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Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions

by S. Sivasubramanian (Tindivanam), R. Sivakumar (Tindivanam), S. Kanas (Rzeszów), and Seong-A Kim (Gyeongju)

Abstract. Let σ denote the class of bi-univalent functions f, that is, both $f(z) = z + a_2z^2 + \cdots$ and its inverse f^{-1} are analytic and univalent on the unit disk. We consider the classes of strongly bi-close-to-convex functions of order α and of bi-close-to-convex functions of order β, which turn out to be subclasses of σ. We obtain upper bounds for $|a_2|$ and $|a_3|$ for those classes. Moreover, we verify Brannan and Clunie's conjecture $|a_2| \le \sqrt{2}$ for some of our classes. In addition, we obtain the Fekete–Szegö relation for these classes.

1. Introduction and motivations. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Further we denote by $\mathcal{S}$ the subclass of functions in $\mathcal{A}$ which are univalent on $\mathbb{U}$, and for $0 \le \beta < 1$, let $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ be the subclasses of $\mathcal{S}$ consisting of *starlike* functions of order β and *convex* functions of order β, respectively. Their analytic descriptions are

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \ (z \in \mathbb{U}) \right\}, \tag{1.2}$$

$$\mathcal{C}(\beta) = \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \ (z \in \mathbb{U}) \right\}. \tag{1.3}$$

The class $\mathcal{C}(0) \equiv \mathcal{C}$ is the class of *convex* univalent functions.

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It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$(f^{-1} \circ f)(z) = z \quad (z \in \mathbb{U}) \tag{1.4}$$

and

$$(f \circ f^{-1})(w) = w \quad (|w| < r_0(f);\ r_0(f) \geq 1/4). \tag{1.5}$$

The inverse function may have an analytic continuation to $\mathbb{U}$, with

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.6}$$

Lewin [L] investigated the class of functions $f \in \mathcal{A}$ such that both f and f^{-1} are normalized univalent functions on $\mathbb{U}$. A function in this class was called *bi-univalent* and the class was denoted by σ. Lewin [L] also showed that $|a_2| \leq 1.51$. Further, Brannan and Clunie [BC] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [N] obtained an exact upper bound $|a_2| = 4/3$ for the subclass σ_1 of σ that consists of all functions that are bi-univalent and their ranges contain the unit disk $\mathbb{U}$. However, the exact upper bound of $|a_2|$ or bounds for $|a_n|$ $(n > 2)$ for functions in the class σ are not known.

Examples of bi-univalent functions are

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\frac{1+z}{1-z}, \quad -\log(1-z)$$

(see also Srivastava et al. [SMG]). However the familiar Koebe function $z/(1-z)^2$ and its rotations are not members of σ.

Brannan and Taha [BT] introduced certain subclasses of σ, similar to the familiar subclasses $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$. They defined that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\sigma[\alpha]$ of *strongly bi-starlike functions of order* α $(0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$f \in \sigma \quad \text{and} \quad |\arg(zf'(z)/f(z))| < \alpha\pi/2 \quad (z \in \mathbb{U};\ 0 < \alpha \leq 1), \tag{1.7}$$

and

$$|\arg(wg'(w)/g(w))| < \alpha\pi/2 \quad (w \in \mathbb{U};\ 0 < \alpha \leq 1), \tag{1.8}$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$.

The classes $\mathcal{S}^*_\sigma(\beta)$ and $\mathcal{C}_\sigma(\beta)$ of *bi-starlike functions of order* β and *bi-convex functions of order* β, corresponding to $\mathcal{S}^*(\beta)$ and $\mathcal{C}(\beta)$ defined by (1.2) and (1.3), were also introduced analogously. Brannan and Taha found non-sharp estimates on $|a_2|$ and $|a_3|$ for functions in $\mathcal{S}^*_\sigma(\beta)$ and in $\mathcal{C}_\sigma(\beta)$ (for details see [BT]). Following Brannan and Taha [BT], many researchers (see [AL$^+$, FA, GG, HW, SMG, XSL, XGS, XXS]) have recently introduced and investigated several interesting subclasses of σ and found non-sharp estimates on the first two Taylor–Maclaurin coefficients.

For $0 \leq \alpha \leq 1$, let $\mathcal{K}_\alpha$ denote the family of analytic functions f of the form (1.1) with $f'(z) \neq 0$ on $\mathbb{U}$ for which there exists a convex function ϕ

such that

$$|\arg(f'(z)/\phi'(z))| < \alpha\pi/2. \tag{1.9}$$

These classes were introduced by Kaplan [Kap] and later studied by Reade [R]. In particular, $\mathcal{K}_0$ is the family of *convex* univalent functions and $\mathcal{K}_1$ is the family of *close-to-convex* functions. Moreover, $\mathcal{K}_{\alpha 1}$ is a proper subclass of $\mathcal{K}_{\alpha 2}$ whenever $\alpha_1 < \alpha_2$. Similarly, the class of *close-to-convex functions of order* β was introduced by the analytic condition [R]

$$\Re(f'(z)/\phi'(z)) > \beta. \tag{1.10}$$

Motivated by the works of Brannan and Taha [BT] and Reade [R], we introduce the following classes:

- $\mathcal{K}_\sigma$: *bi-close-to-convex* functions;
- $\mathcal{K}_\sigma[\alpha]$: *strongly bi-close-to-convex* functions of order α;
- $\mathcal{K}_\sigma(\beta)$: *bi-close-to-convex* functions of order β,

which are analogous to the classes of strongly bi-convex functions of order α and of strongly bi-starlike functions of order α [BT]. Also, we find estimates for $|a_2|$ and $|a_3|$ for functions in these new subclasses. Further we verify Brannan and Clunie's [BC] conjecture $|a_2| \le \sqrt{2}$ for some of our subclasses. In addition, we obtain the Fekete–Szegö inequality for those classes.

Denote also by $\mathcal{P}$ the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ such that $\Re(p(z)) > 0$ in $\mathbb{U}$.

To derive our main result we use the following well known lemmas.

LEMMA 1.1 ([D, MM]). *If $p \in \mathcal{P}$, then $|p_k| \le 2$ for each $k \ge 1$, and*

$$|p_2 - p_1^2/2| \le 2 - |p_1|^2/2. \tag{1.11}$$

LEMMA 1.2 ([LZ1, LZ2]). *If $p \in \mathcal{P}$, then $p_2 = p_1^2 + x(4 - p_1^2)$, and*

$$4p_3 = p_1^3 + 2xp_1(4 - p_1^2) - x^2 p_1(4 - p_1^2) + 2\zeta(1 - |x|^2)(4 - p_1^2) \tag{1.12}$$

for some x, ζ such that $|x|, |\zeta| \le 1$.

LEMMA 1.3 ([Kan]). *If $\phi \in \mathcal{C}$, then for $\lambda \in \mathbb{R}$,*

$$|c_3 - \lambda c_2^2| \le \begin{cases} 1 - \lambda & \text{for } \lambda < 2/3, \\ 1 & \text{for } 2/3 \le \lambda \le 4/3, \\ \lambda - 1 & \text{for } \lambda > 4/3. \end{cases}$$

2. Coefficient bounds for $\mathcal{K}_\sigma[\alpha]$. In the present section, we first find bounds for the first two coefficients of the functions in the class of strongly bi-close-to-convex of order α. Let us begin with the definitions.

DEFINITION 2.1. Let $\mathcal{A}_\sigma(R)$ denote the class of functions of the form (1.1), defined on $|z| < R$, for which the inverse function has an *analytic* continuation to $|z| < R$ with series expansion

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

We call the functions in $\mathcal{A}_\sigma(R)$ *bi-analytic* in $|z| < R$.

When $R = 1$, it will be convenient to omit the reference to the circular domain in Definition 2.1. Therefore, a bi-analytic function will mean a function which is bi-analytic on $\mathbb{U}$. We abbreviate $\mathcal{A}_\sigma(1) = \mathcal{A}_\sigma$.

We note that $\mathcal{A}_\sigma$ is a proper subclass of $\mathcal{A}$.

DEFINITION 2.2. Let $0 \le \alpha \le 1$. A function $f \in \mathcal{A}_\sigma$, given by (1.1), is said to be *strongly bi-close-to-convex* of order α if there exist bi-convex functions ϕ and ψ such that

$$|\arg(f'(z)/\phi'(z))| < \alpha\pi/2 \quad (z \in \mathbb{U}), \tag{2.1}$$

$$|\arg(g'(w)/\psi'(w))| < \alpha\pi/2 \quad (w \in \mathbb{U}). \tag{2.2}$$

Here, g is the analytic continuation of f^{-1} to $\mathbb{U}$. We denote the class of strongly bi-close-to-convex functions of order α by $\mathcal{K}_\sigma[\alpha]$.

Observe that if f is given by (1.1), then

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots , \tag{2.3}$$

and if

$$\phi(z) = z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots , \tag{2.4}$$

then

$$\psi(w) = w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2 c_3 + c_4)w^4 + \cdots . \tag{2.5}$$

Here $\phi^{-1}(w) = \psi(w)$.

We observe that $\mathcal{K}_\sigma[\alpha_1] \subsetneq \mathcal{K}_\sigma[\alpha_2]$ for $\alpha_1 < \alpha_2$. Also, $\mathcal{K}_\sigma[1] \equiv \mathcal{K}_\sigma$ will be called the class of *bi-close-to-convex* functions. Finally, $\mathcal{K}_\sigma[0] \equiv \mathcal{C}_\sigma$ is the class of *bi-convex* functions [BT].

Kaplan [Kap] mentioned that (2.1) and (2.2) might be replaced by

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -\pi\alpha, \quad z = re^{i\theta},$$

and

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{wg''(w)}{g'(w)}\right\} d\theta > -\pi\alpha, \quad w = re^{i\theta}.$$

Here, $\theta_1 < \theta_2 < \theta_1 + 2\pi$ and $0 \le r < 1$.

Now, we first prove the following theorem.

PROPOSITION 2.1. *If f given by* (1.1) *is in the class $\mathcal{K}_\sigma[\alpha]$ where $0 \le \alpha \le 1$, then $f(z)$ is bi-univalent.*

Proof. For $\alpha = 1$, the statement follows from the work of Kaplan [Kap] for close-to-convex functions. When $0 \le \alpha < 1$, we have $\mathcal{K}_\sigma[\alpha] \subsetneq \mathcal{K}_\sigma[1]$, which completes the proof. ∎

By the above proposition, $\mathcal{K}_\sigma[\alpha]$ is a subclass of σ. By a particular choice of $\phi(z)$ in the statement of Definition 2.2, one can obtain the following other subclasses of σ:

- $|\arg(1-z)^2 f'(z)| < \alpha\pi/2$ and $|\arg(1-w)^2 g'(w)| < \alpha\pi/2$;
- $|\arg f'(z)| < \alpha\pi/2$ and $|\arg g'(w)| < \alpha\pi/2$ (studied by Srivastava et al. [SMG]).

THEOREM 2.1. *Let $0 \le \alpha \le 1$, and let f given by* (1.1) *be in the class $\mathcal{K}_\sigma[\alpha]$. Then*

$$|a_2| \le \sqrt{1+2\alpha}, \tag{2.6}$$

$$|a_3| \le 1+2\alpha. \tag{2.7}$$

Proof. From (2.1) and (2.2) we get

$$f'(z) = \phi'(z)[p(z)]^\alpha \tag{2.8}$$

for some $p \in \mathcal{P}$. Similarly, there exists $q \in \mathcal{P}$ such that

$$g'(w) = \psi'(w)[q(w)]^\alpha. \tag{2.9}$$

Now, p, q have series representations

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \tag{2.10}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots. \tag{2.11}$$

Then, from (2.8) and (2.9), we obtain

$$2a_2 = 2c_2 + \alpha p_1, \tag{2.12}$$

$$3a_3 = 3c_3 + 2\alpha c_2 p_1 + \alpha p_2 + \tfrac{1}{2}\alpha(\alpha-1)p_1^2, \tag{2.13}$$

$$-2a_2 = -2c_2 + \alpha q_1, \tag{2.14}$$

$$6a_2^2 - 3a_3 = 6c_2^2 - 3c_3 - 2c_2\alpha q_1 + \alpha q_2 + \tfrac{1}{2}\alpha(\alpha-1)q_1^2. \tag{2.15}$$

From (2.12) and (2.14), we additionally get $p_1 = -q_1$. Now, adding (2.13) and (2.15), we obtain

$$6a_2^2 = 6c_2^2 + 2\alpha c_2(p_1 - q_1) + \alpha\big(p_2 - \tfrac{1}{2}p_1^2 + q_2 - \tfrac{1}{2}q_1^2\big) + \tfrac{1}{2}\alpha^2(p_1^2 + q_1^2). \tag{2.16}$$

Making use of Lemma 1.1 we get

$$\begin{aligned}
6|a_2^2| &\le 6|c_2|^2 + 2\alpha|c_2|\,|p_1 - q_1| \\
&\quad + \alpha\big(2 - \tfrac{1}{2}|p_1|^2 + 2 - \tfrac{1}{2}|q_1|^2\big) + \tfrac{1}{2}\alpha^2(|p_1|^2 + |q_1|^2) \\
&= 6|c_2|^2 + 2\alpha|c_2|\,|p_1 - q_1| + \alpha\big(2 - \tfrac{1}{2}(1-\alpha)|p_1^2| + 2 - \tfrac{1}{2}(1-\alpha)|q_1|^2\big).
\end{aligned}$$

Now, applying the estimate $|c_k| \le 1$, $|p_k| \le 2$ and $|q_k| \le 2$ for $k = 1, 2, \ldots$, we obtain

$$6|a_2^2| \le 6 + 8\alpha + 4\alpha.$$

Therefore, $|a_2^2| \le 1 + 2\alpha$, proving (2.6).

For (2.7), we apply a similar procedure to relation (2.13). ∎

For $\alpha = 0$, we obtain the following corollary from Theorem 2.1.

COROLLARY 2.1. *Let f given by* (1.1) *be in the class $\mathcal{K}_\sigma[0] = \mathcal{C}_\sigma$. Then $|a_2| \le 1$.*

We note that when $0 \le \alpha \le 1/2$, relation (2.6) gives $|a_2| \le \sqrt{2}$, as below. Therefore, Brannan and Clunie's [BC] conjecture holds for the subclasses $\mathcal{K}_\sigma[\alpha]$, $0 \le \alpha \le 1/2$.

COROLLARY 2.2. *Let f given by* (1.1) *be in the class $\mathcal{K}_\sigma[\alpha]$ and $0 \le \alpha \le 1/2$. Then $|a_2| \le \sqrt{2}$.*

THEOREM 2.2. *Let $0 \le \alpha \le 1$, and let f given by* (1.1) *be in the class $\mathcal{K}_\sigma[\alpha]$. Then*

$$|a_3 - \lambda a_2^2| \le \begin{cases} (1-\lambda)\big(1 + \frac{4}{3}\alpha + \frac{1}{3}\alpha M\big) & \text{for } \lambda < 0, \\ (1-\lambda)\big(1 + \frac{4}{3}\alpha\big) + \frac{1}{3}\alpha M & \text{for } 0 \le \lambda < 2/3, \\ 1 + \frac{4}{3}\alpha(1-\lambda) + \frac{1}{3}\alpha M & \text{for } 2/3 \le \lambda < 1, \\ 1 + \frac{4}{3}\alpha(\lambda-1) + \frac{1}{3}\alpha M & \text{for } 1 \le \lambda \le 4/3, \\ (\lambda-1)\big(1 + \frac{4}{3}\alpha\big) + \frac{1}{3}\alpha M & \text{for } 4/3 < \lambda < 2, \\ (\lambda-1)\big(1 + \frac{4}{3}\alpha + \frac{1}{3}\alpha M\big) & \text{for } \lambda \ge 2, \end{cases} \tag{2.17}$$

where

$$M \le 2. \tag{2.18}$$

Proof. Using (2.13) and (2.16) we obtain

$$\begin{aligned} a_3 - \lambda a_2^2 &= c_3 + \tfrac{2}{3}\alpha c_2 p_1 + \tfrac{1}{3}\alpha p_2 + \tfrac{1}{6}\alpha(\alpha-1)p_1^2 \\ &\quad - \lambda\big[c_2^2 + \tfrac{1}{3}\alpha c_2(p_1 - q_1) + \tfrac{1}{6}\alpha(p_2 + q_2) + \tfrac{1}{12}\alpha(\alpha-1)(p_1^2 + q_1^2)\big]. \end{aligned}$$

By the relations $q_1 = -p_1$, $|c_2| \le 1$ and $|p_1| \le 2$ we get from the above

$$\begin{aligned} |a_3 - \lambda a_2^2| \le |c_3 - \lambda c_2^2| + \tfrac{4}{3}\alpha|1-\lambda| + \tfrac{1}{6}\alpha|2-\lambda|\big[\big|p_2 - \tfrac{1}{2}p_1^2\big| + \tfrac{1}{2}\alpha|p_1^2|\big] \\ + \tfrac{1}{6}\alpha|\lambda|\big[\big|q_2 - \tfrac{1}{2}q_1^2\big| + \tfrac{1}{2}\alpha|q_1^2|\big]. \end{aligned}$$

The expressions $\big|p_2 - \frac{1}{2}p_1^2\big| + \frac{1}{2}\alpha|p_1^2|$ and $\big|q_2 - \frac{1}{2}q_1^2\big| + \frac{1}{2}\alpha|q_1^2|$ have the same bounds, so that we obtain

$$\begin{aligned} |a_3 - \lambda a_2^2| \le |c_3 - \lambda c_2^2| + \tfrac{4}{3}\alpha|1-\lambda| \\ + \tfrac{1}{6}\alpha[|2-\lambda| + |\lambda|]\big[\big|p_2 - \tfrac{1}{2}p_1^2\big| + \tfrac{1}{2}\alpha|p_1^2|\big]. \end{aligned}$$

Making use of Lemma 1.3, and proceeding as in the previous theorem yields the assertion. ■

3. Coefficient bounds for $\mathcal{K}_\sigma(\beta)$

DEFINITION 3.1. Let $0 \le \beta < 1$, and let $f \in \mathcal{A}_\sigma$ given by (1.1) be such that $f'(z) \ne 0$ on $\mathbb{U}$. Then f is said to be *bi-close-to convex* of order β if there exist bi-convex functions $\phi, \psi \in \mathcal{C}_\sigma$ such that

$$\Re\left(\frac{f'(z)}{\phi'(z)}\right) > \beta \quad (z \in \mathbb{U}), \tag{3.1}$$

$$\Re\left(\frac{g'(w)}{\psi'(w)}\right) > \beta \quad (w \in \mathbb{U}), \tag{3.2}$$

where g is the analytic continuation of f^{-1} to $\mathbb{U}$. We denote by $\mathcal{K}_\sigma(\beta)$ the class of *bi-close-to-convex* functions of order β.

Let g, ϕ, ψ have Taylor expansions as in (2.3), (2.4) and (2.5). We note that $\mathcal{K}_\sigma(\beta_2) \subsetneq \mathcal{K}_\sigma(\beta_1)$ when $\beta_1 < \beta_2$, and $\mathcal{K}_\sigma(0) = \mathcal{K}_\sigma$, the class of bi-close-to-convex functions.

We first prove the following proposition.

PROPOSITION 3.1. *If f given by* (1.1) *is in the class $\mathcal{K}_\sigma(\beta)$, $0 \le \beta < 1$, then f is bi-univalent.*

Proof. For $\beta = 0$, this follows from the work of Kaplan [Kap] for close-to-convex functions; and for $0 < \beta < 1$, $\mathcal{K}_\sigma(\beta)$ is a subclass of $\mathcal{K}_\sigma(0)$. ■

THEOREM 3.1. *Let f given by* (1.1) *be in the class $\mathcal{K}_\sigma(\beta)$, $0 \le \beta < 1$. Then*

$$|a_2| \le \sqrt{3 - 2\beta}, \tag{3.3}$$

$$|a_3| \le 3 - 2\beta \tag{3.4}$$

and

$$|a_3 - \lambda a_2^2| \le \begin{cases} (1-\lambda)\big(1 + \frac{4}{3}(1-\beta) + \frac{1}{3}N\big) & \text{for } \lambda < 0, \\ (1-\lambda)\big(1 + \frac{4}{3}(1-\beta)\big) + \frac{1}{3}N & \text{for } 0 \le \lambda < 2/3, \\ 1 + \frac{4}{3}(1-\beta)(1-\lambda) + \frac{1}{3}N & \text{for } 2/3 \le \lambda < 1, \\ 1 + \frac{4}{3}(1-\beta)(\lambda-1) + \frac{1}{3}N & \text{for } 1 \le \lambda \le 4/3, \\ (\lambda-1)\big(1 + \frac{4}{3}(1-\beta)\big) + \frac{1}{3}N & \text{for } 4/3 < \lambda < 2, \\ (\lambda-1)\big(1 + \frac{4}{3}(1-\beta) + \frac{1}{3}N\big) & \text{for } \lambda \ge 2, \end{cases} \tag{3.5}$$

where

$$N \le 2(1-\beta). \tag{3.6}$$

Proof. From (3.1) and (3.2) we get

$$\frac{f'(z)}{\phi'(z)} = \beta + (1-\beta)p(z), \qquad \frac{g'(w)}{\psi'(w)} = \beta + (1-\beta)q(w),$$

for some $p, q \in \mathcal{P}$ with series representations (2.10) and (2.11). Hence,

$$f'(z) = \phi'(z)[\beta + (1-\beta)p(z)], \qquad g'(w) = \psi'(w)[\beta + (1-\beta)q(w)]. \tag{3.7}$$

From the two equations in (3.7), we obtain

$$2a_2 = 2c_2 + (1-\beta)p_1, \tag{3.8}$$

$$3a_3 = 3c_3 + 2(1-\beta)c_2p_1 + (1-\beta)p_2, \tag{3.9}$$

$$-2a_2 = -2c_2 + (1-\beta)q_1, \tag{3.10}$$

$$6a_2^2 - 3a_3 = 6c_2^2 - 3c_3 - 2(1-\beta)c_2q_1 + (1-\beta)q_2. \tag{3.11}$$

Then (3.8) and (3.11) yield $q_1 = -p_1$. Adding (3.9) and (3.11), we obtain

$$6a_2^2 = 6c_2^2 + 2(1-\beta)c_2(p_1 - q_1) + (1-\beta)(p_2 + q_2). \tag{3.12}$$

By the relations $q_1 = -p_1$, $|c_k| \le 1$ and Lemma 1.1, we have

$$|a_2|^2 \le 3 - 2\beta.$$

This gives (3.3).

To obtain (3.4), we apply a similar procedure to relation (3.9).

Now, by (3.9) and (3.12), for all real λ,

$$\begin{aligned} a_3 - \lambda a_2^2 = c_3 &+ \tfrac{2}{3}(1-\beta)c_2p_1 + \tfrac{1}{3}(1-\beta)p_2 \\ &- \lambda\big[c_2^2 + \tfrac{1}{3}(1-\beta)c_2(p_1-q_1) + \tfrac{1}{6}(1-\beta)(p_2+q_2)\big]. \end{aligned}$$

Hence,

$$|a_3 - \lambda a_2^2| \le |c_3 - \lambda c_2^2| + \tfrac{4}{3}(1-\beta)|1-\lambda| + \tfrac{1}{3}(1-\beta)[|2-\lambda| + |\lambda|].$$

By Lemma 1.3, we obtain (3.5). ■

COROLLARY 3.1. *Let f given by* (1.1) *be in $\mathcal{K}_\sigma(\beta)$ and $1/2 \le \beta < 1$. Then $|a_2| \le \sqrt{2}$.*

Proof. Obvious from (3.3), since $1/2 \le \beta < 1$. ■

Corollary 3.1 verifies Brannan and Clunie's [BC] conjecture for the subclasses $\mathcal{K}_\sigma(\beta)$, where $1/2 \le \beta < 1$.

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S. Sivasubramanian, R. Sivakumar
Department of Mathematics
University College of Engineering Tindivanam
Anna University
Tindivanam, India
E-mail: sivasaisastha@rediffmail.com
mrsiva75@gmail.com

S. Kanas
Institute of Mathematics
University of Rzeszów
Rzeszów, Poland
E-mail: skanas@ur.edu.pl

Seong-A Kim
Department of Mathematics Education
Dongguk University
Gyeongju, Korea
E-mail: sakim@dongguk.ac.kr

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