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On the Cauchy problem for linear PDEs with retarded arguments at derivatives

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Abstract. We present an existence theorem for the Cauchy problem related to linear partial differential-functional equations of an arbitrary order. The equations considered include the cases of retarded and deviated arguments at the derivatives of the unknown function. In the proof we use Tonelli's constructive method. We also give uniqueness criteria valid in a wide class of admissible functions. We present a set of examples to illustrate the theory.

1. Introduction. We study the Cauchy problem for linear partial differential-functional equations of an arbitrary order. We consider the cases of retarded and deviated arguments at the derivatives of the unknown function. The proof of our existence theorem is based on the following observation: by introducing an additional constant delay one can integrate the equation step by step and then pass to the limit with the added delay. This method or rather "way of thinking" comes from L. Tonelli who first applied it to solve a Volterra integral equation (see [T]).

The situation where functional dependence is not only at the unknown function but also at its derivatives is very difficult, much more so than in the case with no functionals at derivatives. Even a simple equation may cause many mathematical problems. A good example is the equation $D_t u(t,x) = D_x u(t,x/2)$. The method of characteristics does not work here. No existence or uniqueness result for this equation is known in the class of C^1 solutions or in any class of generalized solutions.

The methods which usually work in the theory of functional-differential equations can be applied only to very special cases (see [A, L]). Some nontrivial results can be obtained by considering analytic solutions (see [ALW1, ALW2]) and solutions analytic with respect to the spatial variable (see [AL]). The investigations in [A, ALW1, ALW2, L] are based mainly on

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the Nagumo Lemma, fixed point methods or monotone iterative techniques. The existence of solutions for a majorant problem is usually assumed. The investigation in each case is rather complicated.

The main advantage of the technique presented in this paper is its simplicity and simplicity of the assumptions. Tonelli's method seems to be particularly useful in the study of differential functional equations. For instance, in our problem the procedure does not vary much whether it is applied to equations with functionals at the unknown function or at its derivatives.

Tonneli's constructive method has a long tradition in the study of differential equations. Cinquini [C] gives a wide spectrum of its applications to partial differential equations. In particular the Cauchy problem for first order partial differential equations is studied in [BV]. Conti [Co] deals with the Darboux problem. Quasilinear systems of hyperbolic type are considered in [CC].

Tonelli's method for a nonlinear parabolic differential-functional Cauchy problem is considered in [To]. The functional dependence (only at the unknown function) is in a general form (of a Hale type operator). The existence of viscosity solutions is proved. According to the author's knowledge, the present paper and [To] are the first where Tonnelli's method is applied to differential-functional equations.

Set

$$\Theta = (0,T] \times \mathbb{R}^n, \quad \Theta_0 = (-\infty, 0] \times \mathbb{R}^n, \quad E = (-\infty, T] \times \mathbb{R}^n, \quad T > 0.$$

Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ we define $|m| = m_1 + \cdots + m_n$. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we write $xy = (x_1y_1, \ldots, x_ny_n)$, $x/y = (x_1/y_1, \ldots, x_n/y_n)$, $x^m = x_1^{m_1} \ldots x_n^{m_n}$. We define $B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| \le R\}$ where $R > 0$, $x_0 \in \mathbb{R}^n$ (here $|x|$ is the maximum norm of $x \in \mathbb{R}^n$).

Let $r \in \mathbb{Z}_+$. We consider the Cauchy problem

$$D_t u(t,x) = \sum_{|m| \le r} a_m(t) D^m u(\mu_m(t), \beta_m(t)x + \gamma_m(t)) + f(t,x) \quad \text{in } \Theta, \tag{1.1}$$

$$u(t,x) = \phi(t,x) \quad \text{in } \Theta_0, \tag{1.2}$$

where $a_m, \mu_m : [0,T] \to \mathbb{R}$, $\mu_m(t) \le t$, $\beta_m = (\beta_m^1, \ldots, \beta_m^n)$, $\gamma_m = (\gamma_m^1, \ldots, \gamma_m^n) : [0,T] \to \mathbb{R}^n$, $m \in \mathbb{Z}_+^n$. We write $D^m u = D_x^m u = D_{x_1,\ldots,x_n}^{m_1,\ldots,m_n} u$ for $m \in \mathbb{Z}_+^n$, $D^0 u = u$ and $Du = D^1 u$ $(n = 1)$.

In this paper we consider infinite delay. The case of finite delay, i.e. $t - \tau \le \mu_m(t) \le t$ for some $\tau > 0$, or of no delay, i.e. $\mu_m(t) = t$, can be derived from our model by extending the initial function.

REMARK 1.1. One can obtain similar results for the equation

$$D_t u(t,x) = \sum_{j=1}^{l} \sum_{|m|\le r} a_{j,m}(t) D^m u\big(\mu_{j,m}(t), \beta_{j,m}(t)x + \gamma_{j,m}(t)\big) + f(t,x) \quad \text{in } \Theta,$$

where $a_{j,m}, \mu_{j,m} : [0,T] \to \mathbb{R}$, $\mu_{j,m}(t) \le t$, $\beta_{j,m} = (\beta^1_{j,m}, \ldots, \beta^n_{j,m})$, $\gamma_{j,m} = (\gamma^1_{j,m}, \ldots, \gamma^n_{j,m}) : [0,T] \to \mathbb{R}^n$, $m \in \mathbb{Z}^n_+$, $j = 1, \ldots, l$.

Unless otherwise stated, we will assume:

(H1) $|\beta^i_m(t)| \le 1$ in $[0,T]$, $i = 1, \ldots, n$, $m \in \mathbb{Z}^n_+$, $|m| \le r$.

(H2) There exists a continuous function $x_0(\cdot) : (-\infty, T] \to \mathbb{R}^n$ absolutely continuous in $[0,T]$ such that

$$\beta_m(t) x_0(t) + \gamma_m(t) = x_0(\mu_m(t)) \quad \text{for } t \in [0,T],\ m \in \mathbb{Z}^n_+,\ |m| \le r. \tag{1.3}$$

REMARK 1.2. Under assumption (H1) the condition (1.3) states that

$$|\beta_m(t)x + \gamma_m(t) - x_0(\mu_m(t))| \le |x - x_0(t)| \quad \text{for } (t,x) \in \Theta,\ m \in \mathbb{Z}^n_+,\ |m| \le r.$$

Although (1.3) seems to be rather strong ($x_0(t)$ does not depend on m) it is satisfied for many nontrivial problems. Indeed, (1.1), (1.2) is interesting even if $r = 1, n = 1$, $\gamma_0 = \gamma_1 \equiv 0$ ($x_0 \equiv 0$). In particular the simple equation $D_t u(t,x) = Du(t, \beta x)$, $|\beta| < 1$, is very difficult. This is due to the fact that the functional dependence is at the derivative. In the next few remarks we discuss the condition (1.3).

REMARK 1.3. If $\gamma_m \equiv 0$ for all $m \in \mathbb{Z}^n_+$, $|m| \le r$, then every function $\beta_m(\cdot)$ satisfying (H1) is admissible ($x_0(t) \equiv 0$).

REMARK 1.4. If $\beta_m(t) = (\beta^1_m, \ldots, \beta^n_m)$ and $\gamma_m(t) = (\gamma^1_m, \ldots, \gamma^n_m)$ are constant vector functions, $\mu_m(t) \equiv t$ for $|m| \le r$, then (H1), (H2) are satisfied only if $\beta^i_m = 1$, $\gamma^i_m = 0$ ($x^i_0(t) = x^i_0 \in \mathbb{R}$ is arbitrary) or $\beta^i_m < 1$, $\gamma^i_m \in \mathbb{R}^n$ ($x^i_0(t) = x^i_0 = \gamma^i_m/(\beta^i_m - 1)$), $i = 1, \ldots, n$, $x_0 = (x^1_0, \ldots, x^n_0)$.

REMARK 1.5. By Remarks 1.4 and 1.1 we see that our equation can have any number of elements with no functional dependence ($\beta^i_m \equiv 1$, $\gamma^i_m \equiv 0$ for $i = 1, \ldots, n$, $\mu_m(t) = t$).

REMARK 1.6. If $\mu_m(t) \equiv t$ for $m \in \mathbb{Z}^n_+$, then (H2) means that $x_0(t) = \gamma_m(t)/(\beta_m(t) - 1)$ does not depend on m and it is absolutely continuous in $[0,T]$ (we set $x_0(t) = x_0(0)$ for $t \le 0$).

For a delayed equation we can give a nontrivial example of (1.3).

EXAMPLE 1.7. $n = 1$, $\beta \equiv 1$, $\mu(t) = \mu t$, $\mu < 1$, $\gamma(t) = \sum_{j=1}^{\infty} \gamma_j t^j$, $x_0(t) = \sum_{j=1}^{\infty} \frac{\gamma_j}{\mu^j - 1} t^j$. In a similar way we can treat the case $|\beta| \le 1$.

2. Tonelli's method. Define

$$C^{0,\infty}(E) = \{u \in C(E) : D^m u \in C(E),\ m \in \mathbb{Z}_+^n\},$$
$$C^{1,\infty}(\Theta) = \{u \in C(\Theta) : D_t u,\ D^m u \in C(\Theta),\ m \in \mathbb{Z}_+^n\}.$$

In this section we assume that $\mu_m(t)$, $\beta_m(t)$, $\gamma_m(t)$ are bounded measurable, and $a_m(t)$ are Lebesgue integrable for $m \in \mathbb{Z}_+^n$, $|m| \le r$. Moreover, $f(\cdot, x)$ for fixed $x \in \mathbb{R}^n$ is integrable in $[0,T]$ and $\int_0^t f(s,\cdot)\,ds \in C^{0,\infty}(\Theta)$. Assume that $\phi \in C^{0,\infty}(\Theta_0)$ is locally bounded in x.

Definition 2.1. We say that u is a $C^{0,\infty}$-*solution* of (1.1), (1.2) if u is in $C^{0,\infty}(E)$, and

$$u(t,x) = \phi(0,x) + \int_0^t f(s,x)\,ds + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s) D^m u\big(\mu_m(s), \beta_m(s)x + \gamma_m(s)\big)\Big\}\,ds \quad \text{in } \Theta, \tag{2.1}$$

$$u = \phi \quad \text{in } \Theta_0. \tag{2.2}$$

Of course if there is no delay in (2.1) (i.e. $\mu_m(s) = s$ for each m) the condition (2.2) is superfluous.

We set $f(\cdot,x), a_m, \mu_m, \beta_m, \gamma_m$ equal to zero for $t < 0$, and $\phi(t,x) = \phi(0,x)$ for $t > 0$. Thus we can consider a formulation equivalent to (2.1), (2.2):

$$u(t,x) = \phi(t,x) + \int_0^t f(s,x)\,ds + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s) D^m u(\mu_m(s), \beta_m(s)x + \gamma_m(s))\Big\}\,ds \quad \text{in } E. \tag{2.3}$$

Example 2.2. ($n = 1$, $\tau = 0$, $a_0(t) \equiv 0$, $a_1(t) \equiv 1$). Consider the Cauchy problem

$$D_t u(t,x) = Du(\mu(t), x) \quad \text{in } [0,T]\times\mathbb{R}, \qquad u(0,x) = \phi(x) \quad \text{in } \mathbb{R},$$

where $\mu(t) = 0$ in $[0,c]$, $\mu(t) = t$ in $(c,T]$, $c \in (0,T)$ and $\phi \in C^\infty(\mathbb{R}^n)$.

The solution is $u(t,x) = \phi'(x)t + \phi(x)$ in $[0,c]$, $u(t,x) = c\phi'(t+x-c) + \phi(t+x-c)$ in $[c,T]$. Of course $u \in C^{0,\infty}(E)$ but $u \notin C^{1,\infty}(\Theta)$ if ϕ is not linear.

We will say u is a $C^{1,\infty}$-*solution* of (1.1), (1.2) if $u \in C^{1,\infty}(\Theta) \cap C^{0,\infty}(E)$ and (1.1), (1.2) are satisfied everywhere.

Remark 2.3. It is clear that every $C^{1,\infty}$-solution of (1.1), (1.2) is a $C^{0,\infty}$-solution. Moreover, if a_m, μ_m, β_m, γ_m for $m \in \mathbb{Z}_+^n$, $|m| \le r$, $f(\cdot,x)$ in $[0,T]$ and $\phi(\cdot,x)$ in $(-\infty,0]$ are continuous, then every $C^{0,\infty}$-solution of (1.1), (1.2) is a $C^{1,\infty}$-solution.

REMARK 2.4. If (H1), (H2) hold, we can reduce our problem to the case $\gamma_m(t) \equiv 0$, $|m| \le r$. We set $\bar{u}(t,x) = u(t, x+x_0(t))$. It is not difficult to check that u satisfies (2.1), (2.2) if and only if $\bar{u}$ does with $\bar{\gamma}_m(t) \equiv 0$, $|m| \le r$, $\bar{\phi}(t,x) = \phi(t, x+x_0(t))$ in Θ_0 and $\bar{x}_0(t) \equiv 0$ (see Remark 1.1). If we assume that $x_0(t)$ is continuously differentiable, then the equivalence also holds for $C^{1,\infty}$-solutions.

Let $g \in C^{0,\infty}(E)$. For $R > 0$ and $p = 0, 1, \ldots$ we define

$$L_R^p[g](t) = \sup\{|D^i g(s,x)| : s \le t,\ |x| \le R,\ i \in \mathbb{Z}_+^n,\ |i| \le p\}.$$

Let $\mu(t) = \max\{\mu_m(t) : m \in \mathbb{Z}_+^n,\ |m| \le r\}$ and $a(s) = \sum_{|m|\le r} |a_m(s)|$.

For every bounded and nondecreasing function $v : (-\infty, T] \to \mathbb{R}_+$ we define

$$A[v](t) = \int_0^t a(s) v(\mu(s))\, ds \quad \text{for } t \in (-\infty, T] \tag{2.4}$$

and its iterations

$$A^j[v](t) = \int_0^t a(s) A^{j-1}[v](\mu(s))\, ds \quad \text{for } t \in (-\infty, T],$$

$j = 1, 2, \ldots$, where $A^1[v] = A[v]$, $A^0[v] = v$. Of course, for $j = 2, 3, \ldots$,

$$A^j[v](t) = \int_0^t \int_0^{\mu(s_1)} \ldots \int_0^{\mu(s_{j-1})} a(s_1)a(s_2)\ldots a(s_j) v(\mu(s_j))\, ds_j \ldots ds_2 ds_1$$

(clearly $A^j[v](t) = 0$, $t \in (-\infty, 0]$, $j = 1, 2, \ldots$).

Define $F(t,x) = \int_0^t f(s,x)\, ds$, $(t,x) \in E$. Note that $F(t,x) = 0$ for $t \le 0$.

ASSUMPTION 2.5. *Suppose that $\phi \in C^{0,\infty}(\Theta_0)$, $F \in C^{0,\infty}(E)$, and for every $R > 0$ and every $k \in \mathbb{Z}_+$,*

$$\sum_{j=0}^{\infty} A^j[1](T) L_R^{jr+k}[\phi](0) < \infty, \tag{2.5}$$

$$\sum_{j=0}^{\infty} A^j[L_R^{jr+k}[F]](T) < \infty. \tag{2.6}$$

Notice that (2.5) (resp. (2.6)) does not imply that ϕ (resp. F) is analytic with respect to x.

REMARK 2.6. If μ is nondecreasing and for some $i \ge 1$ the iteration μ^i is not positive in $[0, T]$, then $A^j[v] \equiv 0$ for every v and $j \ge i+1$. In this case all $\phi \in C^{0,\infty}(\Theta_0)$ and $F \in C^{0,\infty}(E)$ satisfy (2.5), (2.6).

A simple example of μ such that Remark 2.6 can be applied is $\mu(t) = t-h$ for a given $h > 0$. We can also consider $\mu(t) = 0$ in $[0, h]$ and $\mu(t) = t - h$ in $(h, T]$ for a given $T > h > 0$. Another example is:

Example 2.7. Set $\mu(t) = 0$ in $[0, \varepsilon]$, $\mu(t) = \alpha t$ in $(\varepsilon, T]$ for $0 < \varepsilon < T$ and $0 < \alpha < 1$. It is easy to show that $\mu^i(t) = 0$ in $[0, \varepsilon\alpha^{1-i}]$, $\mu^i(t) = \alpha^i t$ in $(\varepsilon\alpha^{1-i}, T]$ for $i < 1 + \ln(\varepsilon/T)/\ln\alpha = \varepsilon_0$ and $\mu^i(t) \equiv 0$ for $i \geq \varepsilon_0$.

In order to see that the condition $A^j[v] \equiv 0$ for large j is not necessary to consider nonanalytic data we refer the reader to Section 4 (see Remark 4.10).

Proposition 2.8. *A sufficient condition for* (2.6) *to hold is*

$$\sum_{j=0}^{\infty} A^j[1](T)[L_R^{jr+k}[F]](T) < \infty \tag{2.7}$$

for all $R > 0$ and $k \in \mathbb{Z}_+$.

Proof. This follows easily from the monotonicity of A. ∎

Proposition 2.9. *Sufficient conditions for* (2.5), (2.6), *respectively, are:*

$$\Big[\int_0^T a(s)\,ds\Big] \limsup_{j\to\infty} \sqrt[j]{L_R^{jr}[\phi](0)/j!} < 1, \tag{2.8}$$

$$\Big[\int_0^T a(s)\,ds\Big] \limsup_{j\to\infty} \sqrt[j]{L_R^{jr}[F](T)/j!} < 1, \tag{2.9}$$

for every $R > 0$.

Proof. Suppose that (2.8) and (2.9) hold. First we will show that

$$\Big[\int_0^T a(s)\,ds\Big] \limsup_{j\to\infty} \sqrt[j]{L_R^{jr+k}[\phi](0)/j!} < 1, \tag{2.10}$$

$$\Big[\int_0^T a(s)\,ds\Big] \limsup_{j\to\infty} \sqrt[j]{L_R^{jr+k}[F](T)/j!} < 1, \tag{2.11}$$

for every $R > 0$ and every $k \in \mathbb{Z}_+$.

If $r = 0$, then both limits are zero for every k. If $r > 0$, then our statement follows from

$$\left(\frac{L^{jr+k}}{j!}\right)^{1/j} \leq \left(\frac{L^{(j+k)r}}{j!}\right)^{1/j} = \left[\left(\frac{L^{(j+k)r}}{(j+k)!}\right)^{\frac{1}{j+k}}\right]^{\frac{j+k}{j}} \left(\frac{(j+k)!}{j!}\right)^{1/j},$$

where L^m denotes $L_R^m[\phi](0)$ or $L_R^m[F](T)$ for $m \in \mathbb{N}$.

Let $A_0[v]$ be defined by (2.4) with $\mu(t) = t$. Assume that (2.10), (2.11) hold. The conditions (2.5) follow from the inequality $A^j[1](t) \leq A_0^j[1](t)$ and from the fact that $A_0^j[1](t) = [A_0[1](t)]^j/j!$, $A_0[1](t) = A[1](t) = \int_0^t a(s)\,ds$. In the proof of (2.6) we also use the inequality

$$A^j[L_R^{jr+k}[F]](T) \le A^j[1](T)[L_R^{jr+k}[F]](T),$$

which gives (2.7) and yields (2.6) by Proposition 2.8. ∎

Proposition 2.9 is particularly useful if $\mu(t) = t$. In the case of nontrivial delay it is better to use (2.5), (2.6) since these conditions give more general existence results than (2.8), (2.9) (see examples at the end of the paper).

By the Ascoli–Arzelà lemma and a diagonal argument we can prove

THEOREM 2.10. *Suppose that $g_l \in C(\mathbb{R}^n)$, $l = 1, 2, \ldots$, are equicontinuous and uniformly bounded on $B(R) = \{x \in \mathbb{R}^n : |x| \le R\}$ for every $R > 0$. Then g_l has a locally uniformly convergent subsequence.*

THEOREM 2.11. *Suppose that Assumption* 2.5 *holds. Then there exists at least one $C^{0,\infty}$-solution of* (1.1), (1.2).

Proof. In view of Remark 2.4 we can assume that $\gamma_m \equiv 0$.

Let $\alpha > 0$. Consider

$$(2.12)\qquad u(t,x) = \phi(t,x) + F(t,x) + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s) D^m u(\mu_m(s) - \alpha, \beta_m(s)x) \Big\}\,ds \quad \text{in } E.$$

We solve (2.12) by the method of steps.

Let $N \in \mathbb{N}$ with $(N-1)\alpha < T \le N\alpha$ and $t_j = j\alpha$, $0 \le j < N$, $t_N = T$. Set $E_j = E \cap \{(t,x) : t \le t_j\}$ for $j = 0, 1, \ldots, N$ ($E_0 = \Theta_0$ and $E_N = E$).

We define $u_j : E \to \mathbb{R}$, $j = 0, 1, \ldots, N$, by induction: $u_0 = \phi$ in E,

$$(2.13)\qquad u_j(t,x) = u_0(t,x) + F(t,x) + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s) D^m u_{j-1}(\mu_m(s) - \alpha, \beta_m(s)x) \Big\}\,ds \quad \text{in } E_j$$

and $u_j(t,x) = u(t_j,x)$ in $E \backslash E_j$ for $j = 1, \ldots, N$. It is clear that $u_j = u_{j-1}$ in E_{j-1} and $u_\alpha = u_N$ solves (2.12). Let $k \in \mathbb{Z}_+$. Applying D^i, $i \in \mathbb{Z}_+^n$, $|i| = k$, to both sides in (2.13) we obtain

$$(2.14)\qquad D^i u_j(t,x) = D^i u_0(t,x) + D^i F(t,x) + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s)[\beta_m(s)]^i D^{m+i} u_{j-1}(\mu_m(s) - \alpha, \beta_m(s)x) \Big\}\,ds \quad \text{in } E_j.$$

Since $|[\beta_m(s)]^i| \le 1$, we write

$$|D^i u_j(t,x)| \le |D^i u_0(0,x)| + |D^i F(t,x)| + \int_0^t \Big\{ \sum_{|m|\le r} |a_m(s)|\,|D^{m+i} u_{j-1}(\mu_m(s)-\alpha, \beta_m(s)x)| \Big\}\, ds \quad \text{in } E_j.$$

Fix $R > 0$. Then we define $L_j^p(t) = L_R^p[u_j](t)$, $F^p(t) = L_R^p[F](t)$ and $L_0^p = L_0^p(0) = L_0^p[\phi](0)$, $p \in \mathbb{Z}_+$. Of course $L_j^p(t)$ is nondecreasing in t, j, p and $F^p(t)$ is nondecreasing in t, p. By the standard argument,

$$L_j^k(t) \le L_0^k + F^k(t) + \int_0^t \Big\{ \sum_{|m|\le r} |a_m(s)| L_{j-1}^{|m|+k}(\mu_m(s)-\alpha) \Big\}\, ds \quad \text{in } [0, t_j],$$

and since $\mu_m(s) \le \mu(s)$,

$$L_j^k(t) \le L_0^k + F^k(t) + \int_0^t a(s) L_{j-1}^{r+k}(\mu(s)-\alpha)\, ds \quad \text{in } [0, t_j].$$

Since $L_j^p(t) = L_j^p(t_j)$ for $t \in [t_j, T]$ we obtain

$$L_j^k(t) \le L_0^k + F^k(t) + \int_0^t a(s) L_{j-1}^{r+k}(\mu(s))\, ds \quad \text{in } [0, T].$$

Hence

$$L_j^k(t) \le L_0^k + F^k(t) + A[L_{j-1}^{r+k}](t) \quad \text{in } [0, T].$$

for $j = 1, \ldots, N$. This gives by induction

$$L_N^k(t) \le \sum_{j=0}^N A^j[1](t) L_0^{jr+k} + \sum_{j=0}^N A^j[F^{jr+k}](t) \le \mathcal{L}_R^k(t),$$

where

$$\mathcal{L}_R^k(t) = \sum_{j=0}^\infty A^j[1](t) L_0^{jr+k} + \sum_{j=0}^\infty A^j[F^{jr+k}](t). \tag{2.15}$$

Since $u_\alpha = u_N$ belongs to $C^{0,\infty}(E)$ and satisfies (2.12), we can write

$$D^i u(t,x) = D^i \phi(t,x) + \int_0^t D^i f(s,x)\, ds + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s)[\beta_m(s)]^i D^{m+i} u(\mu_m(s)-\alpha, \beta_m(s)x) \Big\}\, ds \quad \text{in } E \tag{2.16}$$

for $i \in \mathbb{Z}_+^n$. Moreover, all derivatives $D^i u_\alpha$, $|i| = k$, are bounded by $\mathcal{L}_R^k(T)$

for $|x| \le R$. By (2.16) for $|i| = k$ and $|x| \le R$ we obtain

$$|D^i u_\alpha(t,x) - D^i u_\alpha(\bar t,x)| \le$$

$$\Big|\int_{\bar t}^{t} \Big\{ \sum_{|m|\le r} |a_m(s)|\,|D^{m+i} u_\alpha(\mu_m(s)-\alpha, \beta_m(s)x)| \Big\}\,ds\Big| \le L_R^{k+r}(T)\Big|\int_{\bar t}^{t} |a(s)|\,ds\Big|.$$

Now we set $\alpha = \alpha_l$, $l = 1, 2, \ldots$, $\alpha_l \to 0$ as $l \to \infty$. By Theorem 2.10 we find that u_{α_l} converges locally uniformly (taking a subsequence if necessary) with all its derivatives. Next letting $\alpha = \alpha_l \to 0$ in (2.12) we get (2.3) for the limit. ∎

Remark 2.12. If u is a $C^{0,\infty}$-solution of (1.1), (1.2) obtained by Tonelli's method, then $L_R^k[u](t) \le \mathcal{L}_R^k(t)$ where $\mathcal{L}_R^k(t)$ is given by (2.15).

In view of Remark 2.3 we can formulate

Theorem 2.13. *Assume that* $a_m, \mu_m : (0,T) \to \mathbb{R}$, $\beta_m, \gamma_m : (0,T) \to \mathbb{R}^n$, $m \in \mathbb{Z}_+^n$, $|m| \le r$, *are continuous. Suppose that the hypothesis of Theorem* 2.11 *holds with* $x_0(\cdot)$ *continuously differentiable in* $[0,T]$. *Then there exists at least one* $C^{1,\infty}$*-solution of* (1.1), (1.2).

3. Uniqueness of solutions

Theorem 3.1. *Suppose that* u *is a* $C^{0,\infty}$*-solution of* (1.1), (1.2). *Let* (2.5), (2.6) *hold for some* $k \in \mathbb{Z}^+$ *and*

$$\lim_{p\to\infty} A^p[L_R^{pr+k}[u]](T) = 0 \quad \textit{for every } R > 0. \tag{3.1}$$

Then

$$L_R^k[u](T) \le \sum_{j=0}^{\infty} A^j[1](T) L_R^{jr+k}[\phi](0) + \sum_{j=0}^{\infty} A^j[L_R^{jr+k}[F]](T).$$

Proof. Let $u = \phi$ in Θ_0 and $\gamma_m \equiv 0$ (see Remark 2.4). Applying D^i, $i \in \mathbb{Z}_+^n$, to both sides of (2.3) ($\gamma_m \equiv 0$) we get

$$\begin{aligned} D^i u(t,x) &= D^i\phi(t,x) + D^i F(t,x) \\ &\quad + \int_0^t \Big\{ \sum_{|m|\le r} a_m(s)[\beta_m(s)]^i D^{m+i} u(\mu_m(s), \beta_m(s)x) \Big\}\,ds. \end{aligned}$$

This gives

$$\begin{aligned} |D^i u(t,x)| &\le |D^i\phi(t,x)| + |D^i F(t,x)| \\ &\quad + \int_0^t \Big\{ \sum_{|m|\le r} |a_m(s)|\,|D^{m+i} u(\mu_m(s), \beta_m(s)x)| \Big\}\,ds \quad \text{in } \Theta. \end{aligned}$$

Taking the supremum over the past for $x \in B(x_0, R)$, $R > 0$, $|i| \le k$, we obtain

$$L_R^k[u](t) \le L_R^k[\phi](0) + L_R^k[F](T) + \int_0^t a(s) L_R^{r+k}[u](\mu(s))\,ds,$$

and consequently

$$L_R^k[u](t) \le L_R^k[\phi](0) + L_R^k[F](T) + A[L_R^{r+k}[u]](t).$$

Iterating p times we obtain

$$L_R^k[u](t) \le \sum_{j=0}^{p-1} A^j[1](T) L_R^{jr+k}[\phi](0) + \sum_{j=0}^{p-1} A^j[L_R^{jr+k}[F]](T) + A^p[L_R^{rp+k}[u]](t).$$

By letting $p \to \infty$ we complete the proof. ∎

Theorem 3.2. *Suppose that* $\phi \in C^{0,\infty}(\Theta_0)$ *and* $F \in C^{0,\infty}(E)$. *Then there exists at most one* $C^{0,\infty}$*-solution of* (1.1), (1.2) *satisfying* (3.1) *with* $k = 0$.

Proposition 3.3. *If* u *is a* $C^{0,\infty}$*-solution of* (1.1), (1.2) *obtained by Tonelli's method under Assumption* 2.5, *then* u *satisfies* (3.1) *for every* k.

Proof. By Remark 2.12 we obtain

$$\begin{aligned} A^p[L^{pr+k}[u]](t) &\le A^p\Big[\sum_{j=0}^{\infty} L_0^{jr+pr+k} A^j[1]\Big](t) + A^p\Big[\sum_{j=0}^{\infty} A^j[F^{jr+pr+k}]\Big](t) \\ &= \sum_{j=0}^{\infty} L_0^{(j+p)r+k} A^{j+p}[1](t) + \sum_{j=0}^{\infty} A^{j+p}[F^{(j+p)r+k}](t) \\ &= \sum_{j=p}^{\infty} L_0^{jr+k} A^j[1](t) + \sum_{j=p}^{\infty} A^j[F^{jr+k}](t). \end{aligned}$$

By applying Assumption 2.5 we complete the proof. ∎

Remark 3.4. Suppose that $L_R^k[\phi](0) = L^k[\phi](0)$ and $L_R^k[F](t) = L^k[F](t)$ for every $k \in \mathbb{Z}_+$ and $t \in [0, T]$ (i.e. ϕ, F have spatial derivatives bounded in Θ_0 and in $(-\infty, t] \times \mathbb{R}^n$, respectively). Then we can drop assumption (H2). Theorems 2.11 and 3.2 give existence and uniqueness results in the class of $C^{0,\infty}$ functions with spatial derivatives bounded in $(-\infty, t] \times \mathbb{R}^n$.

4. Examples. Let $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ and $p > 0$ (i.e. $p_i > 0$ for $i = 1, \ldots, n$). Define

$$\mathbb{X}_{\mathrm{loc}}^{0,\infty}(E, p) = \big\{u \in C^{0,\infty}(E) : \forall_{R>0}\, \exists_{M_R \ge 0}\, \forall_{m \in \mathbb{Z}_+^n}\ |D^m u(t,x)| \le p^m M_R \text{ in } (-\infty, T] \times B(R)\big\},$$

$$\mathbb{X}_{\mathrm{loc}}^{0,\infty}(\Theta_0, p) = \big\{u \in C^{0,\infty}(\Theta_0) : \forall_{R>0}\, \exists_{M_R \ge 0}\, \forall_{m \in \mathbb{Z}_+^n}\ |D^m u(t,x)| \le p^m M_R \text{ in } (-\infty, 0] \times B(R)\big\},$$

where $B(R) = \{x \in \mathbb{R}^n : |x| \le R\}$ and $p^m = p_1^{m_1} \ldots p_n^{m_n}$.

REMARK 4.1. It is easy to verify that $u \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ if and only if $u(t,\cdot)$ is an entire real function for each t and $\exists_{M\geq 0}\ \forall_{m\in\mathbb{Z}^n_+}\ |D^m u(t,0)| \leq p^m M$. Moreover, if $u \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$, then $|D^m u(t,x)| \leq p^m M e^{\sum_{i=1}^n p_i|x_i|}$ for $x \in \mathbb{R}^n$, $j = 0,1,\ldots$, uniformly in t.

It is not difficult to show that if $u, v \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$, then $u+v \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ and $uv \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,2p)$.

EXAMPLE 4.2 ($n = 1$). Let $P(t,x)$, $Q(t,x)$ be two polynomials with respect to x continuous in t and $\alpha(t), \beta(t)$ be continuous scalar functions. Every function of the type $e^{\alpha(t)x}[P(t,x)\cos\beta(t)x + Q(t,x)\sin\beta(t)x]$ or a finite sum of such functions belongs to $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$.

EXAMPLE 4.3. The function $\phi(t,x) = e^{\sum_{i=1}^n x_i^2}$ does not belong to $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ for any p.

We define

$$\mathbb{X}^{0,\infty}(E,p) = \{u \in C^{0,\infty}(E) : \exists_{M\geq 0}\ \forall_{m\in\mathbb{Z}^n_+}\ |D^m u(t,x)| \leq p^m M \text{ in } E\},$$
$$\mathbb{X}^{0,\infty}(\Theta_0,p) = \{u \in C^{0,\infty}(\Theta_0) : \exists_{M\geq 0}\ \forall_{m\in\mathbb{Z}^n_+}\ |D^m u(t,x)| \leq p^m M \text{ in } \Theta_0\}.$$

THEOREM 4.4. *Suppose that* (H1) *and* (H2) *hold, and* $\phi \in \mathbb{X}^{0,\infty}_{\text{loc}}(\Theta_0,p)$, $F \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$. *Then there exists a unique solution of* (1.1), (1.2) *in* $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$. *Suppose that* (H1) *holds,* $\phi \in \mathbb{X}^{0,\infty}(\Theta_0,p)$, $F \in \mathbb{X}^{0,\infty}(E,p)$. *Then there exists a unique solution of* (1.1), (1.2) *in* $\mathbb{X}^{0,\infty}(E,p)$.

Proof. For $\phi \in \mathbb{X}^{0,\infty}_{\text{loc}}(\Theta_0,p)$ and $F \in \mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ we have $L^k_R[\phi](0) \leq M_R p^k$ and $L^k_R[F](T) \leq M_R p^k$. This gives (2.5), (2.6), and by Theorems 2.11 and 3.2 the existence of a unique solution u in the class of $C^{0,\infty}$ functions satisfying (3.1) with $k = 0$. By Remark 2.12 we can verify that u is in $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$. Since all elements of $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ satisfy (3.1), the proof of the first part is complete. The second part can be proved similarly (see Remark 3.4). ■

REMARK 4.5. Since $\mathbb{X}^{0,\infty}(E,p)$ is in fact a Bernstein class of real analytic functions considered in [RW], Theorem 4.4 (its second part) extends the main result of [RW] to the class of functional differential equations. In this case we can follow the contraction argument used in [RW] instead of Tonelli's method. We use the fact that the space $\mathbb{X}^{0,\infty}(E,p)$ equipped with the supremum norm is complete. It seems to be difficult to adopt this reasoning to the space $\mathbb{X}^{0,\infty}_{\text{loc}}(E,p)$ or all the more to the function space defined by the condition (2.6).

The following examples show that our main existence result in Theorem 2.11 is not limited to the case given in Theorem 4.4.

EXAMPLE 4.6. Consider the problem ($n=1$, $r=1$, $T>0$)

$$D_t u(t,x) = a(t) Du(\mu(t), \beta(t)x+\gamma(t)) \quad \text{in } \Theta, \qquad u(0,x) = e^{x^2}, \quad x \in \mathbb{R}. \tag{4.1}$$

For simplicity we set $f \equiv 0$.

Based on the formula

$$\phi^{(j+1)}(x) = 2x\phi^{(j)}(x) + 2j\phi^{(j-1)}(x) \tag{4.2}$$

for $\phi(x) = e^{x^2}$ we see that $L_R^j[\phi](0) = \phi^{(j)}(R)$ and

$$\lim_{j\to\infty} \sqrt[j]{L_R^j[\phi](0)/j!} = 0.$$

This in view of Proposition 2.9 gives a global existence result.

EXAMPLE 4.7. Consider the problem ($n=1$, $r=2$)

$$D_t u(t,x) = a(t) D^2 u(\mu(t), \beta(t)x+\gamma(t)) \text{ in } \Theta, \quad u(0,x) = e^{x^2}, \ x \in \mathbb{R}. \tag{4.3}$$

Setting $2j+1$, $2j$, $2j-1$ in (4.2) we get

$$\phi^{(2j+2)}(x) = (4x^2+8j+2)\phi^{(2j)}(x) - 8j(2j-1)\phi^{(2j-2)}(x)$$

and

$$a_j = \frac{4x^2+8j+2}{j+1} - \frac{16j-8}{j+1}\,\frac{1}{a_{j-1}}$$

where $a_j = \frac{1}{j+1}\frac{\phi^{2j+2}(x)}{\phi^{2j}(x)}$. This gives

$$\lim_{j\to\infty} \sqrt[j]{L_R^{2j}[\phi](0)/j!} = 4. \tag{4.4}$$

Hence in view of Proposition 2.9 a solution exists in the interval $[0,T]$ where T is such that $\int_0^T a(s)\,ds < 1/4$. This result corresponds to a well known property of the heat equation (nonfunctional case $\mu(t) \equiv t$, $\beta(t) \equiv 1$, $\gamma(t) \equiv 0$) which states that solutions bounded by Me^{cx^2} exist only in $[0,1/4)$ (see [F]).

The next example shows that conditions (2.5), (2.6) are more general than (2.8), (2.9).

EXAMPLE 4.8. Consider the problem ($n=1$, $r=2$, $T=1$)

$$D_t u(t,x) = D^2 u(t^\alpha, \beta(t)x) \quad \text{in } \Theta, \quad u(0,x) = e^{x^2}, \quad x \in \mathbb{R}, \tag{4.5}$$

where $\alpha > 1$. We will show that (4.5) has a solution in $[0,1]$. Indeed, in this case

$$A^j[1](t) = \frac{1}{(\alpha+1)\ldots(\alpha^{j-1}+\cdots+1)} t^{\alpha^{j-1}+\cdots+1}$$

and

$$\lim_{j\to\infty} \sqrt[j]{A^j[1](t)j!} = \lim_{j\to\infty} (1-\alpha)\frac{j+1}{1-\alpha^{j+1}} t^{\alpha^j} = 0,$$

for $t \in [0,1]$. Thus in view of (4.4) and Cauchy's root test the condition (2.5) holds.

EXAMPLE 4.9. Consider the problem ($n = 1$, $r = 2$, $T > 0$)

$$D_t u(t,x) = D^2 u(\alpha t, \beta(t)x) \quad \text{in } \Theta, \quad u(0,x) = e^{x^2}, \quad x \in \mathbb{R}, \tag{4.6}$$

where $0 < \alpha < 1$. We have $A^j[1](t) = \alpha^{j(j-1)/2} t^j / j!$. Since

$$\lim_{j\to\infty} \sqrt[j]{A^j[1](t) j!} = \lim_{j\to\infty} t\alpha^{(j-1)/2} = 0,$$

by the same argument as in Example 4.8 the solution exists in $[0,T]$ for all $T > 0$.

REMARK 4.10 (nonanalytic solutions). Consider the problem of Example 4.9 with a new initial function ϕ. Suppose that ϕ is in the Gevrey class of order $\sigma > 1$ (see [G]). Then for every $R > 0$, $L_R^j[\phi](0) \le M_R^{1+j}(j!)^\sigma$ where $M_R \ge 0$. It is not difficult to verify that (2.5) holds for every $R > 0$ and $k \in \mathbb{Z}_+$.

5. Remarks. If we drop the assumption $\mu(t) \le t$ the uniqueness may fail.

EXAMPLE 5.1 (many solutions in the case of an advanced argument). Consider the problem

$$D_t u = 2Du(\sqrt{t}, x), \quad (t,x) \in [0,T] \times \mathbb{R}, \quad u(0,x) = 0, \quad x \in \mathbb{R}. \tag{5.1}$$

A function $u(t,x) = Ct^2 e^x$ is a solution of (5.1) for every $C \in \mathbb{R}$.

It is clear that Tonelli's method works for C^∞ functions in x. Indeed, it follows from (2.13) that the initial function has to be infinitely differentiable. It is not surprising that this method may not work for a function which is smooth but not analytic in x (in a neighborhood of zero). This can be easily seen even in the nonfunctional case.

EXAMPLE 5.2. Consider the problem ($r = 1$, $n = 1$),

$$D_t u(t,x) = Du(t,x) \quad \text{in } \Theta, \quad u(0,x) = \phi(x) \quad \text{in } \mathbb{R} = \Theta_0. \tag{5.2}$$

Let $\phi \in C^\infty(\mathbb{R})$ be such that $D^k\phi(0) = 0$ for all $k \in \mathbb{Z}_+$ and $\phi > 0$ elsewhere (an example is $\phi(x) = e^{-1/x^2}$, $x \ne 0$, $\phi(0) = 0$). It is not difficult to check that after applying our method we get

$$D^k u(t,x) = D^k\phi(x) + \int_0^t D^{k+1} u(s-\alpha, x)\, ds \quad \text{in } \Theta.$$

Moreover, for all α the solution u_α has the property $u_\alpha(t,0) = 0$ (we see this step by step). On the other hand we know, by the method of characteristics, that the unique solution of (5.2) is given by $\tilde{u}(t,x) = \phi(t+x)$. Since $\tilde{u}(t,0) = \phi(t) > 0$ for $t > 0$, none of the subsequences of u_α converges to $\tilde{u}$.

Remark 5.3. All the results in this paper can be extended to strongly coupled systems.

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