Invariance of domain in o-minimal structures

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Abstract. The aim of this paper is to prove the theorem on invariance of domain in an arbitrary o-minimal structure. We do not make use of the methods of algebraic topology and the proof is based merely on some basic facts about cells and cell decompositions.

1. Preliminaries. For the convenience of the reader we recall some notions and facts on o-minimal structures (cf. [1]).

(1.1) A linearly ordered set $R$ is called dense if for all $a, b \in R$ with $a < b$ there is $c \in R$ such that $a < c < b$. Let $(R, <)$ be a dense linearly ordered set without endpoints, that is, $R$ has no largest or smallest element. We add two endpoints $-\infty, +\infty$ satisfying $-\infty < a < +\infty$ for all $a \in R$ and define open and closed intervals respectively by

\[
(a, b) = \{x \in R \mid a < x < b\} \quad \text{where } -\infty \leq a < b \leq +\infty, \\
[a, b] = \{x \in R \mid a \leq x \leq b\} \quad \text{where } -\infty < a \leq b < +\infty.
\]

We take $R$ equipped with the order topology. By an open box in $R^n$ we will mean the cartesian product of $n$ open intervals. Then open boxes form the base of a topology in $R^n$.

(1.2) Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints.

Definition. An o-minimal structure on $R$ is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n \in \mathbb{N}}$ such that for each $n$:

1. $\mathcal{S}_n$ is a boolean algebra of subsets of $R^n$;
2. If $A \in \mathcal{S}_n$, then $A \times R$ and $R \times A$ belong to $\mathcal{S}_{n+1}$;
3. $\{(x_1, \ldots, x_n) \in R^n \mid x_1 = x_n\} \in \mathcal{S}_n$;
4. If $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi : R^{n+1} \to R^n$ is the projection on the first $n$ coordinates;

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5. \((x, y) \in \mathbb{R}^2 \mid x < y\) \in \mathcal{S}_2; \\
6. The sets in \(\mathcal{S}_1\) are exactly finite unions of intervals and points.

(1.3) We shall now recall an example of an \(o\)-minimal structure on the ordered set \(\mathbb{R}\) of real numbers. A \textit{semialgebraic subset} of \(\mathbb{R}^n\) is a subset defined by a finite system of polynomial equations and inequalities with real coefficients. By the Tarski–Seidenberg theorem the semialgebraic sets satisfy axiom 4 in the definition above. The remaining axioms are obviously fulfilled, so the semialgebraic sets form an \(o\)-minimal structure.

(1.4) From now on we fix an \(o\)-minimal structure \(\mathcal{S}\) on \(\mathbb{R}\). Let \(A \subset \mathbb{R}^n\) and \(f : A \to \mathbb{R}^m\). We say \(A\) is \textit{definable} if \(A \in \mathcal{S}_n\). We say the map \(f\) is definable if its graph \(\Gamma(f) \subset \mathbb{R}^{n+m}\) is definable. If \(f\) is definable, then the domain \(A\) of \(f\) and its image \(f(A)\) are also definable. Moreover, if \(f\) is injective, its inverse \(f^{-1}\) is definable.

(1.5) If \(A \subset \mathbb{R}^n\) is definable, so are its closure, interior, boundary and frontier, where the frontier of \(A\) is \(\partial A = \overline{A} \setminus A\).

(1.6) A set \(X \subset \mathbb{R}^n\) is called \textit{definably connected} if \(X\) is definable and \(X\) is not the union of two disjoint nonempty definable open subsets of \(X\). Of course, an interval is definably connected. A \textit{definably connected component} of a definable set \(X \subset \mathbb{R}^n\) is, by definition, a maximal definably connected subset of \(X\).

(1.7) For each definable set \(X \subset \mathbb{R}^n\) we put
\[
C(X) = \{ f : X \to \mathbb{R} \mid f \text{ is definable and continuous} \}.
\]
Let \(C_\infty(X) = C(X) \cup \{-\infty, +\infty\}\), where we regard \(-\infty\) and \(+\infty\) as constant functions on \(X\). For \(f, g\) in \(C_\infty(X)\) we write \(f < g\) if \(f(x) < g(x)\) for all \(x \in X\), and in this case we put
\[
(f, g) = \{(x, r) \in X \times \mathbb{R} \mid f(x) < r < g(x)\}.
\]

(1.8) \textbf{Definition.} Let \((i_1, \ldots, i_n)\) be a sequence of zeros and ones of length \(n\). An \((i_1, \ldots, i_n)\)-cell is a definable subset of \(\mathbb{R}^n\) obtained by induction on \(n\) as follows:
- \(a (0)\)-cell is a point \(\{r\} \subset \mathbb{R}\), a \((1)\)-cell is an interval \((a, b) \subset \mathbb{R}\);
- suppose \((i_1, \ldots, i_n)\)-cells are already defined; then an \((i_1, \ldots, i_n, 0)\)-cell is the graph \(\Gamma(f)\) of a function \(f \in C(X)\), where \(X\) is an \((i_1, \ldots, i_n)\)-cell; further, an \((i_1, \ldots, i_n, 1)\)-cell is a set \((f, g)\), where \(f, g \in C_\infty(X)\), \(f < g\) and \(X\) is an \((i_1, \ldots, i_n)\)-cell.

A \textit{cell} in \(\mathbb{R}^n\) is an \((i_1, \ldots, i_n)\)-cell for some sequence \((i_1, \ldots, i_n)\). We call \((1, \ldots, 1)\)-cells \textit{open cells}.

(1.9) \textbf{Definition.} A \textit{cell decomposition} of \(\mathbb{R}^n\) is a partition of \(\mathbb{R}^n\) into finitely many cells defined inductively as follows:
• a cell decomposition of $R^1 = R$ is a collection
  \[ \{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\} \]
  where $a_1 < \ldots < a_k$ are points in $R$;
• a cell decomposition of $R^{n+1}$ is a finite partition of $R^n$ into cells $A$
  such that the set of projections $\pi(A)$ is a cell decomposition of $R^n$.

(1.10) CELL DECOMPOSITION THEOREM [1]. Given a finite collection of
definable sets in $R^n$ there is a cell decomposition of $R^n$ partitioning each of
them.

(1.11) We define the dimension of a nonempty definable set $X \subset R^n$ by
\[ \dim X = \max\{i_1 + \ldots + i_n \mid X \text{ contains an } (i_1, \ldots, i_n)\text{-cell}\}. \]
To the empty set we assign the dimension $-\infty$.
A nonempty definable set $X \subset R^n$ is called purely $k$-dimensional if for
each nonempty definable open subset $U \subset X$ we have $\dim U = \dim X = k$.

(1.12) THEOREM [1]. Let $X$ be a nonempty definable subset of $R^n$. Then
\[ \dim \partial X < \dim X. \]

(1.13) A stratification of $R^n$ is a partition $T$ of $R^n$ into finitely many
cells such that for each cell $A \in T$, $\partial A$ is a union of cells of $T$.

(1.14) THEOREM [1]. Given any definable sets $A_1, \ldots, A_k \subset R^n$ there is
a stratification of $R^n$ partitioning each of $A_1, \ldots, A_k$.

(1.15) THEOREM [2]. If $f : X \to R^n$ is a definable map on a closed
bounded set $X \subset R^m$, then $f(X)$ is closed and bounded in $R^n$.

(1.16) THEOREM [1]. If $X \subset R^n$, $Y \subset R^m$ are definable and there is a
definable bijection between $X$ and $Y$, then $\dim X = \dim Y$.

(1.17) COROLLARY. Let $f : A_1 \to A_2$ be a definable continuous bijection,
where $A_1$ is a purely $k$-dimensional subset of $R^n$ and $A_2 \subset R^m$. Then $A_2$
is a purely $k$-dimensional subset of $R^m$.

2. Some properties of cells. In this section we prove some lemmas
on cells. In the proofs we assume that if a cell $\theta$ is of the form $(f, g)$, then
$f, g \in C(\pi(\theta))$. We do not consider cells of the form $(-\infty, f)$, $(f, +\infty)$,
$X \times R$, because the corresponding proofs are simple modifications of the
cases handled below.

(2.1) LEMMA. If $G$ is an open cell in $R^n$, then $\text{int } G = G$.

Proof. The lemma is clear for $n = 1$. Let $n > 1$ and assume inductively
the lemma holds for $n - 1$. Take $G = (f, g)$ with $f, g \in C(\Omega)$, where $\Omega$
is an open cell in $R^{n-1}$. We have $\overline{G} = [\overline{G} \cap ((\overline{\Omega} \setminus \Omega) \times R)] \cup G \cup \Gamma(f) \cup \Gamma(g)$. If
$x \in \text{int } G$, then $x \in W \subset \overline{G}$, where $W$ is an open set in $R^n$. Clearly, $\pi(W)$
is open in $R^{n-1}$ and $\pi(x) \in \pi(W) \subset \Omega$, so $\pi(x) \in \text{int} \Omega$. By the induction hypothesis $\pi(x) \in \Omega$. Therefore $x \in G$ and $\text{int} G \subset G$. The reverse inclusion is trivial.

(2.2) **Lemma.** Let $\theta \subset R^n$ be a cell with $\dim \theta = n - 1$ and let $G$ be an open cell such that $\theta \subset \partial G$. Then there is a definable set $Z \subset \theta$ such that $\dim Z < n - 1$ and for each $a \in \theta \setminus Z$ there exist arbitrarily small open boxes $K$ satisfying the following condition: $a \in K$ and $K \setminus \theta$ has exactly two definably connected components $K_1, K_2$, both open in $R^n$ and $K_1 \subset G$, $K_2 \cap G = \emptyset$.

**Proof.** By induction on $n$. The case $n = 1$ is obvious. Suppose $n > 1$ and that the lemma holds for $n - 1$.

**Case 1:** The cell $\theta$ is of the form $(\varphi, \psi)$ with $\varphi, \psi \in C(\pi(\theta))$. One checks easily that $\pi(\theta) \subset \partial \pi(G)$. By the induction hypothesis applied to the cells $\pi(\theta)$ and $\pi(G)$ there is a definable set $Z' \subset \pi(\theta)$ satisfying the required conditions. We put $Z = \theta \cap [\Gamma(f) \cup \Gamma(g) \cup (Z' \times R)]$ with $f, g \in C(\pi(G))$ such that $G = (f, g)$. Clearly, $Z = \theta \cap [\partial \Gamma(f) \cup \partial \Gamma(g) \cup (Z' \times R)]$, so $\dim Z < n - 1$. Let $a \in \theta \setminus Z$ and let $L$ be an open box such that $a \in L$ and $L \setminus [\Gamma(f) \cup \Gamma(g)] = \emptyset$. We may assume that $L$ is arbitrarily small and $\pi(L \setminus \theta) \cap \pi(\theta) = \emptyset$ (replacing $L$ by a smaller box, if necessary). We have $\pi(a) \in \pi(\theta) \setminus Z'$, so there is an open box $K'$ in $R^{n-1}$ such that $\pi(a) \in K'$ and $K' \setminus \pi(\theta)$ has exactly two definably connected components $K'_1, K'_2$, both open in $R^{n-1}$ and $K'_1 \subset \pi(G)$, $K'_2 \cap \pi(G) = \emptyset$. Since $K'$ can be chosen arbitrarily small, we may assume that $K' \subset \pi(L)$. We put $K = K' \times I$, $K_1 = K'_1 \times I$, $K_2 = K'_2 \times I$, where $I$ is an interval such that $L = \pi(L) \times I$. One easily verifies that the box $K$ has the required property.

**Case 2:** The cell $\theta$ is of the form $\Gamma(\varphi)$, $\varphi \in C(\pi(\theta))$. We have $G = (f, g)$ with $f, g \in C(\pi(G))$ and $\pi(\theta), \pi(G)$ are open cells in $R^{n-1}$. Note that $\pi(\theta) \subset \text{int} \pi(G)$. Hence by (2.1), $\pi(\theta) \subset \pi(G)$ and either $\varphi = f|\pi(\theta)$ or $\varphi = g|\pi(\theta)$. The rest of the proof is now straightforward. In this case we can take $Z = \emptyset$.

(2.3) **Remark.** In general, we cannot expect to have $Z = \emptyset$ in the previous lemma. Consider a semialgebraic map $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ such that $\Gamma(f) = \Gamma(g_1) \cup \Gamma(g_2)$ with

$$
g_1 : [0, +\infty) \times (0, +\infty) \ni (x, y) \mapsto x/y \in \mathbb{R},
$$

$$
g_2 : (-\infty, 0] \times (0, +\infty) \ni (x, y) \mapsto -x/y \in \mathbb{R}.
$$

We put $G = (-\infty, f)$ and $\theta = \mathbb{R} \times \{0\} \times \mathbb{R}$.

(2.4) **Lemma.** Let $G$ and $H$ be disjoint open cells in $R^n$ and let $\theta$ be a cell of dimension $n - 1$ contained in $\partial G \cap \partial H$. Then there is a definable
set $Z \subset \theta$ such that $\dim Z < n - 1$ and $(\theta \setminus Z) \cup G \cup H$ is open in $R^n$. In particular $\theta \setminus Z \subset \text{int} \overline{G \cup H}$.

**Proof.** Applying Lemma (2.2) twice: to the cells $\theta, G$ and then to $\theta, H$, we obtain definable sets $Z_1, Z_2$. We put $Z = \theta \cap (Z_1 \cup Z_2)$. One checks easily that the set $Z$ satisfies the requirements.

(2.5) **Lemma.** Let $T$ be an open cell in $R^n$ and let $\theta$ be a cell of dimension $n - 1$ contained in $\partial T$. Then there are no disjoint open cells $G, H$ contained in $T$ such that $\partial G \setminus \partial H$.

**Proof.** Suppose there are disjoint open cells $G, H$ contained in $T$ such that $\partial G \setminus \partial H$. By (2.4), $\theta \cap \text{int} \overline{G \cup H} \neq \emptyset$ and hence $\theta \cap \text{int} \overline{T} \neq \emptyset$. This is a contradiction, because $\text{int} \overline{T} = T$.

(2.6) **Lemma.** Let $V$ be an open box in $R^n$ and let $\Omega \subset V$ be an open definable set which is not definably connected. Suppose $\mathcal{C}$ is a cell decomposition of $R^n$ partitioning $V$ and $\Omega$. Then there is a cell $\theta \in \mathcal{C}$ of dimension $n - 1$ such that $\theta \subset V \setminus \Omega$.

**Proof.** By induction on $n$, the case $n = 1$ being trivial. Suppose the lemma holds for $n - 1$, where $n > 1$. We have $V = V' \times I$ with $V'$ a box in $R^{n-1}$ and $I$ an interval. By the assumption $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1, \Omega_2$ nonempty definable open subsets of $R^n$ such that $\Omega_1 \cap \Omega_2 = \emptyset$.

CASE 1: $\pi(\Omega_1) \cap \pi(\Omega_2) \neq \emptyset$. There is a cell $\theta' \in \mathcal{C}'$ of dimension $n - 1$ such that $\theta' \subset \pi(\Omega_1) \cap \pi(\Omega_2)$, where $\mathcal{C}'$ denotes the corresponding cell decomposition of $R^{n-1}$. Clearly, for some cell $\theta \in \mathcal{C}$ of dimension $n - 1$ we have $\theta \subset V \setminus \Omega$.

CASE 2: $\pi(\Omega_1) \cap \pi(\Omega_2) = \emptyset$. Applying the induction hypothesis to $V'$ and $\pi(\Omega)$ we get a cell $\theta' \subset V'$ of dimension $n - 2$ such that $\theta' \cap \pi(\Omega) = \emptyset$. There is a cell $\theta \in \mathcal{C}$ of dimension $n - 1$ such that $\pi(\theta) = \theta'$ and $\theta \subset V$. Obviously, $\theta \cap \Omega = \emptyset$.

3. Main result

**Theorem.** Let $\Omega_1$ be an open definable subset of $R^n$ and let $f : \Omega_1 \rightarrow \Omega_2$ be a definable homeomorphism onto a definable set $\Omega_2 \subset R^n$. Then $\Omega_2$ is open in $R^n$.

**Proof.** Given $a \in \Omega_1$ we have to show that $b = f(a) \in \text{int} \Omega_2$. Let $K$ be a closed box in $R^n$ (i.e. a cartesian product of $n$ closed intervals) such that $a \in \text{int} K$ and $K \subset \Omega_1$. Note first that by (1.15), $f(K)$ is closed and bounded in $R^n$.

We take a stratification $\mathcal{C}$ of $R^n$ that partitions $f(K)$. Let $\mathcal{A}$ be a collection of all open cells $A \in \mathcal{C}$ such that $b \in \overline{A}$. We define $\mathcal{F} = \{ A \in \mathcal{A} |$
A \subset f(K)$. Since $f(K)$ is a purely $n$-dimensional set (cf. (1.17)), it follows that $\mathcal{F} \neq \emptyset$. Suppose now that $b \not\in \text{int } f(K)$. Note that then $A \setminus \mathcal{F} \neq \emptyset$.

**Claim.** There exist $G \in \mathcal{F}$, $H \in A \setminus \mathcal{F}$ and a cell $\theta \in \mathcal{C}$ of dimension $n-1$ such that $b \in \overline{\theta}$ and $\theta \subset \partial G \cap \partial H$.

To see this, let $V$ be an open box containing $b$ and disjoint from any cell $A \in \mathcal{C}$ such that $b \not\in \overline{A}$. We put $V_1 = \bigcup \{A \mid A \in \mathcal{F}\}$, $V_2 = \bigcup \{A \mid A \in A \setminus \mathcal{F}\}$. By (2.6), $\dim (V \setminus \Omega) = n-1$, where $\Omega = V \cap (\text{int } V_1 \cup \text{int } V_2)$. Let $\theta \in \mathcal{C}$ be a cell such that $\dim [(V \setminus \Omega) \cap \theta] = n-1$. Suppose the cell $\theta$ does not have the required property. Then $\theta \subset \partial G_1 \cap \partial G_2$, where $G_1 \neq G_2$ and $G_1, G_2 \in \mathcal{F}$ or $G_1, G_2 \in A \setminus \mathcal{F}$. By Lemma (2.4), $\dim [(V \setminus \Omega) \cap \theta] < n-1$ and this is a contradiction.

Using (2.4) one easily checks that $\theta \cap \partial A = \emptyset$ whenever $A \in \mathcal{F}$ and $A \neq G$. Let $\mathcal{C}'$ be a stratification of $\mathbb{R}^n$ partitioning $f^{-1}(A)$ for all $A \in \mathcal{C}$. So $f^{-1}(\theta)$ is a purely $(n-1)$-dimensional set (cf. (1.17)), hence there is a cell $\theta' \in \mathcal{C}'$ of dimension $n-1$ such that $a \in \overline{\theta'}$ and $\theta' \subset f^{-1}(\theta)$. Clearly, $\theta' \subset \partial G'$ for some open cell $G' \subset f^{-1}(G)$. Now we use the fact that $a \in \text{int } K$ to conclude that there is an open cell $B \in \mathcal{C}'$ such that $B \neq G'$, $B \subset K$ and $\theta' \subset \partial B$ (cf. (2.2)). It is routine to check that $B \subset f^{-1}(G)$. Taking a stratification of $\mathbb{R}^n$ partitioning $f(\theta')$, $f(G')$, $f(B)$ we obtain a contradiction with Lemma (2.5).

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