is also a solution of (7). ∎

Remark 5. The most general result in case $E\equiv\mathbb{R}^n$ has been obtained in [2]. The authors consider an abstract problem and use our approach (Bressan–Colombo result for Γ^N-continuous selections). As corollaries the existence of solutions under mixed semicontinuity conditions is obtained in several cases, including the case considered here (without constraints), boundary valued problems, integral inclusions. The approach, however, does not hold in the case of state constraints and is applied only for finite-dimensional spaces.

Our problem is also considered in another recent paper [12]. However, the main tool there is to prove the existence of appropriate selections for multimaps with decomposable values (Fryszkowski and Bressan–Colombo results). It would be interesting to compare our results with more general results which can be obtained with the help of Theorem 2.2 of [12].

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