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# Invariance of domain in o-minimal structures

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**Abstract.** The aim of this paper is to prove the theorem on invariance of domain in an arbitrary o-minimal structure. We do not make use of the methods of algebraic topology and the proof is based merely on some basic facts about cells and cell decompositions.

**1. Preliminaries.** For the convenience of the reader we recall some notions and facts on o-minimal structures (cf. [1]).

(1.1) A linearly ordered set $R$ is called *dense* if for all $a, b \in R$ with $a < b$ there is $c \in R$ such that $a < c < b$. Let $(R, <)$ be a dense linearly ordered set without endpoints, that is, $R$ has no largest or smallest element. We add two endpoints $-\infty, +\infty$ satisfying $-\infty < a < +\infty$ for all $a \in R$ and define open and closed intervals respectively by

$$(a, b) = \{x \in R \mid a < x < b\} \quad \text{where } -\infty \le a < b \le +\infty,$$
$$[a, b] = \{x \in R \mid a \le x \le b\} \quad \text{where } -\infty < a < b < +\infty.$$

We take $R$ equipped with the order topology. By an *open box* in $R^n$ we will mean the cartesian product of $n$ open intervals. Then open boxes form the base of a topology in $R^n$.

(1.2) Let $(R, <)$ be a dense linearly ordered nonempty set without endpoints.

Definition. An *o-minimal structure* on $R$ is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each $n$:

1. $\mathcal{S}_n$ is a boolean algebra of subsets of $R^n$;
2. If $A \in \mathcal{S}_n$, then $A \times R$ and $R \times A$ belong to $\mathcal{S}_{n+1}$;
3. $\{(x_1, \ldots, x_n) \in R^n \mid x_1 = x_n\} \in \mathcal{S}_n$;
4. If $A \in \mathcal{S}_{n+1}$, then $\pi(A) \in \mathcal{S}_n$, where $\pi : R^{n+1} \to R^n$ is the projection on the first $n$ coordinates;

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5. $\{(x,y) \in R^2 \mid x < y\} \in \mathcal{S}_2$;
6. The sets in $\mathcal{S}_1$ are exactly finite unions of intervals and points.

(1.3) We shall now recall an example of an o-minimal structure on the ordered set $\mathbb{R}$ of real numbers. A *semialgebraic subset* of $\mathbb{R}^n$ is a subset defined by a finite system of polynomial equations and inequalities with real coefficients. By the Tarski–Seidenberg theorem the semialgebraic sets satisfy axiom 4 in the definition above. The remaining axioms are obviously fulfilled, so the semialgebraic sets form an o-minimal structure.

(1.4) From now on we fix an o-minimal structure $\mathcal{S}$ on $R$. Let $A \subset R^n$ and $f : A \to R^m$. We say $A$ is *definable* if $A \in \mathcal{S}_n$. We say the map $f$ is definable if its graph $\Gamma(f) \subset R^{n+m}$ is definable. If $f$ is definable, then the domain $A$ of $f$ and its image $f(A)$ are also definable. Moreover, if $f$ is injective, its inverse $f^{-1}$ is definable.

(1.5) If $A \subset R^n$ is definable, so are its closure, interior, boundary and frontier, where the frontier of $A$ is $\partial A = \overline{A} \setminus A$.

(1.6) A set $X \subset R^n$ is called *definably connected* if $X$ is definable and $X$ is not the union of two disjoint nonempty definable open subsets of $X$. Of course, an interval is definably connected. A *definably connected component* of a definable set $X \subset R^n$ is, by definition, a maximal definably connected subset of $X$.

(1.7) For each definable set $X \subset R^n$ we put

$$C(X) = \{f : X \to R \mid f \text{ is definable and continuous}\}.$$

Let $C_\infty(X) = C(X) \cup \{-\infty, +\infty\}$, where we regard $-\infty$ and $+\infty$ as constant functions on $X$. For $f, g$ in $C_\infty(X)$ we write $f < g$ if $f(x) < g(x)$ for all $x \in X$, and in this case we put

$$(f,g) = \{(x,r) \in X \times R \mid f(x) < r < g(x)\}.$$

(1.8) Definition. Let $(i_1, \ldots, i_n)$ be a sequence of zeros and ones of length $n$. An $(i_1, \ldots, i_n)$-*cell* is a definable subset of $R^n$ obtained by induction on $n$ as follows:

- a (0)-cell is a point $\{r\} \subset R$, a (1)-cell is an interval $(a,b) \subset R$;
- suppose $(i_1, \ldots, i_n)$-cells are already defined; then an $(i_1, \ldots, i_n, 0)$-cell is the graph $\Gamma(f)$ of a function $f \in C(X)$, where $X$ is an $(i_1, \ldots, i_n)$-cell; further, an $(i_1, \ldots, i_n, 1)$-cell is a set $(f,g)$, where $f, g \in C_\infty(X)$, $f < g$ and $X$ is an $(i_1, \ldots, i_n)$-cell.

A *cell* in $R^n$ is an $(i_1, \ldots, i_n)$-cell for some sequence $(i_1, \ldots, i_n)$. We call $(1, \ldots, 1)$-cells *open cells.*

(1.9) Definition. A *cell decomposition* of $R^n$ is a partition of $R^n$ into finitely many cells defined inductively as follows:

• a cell decomposition of $R^1 = R$ is a collection

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}$$

where $a_1 < \ldots < a_k$ are points in $R$;

• a cell decomposition of $R^{n+1}$ is a finite partition of $R^n$ into cells $A$ such that the set of projections $\pi(A)$ is a cell decomposition of $R^n$.

(1.10) Cell Decomposition Theorem [1]. *Given a finite collection of definable sets in $R^n$ there is a cell decomposition of $R^n$ partitioning each of them.*

(1.11) We define the *dimension* of a nonempty definable set $X \subset R^n$ by

$$\dim X = \max\{i_1 + \ldots + i_n \mid X \text{ contains an } (i_1, \ldots, i_n)\text{-cell}\}.$$

To the empty set we assign the dimension $-\infty$.

A nonempty definable set $X \subset R^n$ is called *purely $k$-dimensional* if for each nonempty definable open subset $U \subset X$ we have $\dim U = \dim X = k$.

(1.12) Theorem [1]. *Let $X$ be a nonempty definable subset of $R^n$. Then* $\dim \partial X < \dim X$.

(1.13) A *stratification* of $R^n$ is a partition $\mathcal{T}$ of $R^n$ into finitely many cells such that for each cell $A \in \mathcal{T}$, $\partial A$ is a union of cells of $\mathcal{T}$.

(1.14) Theorem [1]. *Given any definable sets $A_1, \ldots, A_k \subset R^n$ there is a stratification of $R^n$ partitioning each of $A_1, \ldots, A_k$.*

(1.15) Theorem [2]. *If $f : X \to R^n$ is a definable map on a closed bounded set $X \subset R^m$, then $f(X)$ is closed and bounded in $R^n$.*

(1.16) Theorem [1]. *If $X \subset R^n$, $Y \subset R^m$ are definable and there is a definable bijection between $X$ and $Y$, then* $\dim X = \dim Y$.

(1.17) Corollary. *Let $f : A_1 \to A_2$ be a definable continuous bijection, where $A_1$ is a purely $k$-dimensional subset of $R^n$ and $A_2 \subset R^m$. Then $A_2$ is a purely $k$-dimensional subset of $R^m$.*

**2. Some properties of cells.** In this section we prove some lemmas on cells. In the proofs we assume that if a cell $\theta$ is of the form $(f, g)$, then $f, g \in C(\pi(\theta))$. We do not consider cells of the form $(-\infty, f)$, $(f, +\infty)$, $X \times R$, because the corresponding proofs are simple modifications of the cases handled below.

(2.1) Lemma. *If $G$ is an open cell in $R^n$, then* $\operatorname{int} \overline{G} = G$.

*Proof.* The lemma is clear for $n = 1$. Let $n > 1$ and assume inductively the lemma holds for $n - 1$. Take $G = (f, g)$ with $f, g \in C(\Omega)$, where $\Omega$ is an open cell in $R^{n-1}$. We have $\overline{G} = [\overline{G} \cap ((\overline{\Omega} \setminus \Omega) \times R)] \cup G \cup \Gamma(f) \cup \Gamma(g)$. If $x \in \operatorname{int} \overline{G}$, then $x \in W \subset \overline{G}$, where $W$ is an open set in $R^n$. Clearly, $\pi(W)$

is open in $R^{n-1}$ and $\pi(x) \in \pi(W) \subset \overline{\Omega}$, so $\pi(x) \in \operatorname{int} \overline{\Omega}$. By the induction hypothesis $\pi(x) \in \Omega$. Therefore $x \in G$ and $\operatorname{int} \overline{G} \subset G$. The reverse inclusion is trivial.

(2.2) Lemma. *Let $\theta \subset R^n$ be a cell with* $\dim \theta = n-1$ *and let $G$ be an open cell such that $\theta \subset \partial G$. Then there is a definable set $Z \subset \theta$ such that* $\dim Z < n-1$ *and for each $a \in \theta \setminus Z$ there exist arbitrarily small open boxes $K$ satisfying the following condition: $a \in K$ and $K \setminus \theta$ has exactly two definably connected components $K_1, K_2$, both open in $R^n$ and $K_1 \subset G$, $K_2 \cap G = \emptyset$.*

*Proof.* By induction on $n$. The case $n = 1$ is obvious. Suppose $n > 1$ and that the lemma holds for $n-1$.

Case 1: The cell $\theta$ is of the form $(\varphi, \psi)$ with $\varphi, \psi \in C(\pi(\theta))$. One checks easily that $\pi(\theta) \subset \partial\pi(G)$. By the induction hypothesis applied to the cells $\pi(\theta)$ and $\pi(G)$ there is a definable set $Z' \subset \pi(\theta)$ satisfying the required conditions. We put $Z = \theta \cap [\overline{\Gamma(f)} \cup \overline{\Gamma(g)} \cup (Z' \times R)]$ with $f, g \in C(\pi(G))$ such that $G = (f, g)$. Clearly, $Z = \theta \cap [\partial\Gamma(f) \cup \partial\Gamma(g) \cup (Z' \times R)]$, so $\dim Z < n-1$. Let $a \in \theta \setminus Z$ and let $L$ be an open box such that $a \in L$ and $L \cap [\Gamma(f) \cup \Gamma(g)] = \emptyset$. We may assume that $L$ is arbitrarily small and $\pi(L \setminus \theta) \cap \pi(\theta) = \emptyset$ (replacing $L$ by a smaller box, if necessary). We have $\pi(a) \in \pi(\theta) \setminus Z'$, so there is an open box $K'$ in $R^{n-1}$ such that $\pi(a) \in K'$ and $K' \setminus \pi(\theta)$ has exactly two definably connected components $K_1'$, $K_2'$, both open in $R^{n-1}$ and $K_1' \subset \pi(G)$, $K_2' \cap \pi(G) = \emptyset$. Since $K'$ can be chosen arbitrarily small, we may assume that $K' \subset \pi(L)$. We put $K = K' \times I$, $K_1 = K_1' \times I$, $K_2 = K_2' \times I$, where $I$ is an interval such that $L = \pi(L) \times I$. One easily verifies that the box $K$ has the required property.

Case 2: The cell $\theta$ is of the form $\Gamma(\varphi)$, $\varphi \in C(\pi(\theta))$. We have $G = (f, g)$ with $f, g \in C(\pi(G))$ and $\pi(\theta), \pi(G)$ are open cells in $R^{n-1}$. Note that $\pi(\theta) \subset \operatorname{int} \overline{\pi(G)}$. Hence by (2.1), $\pi(\theta) \subset \pi(G)$ and either $\varphi = f|\pi(\theta)$ or $\varphi = g|\pi(\theta)$. The rest of the proof is now straightforward. In this case we can take $Z = \emptyset$.

(2.3) Remark. In general, we cannot expect to have $Z = \emptyset$ in the previous lemma. Consider a semialgebraic map $f : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ such that $\Gamma(f) = \Gamma(g_1) \cup \Gamma(g_2)$ with

$$g_1 : [0, +\infty) \times (0, +\infty) \ni (x, y) \mapsto x/y \in \mathbb{R},$$
$$g_2 : (-\infty, 0] \times (0, +\infty) \ni (x, y) \mapsto -x/y \in \mathbb{R}.$$

We put $G = (-\infty, f)$ and $\theta = \mathbb{R} \times \{0\} \times \mathbb{R}$.

(2.4) Lemma. *Let $G$ and $H$ be disjoint open cells in $R^n$ and let $\theta$ be a cell of dimension $n-1$ contained in $\partial G \cap \partial H$. Then there is a definable* *set $Z \subset \theta$ such that* $\dim Z < n-1$ *and* $(\theta \setminus Z) \cup G \cup H$ *is open in* $R^n$*. In particular* $\theta \setminus Z \subset \operatorname{int} \overline{G \cup H}$*.*

*Proof.* Applying Lemma (2.2) twice: to the cells $\theta, G$ and then to $\theta, H$, we obtain definable sets $Z_1, Z_2$. We put $Z = \theta \cap (\overline{Z_1 \cup Z_2})$. One checks easily that the set $Z$ satisfies the requirements.

(2.5) LEMMA. *Let $T$ be an open cell in $R^n$ and let $\theta$ be a cell of dimension $n-1$ contained in $\partial T$. Then there are no disjoint open cells $G, H$ contained in $T$ such that $\theta \subset \partial G \cap \partial H$.*

*Proof.* Suppose there are disjoint open cells $G, H$ contained in $T$ such that $\theta \subset \partial G \cap \partial H$. By (2.4), $\theta \cap \operatorname{int} \overline{G \cup H} \neq \emptyset$ and hence $\theta \cap \operatorname{int} \overline{T} \neq \emptyset$. This is a contradiction, because $\operatorname{int} \overline{T} = T$.

(2.6) LEMMA. *Let $V$ be an open box in $R^n$ and let $\Omega \subset V$ be an open definable set which is not definably connected. Suppose $\mathcal{C}$ is a cell decomposition of $R^n$ partitioning $V$ and $\Omega$. Then there is a cell $\theta \in \mathcal{C}$ of dimension $n-1$ such that $\theta \subset V \setminus \Omega$.*

*Proof.* By induction on $n$, the case $n = 1$ being trivial. Suppose the lemma holds for $n-1$, where $n > 1$. We have $V = V' \times I$ with $V'$ a box in $R^{n-1}$ and $I$ an interval. By the assumption $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1, \Omega_2$ nonempty definable open subsets of $R^n$ such that $\Omega_1 \cap \Omega_2 = \emptyset$.

CASE 1: $\pi(\Omega_1) \cap \pi(\Omega_2) \neq \emptyset$. There is a cell $\theta' \in \mathcal{C}'$ of dimension $n-1$ such that $\theta' \subset \pi(\Omega_1) \cap \pi(\Omega_2)$, where $\mathcal{C}'$ denotes the corresponding cell decomposition of $R^{n-1}$. Clearly, for some cell $\theta \in \mathcal{C}$ of dimension $n-1$ we have $\theta \subset V \setminus \Omega$.

CASE 2: $\pi(\Omega_1) \cap \pi(\Omega_2) = \emptyset$. Applying the induction hypothesis to $V'$ and $\pi(\Omega)$ we get a cell $\theta' \subset V'$ of dimension $n-2$ such that $\theta' \cap \pi(\Omega) = \emptyset$. There is a cell $\theta \in \mathcal{C}$ of dimension $n-1$ such that $\pi(\theta) = \theta'$ and $\theta \subset V$. Obviously, $\theta \cap \Omega = \emptyset$.

## 3. Main result

THEOREM. *Let $\Omega_1$ be an open definable subset of $R^n$ and let $f : \Omega_1 \to \Omega_2$ be a definable homeomorphism onto a definable set $\Omega_2 \subset R^n$. Then $\Omega_2$ is open in $R^n$.*

*Proof.* Given $a \in \Omega_1$ we have to show that $b = f(a) \in \operatorname{int} \Omega_2$. Let $K$ be a closed box in $R^n$ (i.e. a cartesian product of $n$ closed intervals) such that $a \in \operatorname{int} K$ and $K \subset \Omega_1$. Note first that by (1.15), $f(K)$ is closed and bounded in $R^n$.

We take a stratification $\mathcal{C}$ of $R^n$ that partitions $f(K)$. Let $\mathcal{A}$ be a collection of all open cells $A \in \mathcal{C}$ such that $b \in \overline{A}$. We define $\mathcal{F} = \{A \in \mathcal{A} \mid$

$A \subset f(K)\}$. Since $f(K)$ is a purely $n$-dimensional set (cf. (1.17)), it follows that $\mathcal{F} \neq \emptyset$. Suppose now that $b \notin \operatorname{int} f(K)$. Note that then $\mathcal{A} \setminus \mathcal{F} \neq \emptyset$.

Claim. *There exist $G \in \mathcal{F}$, $H \in \mathcal{A} \setminus \mathcal{F}$ and a cell $\theta \in \mathcal{C}$ of dimension $n-1$ such that $b \in \overline{\theta}$ and $\theta \subset \partial G \cap \partial H$.*

To see this, let $V$ be an open box containing $b$ and disjoint from any cell $A \in \mathcal{C}$ such that $b \notin \overline{A}$. We put $V_1 = \bigcup\{A \mid A \in \mathcal{F}\}$, $V_2 = \bigcup\{A \mid A \in \mathcal{A} \setminus \mathcal{F}\}$. By (2.6), $\dim(V \setminus \Omega) = n-1$, where $\Omega = V \cap (\operatorname{int} \overline{V}_1 \cup \operatorname{int} \overline{V}_2)$. Let $\theta \in \mathcal{C}$ be a cell such that $\dim[(V \setminus \Omega) \cap \theta] = n-1$. Suppose the cell $\theta$ does not have the required property. Then $\theta \subset \partial G_1 \cap \partial G_2$, where $G_1 \neq G_2$ and $G_1, G_2 \in \mathcal{F}$ or $G_1, G_2 \in \mathcal{A} \setminus \mathcal{F}$. By Lemma (2.4), $\dim[(V \setminus \Omega) \cap \theta] < n-1$ and this is a contradiction.

Using (2.4) one easily checks that $\theta \cap \partial A = \emptyset$ whenever $A \in \mathcal{F}$ and $A \neq G$. Let $\mathcal{C}'$ be a stratification of $R^n$ partitioning $f^{-1}(A)$ for all $A \in \mathcal{C}$. So $f^{-1}(\theta)$ is a purely $(n-1)$-dimensional set (cf. (1.17)), hence there is a cell $\theta' \in \mathcal{C}'$ of dimension $n-1$ such that $a \in \overline{\theta'}$ and $\theta' \subset f^{-1}(\theta)$. Clearly, $\theta' \subset \partial G'$ for some open cell $G' \subset f^{-1}(G)$. Now we use the fact that $a \in \operatorname{int} K$ to conclude that there is an open cell $B \in \mathcal{C}'$ such that $B \neq G'$, $B \subset K$ and $\theta' \subset \partial B$ (cf. (2.2)). It is routine to check that $B \subset f^{-1}(G)$. Taking a stratification of $R^n$ partitioning $f(\theta'), f(G'), f(B)$ we obtain a contradiction with Lemma (2.5).

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