Collapse of warped submersions

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Abstract. We generalize the concept of warped manifold to Riemannian submersions \( \pi : M \rightarrow B \) between two compact Riemannian manifolds \( (M, g_M) \) and \( (B, g_B) \) in the following way. If \( f : B \rightarrow (0, \infty) \) is a smooth function on \( B \) which is extended to a function \( \tilde{f} = f \circ \pi \) constant along the fibres of \( \pi \) then we define a new metric \( g_f \) on \( M \) by

\[
g_f|_{\mathcal{H} \times \mathcal{H}} = g_M|_{\mathcal{H} \times \mathcal{H}}, \quad g_f|_{\mathcal{V} \times T\tilde{M}} = f^2 g_M|_{\mathcal{V} \times T\tilde{M}},
\]

where \( \mathcal{H} \) and \( \mathcal{V} \) denote the bundles of horizontal and vertical vectors. The manifold \( (M, g_f) \) obtained that way is called a warped submersion. The function \( f \) is called a warping function.

We show a necessary and sufficient condition for convergence of a sequence of warped submersions to the base \( B \) in the Gromov–Hausdorff topology. Finally, we consider an example of a sequence of warped submersions which does not converge to its base.

1. Introduction

1.1. Riemannian submersion. Recall that a mapping \( \pi : M \rightarrow B \) between two Riemannian manifolds \( (M, g_M) \) and \( (B, g_B) \), \( \dim B \leq \dim M \), is called a Riemannian submersion if it has maximal rank, and \( g_M(u, w) = g_B(\pi_*u, \pi_*w) \) for any vectors \( u, w \in (\text{Ker} \pi_*)^\perp \). We denote by \( \mathcal{V}(x) = \text{Ker} \pi_{*x} (\mathcal{H}(x) = (\text{Ker} \pi_{*x})^\perp \) resp.) the subspace of vertical (horizontal) vectors.

Lemma 1. Let \( \pi : M \rightarrow B \) be a Riemannian submersion, where \( M, B \) are compact Riemannian manifolds. The function \( \tilde{d} : B \ni x \mapsto \text{diam}^M(\pi^{-1}(x)) \) is continuous.

Proof. Let \( \varepsilon > 0 \) and \( x_0 \in B \). Since \( \pi \) is continuous, there exist points \( y_1, y_2 \in \pi^{-1}(x_0) \) such that \( d_M(y_1, y_2) = \text{diam}^M(\pi^{-1}(x_0)) \).

Let \( x \in B(x_0, \varepsilon/2) \subset B \) and let \( \gamma : [0, \delta] \rightarrow B \), \( \delta > 0 \), be a geodesic curve such that \( \gamma(0) = x \), \( \gamma(\delta) = x_0 \), \( l(\gamma) = d_B(x, x_0) \). Denote by \( \gamma_i, i = 1, 2 \), the horizontal lifts of \( \gamma \) such that \( \gamma_i(\delta) = y_i \). It is clear that \( l(\gamma_i) = l(\gamma) < \varepsilon/2 \).

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Hence
\[
\text{(1)} \quad \text{diam}^M(π^{-1}(x_0)) = d_M(y_1, y_2) \\
\leq l(γ_1) + d_M(γ_1(0), γ_2(0)) + l(γ_2) \\
\leq ε + \text{diam}^M(π^{-1}(x)).
\]

In the same way we show that
\[
\text{(2)} \quad \text{diam}^M(π^{-1}(x)) \leq ε + \text{diam}^M(π^{-1}(x_0)).
\]
Formulae (1) and (2) imply the continuity. ■

As a result, in further considerations we can assume that
\[
\text{diam}^M(π^{-1}(z)) \leq 1
\]
for any z ∈ B.

1.2. Gromov–Hausdorff topology. The Gromov–Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as
\[
\text{(3)} \quad d_{GH}(X, Y) := \inf\{\bar{d}_H(X, Y) : \bar{d} \text{ is an admissible metric on } X ∩ Y\}.
\]
An admissible metric on X ∩ Y is a metric that is an extension of d_X and d_Y. Such a metric always exists, e.g.,
\[
\bar{d}|_{X \times X} ≡ d_X, \quad \bar{d}|_{Y \times Y} ≡ d_Y,
\]
\[
\bar{d}(x, y) = \max\{\text{diam}(X), \text{diam}(Y)\}, \quad x \in X, y \in Y.
\]

In [1] it is shown that (3) defines a metric on the set of isometry classes of compact metric spaces. In further considerations we will need the following two facts.

**Lemma 2** (Gromov). If (X, d_X) and (Y, d_Y) are compact metric spaces and
\[
A = \{x_1, \ldots, x_k\} \subset X, \quad B = \{y_1, \ldots, y_k\} \subset Y
\]
are ε-nets on X and Y, respectively, and if
\[
|d_X(x_i, x_j) - d_Y(y_i, y_j)| ≤ ε, \quad 1 ≤ i, j ≤ k,
\]
then \(d_{GH}(X, Y) ≤ 3ε\).

A proof can be found in [3].

**Theorem 1.** Let \(((X_i, d_{X_i}))_{i \in \mathbb{N}}, (Y, d_Y)\) be compact metric spaces. If \(X_i \to Y\) in the Gromov–Hausdorff topology then for any \(η > 0\) and for any \(η\)-net \(A = \{y_1, \ldots, y_l\}\) on X there exists a sequence of 2\(η\)-nets \(A^i = \{x_{i1}, \ldots, x_{il}\}\) on \(X_i\) such that \(A\) is a quasi-isometric limit of \(A^i\), i.e. for any \(j, k \in \{1, \ldots, l\}\),
\[
|d_Y(y_j, y_k) - d_{X_i}(x_{ij}, x_{ik})| \to 0 \quad \text{as } i \to \infty.
\]

A proof can be found in [1].
1.3. Warped submersion. Let \((M, g_M), (B, g_B)\) be compact Riemannian manifolds, \(\pi : M \to B\) a Riemannian submersion, and \(f : B \to (0, \infty)\) a \(C^\infty\)-function on \(B\). Then \(\tilde{f} = f \circ \pi\) is a smooth function on \(M\) constant along the fibres of \(\pi\). Denote by \(g_f\) the metric on \(M\) given by
\[
g_f|_{\mathcal{H} \times \mathcal{H}} = g_M|_{\mathcal{H} \times \mathcal{H}}, \quad g_f|_{\mathcal{V} \times \mathcal{T}_M} = \tilde{f}^2 g_M|_{\mathcal{V} \times \mathcal{T}_M}.
\]
The manifold \(M\) with metric \(g_f\) will be called a \textit{warped submersion} and denoted by \(M_f\). The function \(f\) will be called a \textit{warping function}.

2. Main results. Let \((f_n : B \to (0, \infty))_{n \in \mathbb{N}}\) be a sequence of smooth warping functions uniformly bounded on \(B\) by a constant \(C\). We ask what should be assumed about \((f_n)\) to ensure that the manifold \(B\) is the limit of \(M_{f_n}\) in the Gromov–Hausdorff topology.

\textbf{Theorem 2.} \(M_{f_n} \to (B, g_B)\) in the Gromov–Hausdorff topology if and only if for any \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that for all \(n > N\) there exists an \(\varepsilon\)-net \(A^n \subset B\) such that
\[
f_n|_{A^n} < \varepsilon.
\]

\textbf{Proof.} \(\Leftarrow\) Let \(\eta > 0\) and \(n > N\). Let \(A^n = \{y_1, \ldots, y_k\}\) be an \(\eta\)-net on \(B\) such that \(f_n|_{A^n} < \eta\). Select points \(x_i \in M_{f_n}, i \in \{1, \ldots, k\}\), in such a way that \(\pi(x_i) = y_i\). Note that the set \(\{x_i\}_{i \in \{1, \ldots, k\}}\) is a \(2\eta\)-net on \(M_{f_n}\). Indeed, let \(y \in M_{f_n}\). There exists \(j \in \{1, \ldots, k\}\) such that \(d_B(\pi(y), y_j) < \eta\). Let \(\gamma : [0, \delta] \to B\) be a minimal geodesic curve joining \(\pi(y)\) and \(y_j\) and \(\tilde{\gamma}\) its horizontal lift such that \(\tilde{\gamma}(0) = y\). We have
\[
d_{M_{f_n}}(y, x_j) \leq l(\tilde{\gamma}) + \text{diam}^{M_{f_n}}(\pi^{-1}(y_j)) \leq l(\gamma) + \eta < 2\eta.
\]
Moreover, for all \(i, j \in \{1, \ldots, k\}\),
\[
d_B(y_i, y_j) \leq d_{M_{f_n}}(x_i, x_j).
\]
Furthermore, if \(\gamma : [0, \delta] \to B\) is a minimal geodesic curve joining \(x_i\) to \(x_j\) and \(\tilde{\gamma}\) its horizontal lift such that \(\tilde{\gamma}(0) = x_i\) then
\[
d_{M_{f_n}}(x_i, x_j) \leq l(\tilde{\gamma}) + d_{M_{f_n}}(\tilde{\gamma}(\delta), x_j) \leq d_B(y_i, y_j) + \eta.
\]
Hence, from (4) and (5), \(|d_B(y_i, y_j) - d_{M_{f_n}}(x_i, x_j)| < 2\eta\) for all \(i, j \in \{1, \ldots, k\}\). Lemma 2 gives us the statement.

\(\Rightarrow\) Suppose that there exists \(\varepsilon_0 > 0\) and a sequence \(n_k \to \infty\) such that for any \(k \in \mathbb{N}\) and any \(\varepsilon_0\)-net \(A \subset B\) there exists \(x \in A\) such that \(f_{n_k}(x) \geq \varepsilon_0\). It is obvious that there exist \(E_0 > 0\) and \(y_0 \in B\) such that \(f_{n_k}|_{B(y_0, E_0)} \geq \varepsilon_0\) for all \(k \in \mathbb{N}\).

Now, suppose that \(M_{f_{n_k}} \to B\) in the Gromov–Hausdorff topology. By Theorem 1, for any \(\eta\)-net \(A = \{y_1, \ldots, y_l\} \subset B\) there exists a sequence of \(2\eta\)-nets \(A_{n_k} = \{x_1^{n_k}, \ldots, x_l^{n_k}\} \subset M_{f_{n_k}}\) such that \(A\) is a quasi-isometric limit.
of $A^n_k$. Moreover, if $A$ is minimal and $\eta$ is small enough,

$$l \min \text{vol}^B B(x, \eta/4) \leq \text{vol} B,$$

$$l \max \text{vol}^{M_{f_{n_k}}} B(x, 2\eta) \geq \text{vol} M_{f_{n_k}}.$$  

Recall that for any compact manifold $\widetilde{M}$ there exists $\widetilde{\eta} > 0$ and a constant $\widetilde{C} \geq 1$ such that for all $\eta < \widetilde{\eta}$ and $x \in \widetilde{M},$

$$\frac{1}{C} \eta^{\dim \widetilde{M}} \leq \text{vol} B(x, \eta) \leq \widetilde{C} \eta^{\dim \widetilde{M}}.$$  

Hence, by (6) and (7),

$$0 < c_0^{\dim M - \dim B} \cdot \text{vol}^M \pi^{-1}(B(y_0, E_0)) \leq \text{vol} M_{f_n}$$

$$\leq \text{vol} B \frac{\max \text{vol}^{M_{f_{n_k}}} B(x, 2\eta)}{\min \text{vol}^B B(x, \eta/4)} \leq \text{vol} B \frac{C_{MC}^{\dim M - \dim B} \cdot (2\eta)^{\dim M}}{(\eta/4)^{\dim B}}.$$  

Hence $M_{f_{n_k}}$ cannot converge to $M$. This yields our statement.  

3. Examples. Let $U \subset B$ be an open set and let $f : B \rightarrow [0, \infty)$ be a function such that $f|_U \equiv 1$ and $f|_{B \setminus U} \equiv 0$. Let $(f_n : B \rightarrow (0, \infty))_{n \in \mathbb{N}}$ be a sequence of smooth functions on $B$ such that

$$f_n|_{U \setminus B(\partial U, 1/n)} \equiv 1, \quad f_n|_{B \setminus U} \equiv 1/n, \quad f_n \leq 1.$$  

It is obvious that $f_n \rightarrow f$. Moreover, the condition of Theorem 2 does not hold, so the limit of the sequence $M_{f_n}$ cannot be $B$. We then ask what the limit of $M_{f_n}$ is (if it exists).

Let $\sim$ be the equivalence relation on $M$ given by

$$x \sim y \Leftrightarrow (\pi(x) = \pi(y) \text{ and } \pi(x) \in B \setminus U) \text{ or } (x = y \text{ and } \pi(x) \in U)$$

Let $\gamma^y_x : [0, \delta^y_x] \rightarrow B$ be a minimal geodesic curve joining $x, y \in B$. Let us set $X = M/\sim$ and define $\rho : X \times X \rightarrow [0, \infty)$ as follows. If all $\gamma^x_{\pi(y)}$ are contained in $U$ then

$$\rho(x, y) = \min\{\min_{z \in \partial U} \{d_B(\pi(x), z) + d_B(z, \pi(y))\}, d_M(x, y)\};$$  

if not,

$$\rho(x, y) = d_B(\pi(x), \pi(y)).$$

It is easy to show that $\rho$ is a metric on $X$. This follows immediately from the fact that

$$d_M(x, y) \geq d_B(\pi(x), \pi(y))$$

and $d_B$ and $d_M$ are metrics on $B$ and $M$ respectively.

Now we can prove the following theorem.
Theorem 3. $M_{f_n} \to (X, \varrho)$ as $n \to \infty$ in the Gromov–Hausdorff topology.

Proof. Let $\eta > 0$ and let $E = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_l\}$ be an $\eta$-net on $X$ such that
\[
\{x_1, \ldots, x_k\} \subset \pi^{-1}(U) / \sim, \quad \{x_{k+1}, \ldots, x_l\} \subset \pi^{-1}(B \setminus U) / \sim.
\]
By (8) there exists $N \in \mathbb{N}$ such that for all $n > N$,
\[
\text{diam}^{M_{f_n}}([x]) < \eta \quad \text{for } x \in M \setminus \pi^{-1}(U),
\]
\[
\text{diam}^{M_{f_n}}(\pi^{-1}(\pi(x_j))) = 1 \quad \text{for } j = 1, \ldots, k.
\]
Let $n > N$ and let $E^n = \{y_1, \ldots, y_l\}$ be such that
\[
y_i = x_i \quad \text{for } i = 1, \ldots, l.
\]
The set $E^n$ is a $2\eta$-net on $M_{f_n}$. Indeed, let $y \in M_{f_n}$. There exists $j \in \{1, \ldots, l\}$ such that $\varrho([y]_\sim, x_j) < \eta$. We consider the following cases:

1. $y \in \pi^{-1}(U)$ and $j \in \{1, \ldots, k\}$,
2. $y \in \pi^{-1}(U)$ and $j \in \{k + 1, \ldots, l\}$,
3. $y \in \pi^{-1}(B \setminus U)$ and $j \in \{1, \ldots, k\}$,
4. $y \in \pi^{-1}(B \setminus U)$ and $j \in \{k + 1, \ldots, k\}$.

We only handle the first case. The others are similar. Let $y \in \pi^{-1}(U)$, $j \in \{1, \ldots, k\}$. If any minimal geodesic curve $\gamma_{\pi(y)} \subset U$ then, since $[y]_\sim = \{y\}$ and (9),
\[
\varrho([y]_\sim, x_j) = \min \{ \min_{z \in \partial U} \{ d_B(\pi(y), z) + d_B(z, \pi(x_j)) \} , d_M(y, x_j) \}.
\]
If $\varrho([y]_\sim, x_j) = d_M(y, x_j)$ then
\[
d_{M_{f_n}}(y, y_j) \leq d_M(y, y_j) = \varrho([y]_\sim, x_j) < 2\eta.
\]
Else if $\varrho([y]_\sim, x_j) = \min_{z \in \partial U} \{ d_B(\pi(y), z) + d_B(z, \pi(x_j)) \}$ then
\[
\varrho([y]_\sim, x_j) = \min \{ \min_{z \in \partial U} \{ d_B(\pi(y), z) + d_B(z, \pi(x_j)) \} \}
\]
and for some $z_0 \in \partial U$,
\[
d_{M_{f_n}}(y, y_j) \leq d_B(\pi(y), z_0) + d_B(z_0, \pi(y_j)) + \text{diam}(\pi^{-1}(z_0))
\]
\[
= d_B(\pi([y]_\sim), z_0) + d_B(z_0, \pi(x_j)) + \eta
\]
\[
= \varrho([y]_\sim, x_j) + \eta < 2\eta.
\]
Furthermore, for any $i, j \in \{1, \ldots, l\}$, we have
\[
|\varrho(x_i, x_j) - d_{M_{f_n}}(y_i, y_j)| < 2\eta.
\]
Indeed, if $k + 1 \leq i \leq l$, $j \in \{1, \ldots, l\}$ then
\[
\varrho(x_i, x_j) = d_B(\pi(x_i), \pi(x_j)) \leq d_{M_{f_n}}(y_i, y_j) + \eta
\]
and as above
\begin{equation}
(12) \quad d_{Mf_n}(y_i, y_j) \leq d_M(\pi(y_i), \pi(y_j)) + \text{diam}^{Mf_n}(\pi^{-1}(\pi(y_j))) \\
\leq \varrho(x_i, x_j) + \eta.
\end{equation}

Let $1 < i, j \leq k$. Suppose that there exists a geodesic curve
\[
\gamma_{\pi(x_i)}^{\pi(x_j)} : [0, \delta_{\pi(x_j)}] \to B
\]
not contained in $U$. Then $\varrho(x_i, x_j) = d_B(\pi(x_i), \pi(x_j))$ and there exists $t_0 \in [0, \delta_{\pi(x_j)}]$ such that $\gamma(t_0) \not\in U$. Hence $\text{diam}^{Mf_n}(\pi^{-1}(\gamma(t_0))) < \eta$. Moreover,
\begin{equation}
(13) \quad \varrho(x_i, x_j) = d_B(y_i, y_j) \leq d_{Mf_n}(y_i, y_j) \leq d_{Mf_n}(y_i, y_j) + \eta
\end{equation}
and
\begin{equation}
(14) \quad d_{Mf_n}(y_i, y_j) \leq d_B(\pi(y_i), \gamma(t_0)) + d_B(\gamma(t_0), y_j) + \text{diam}^{Mf_n} \pi^{-1}(\gamma(t_0)) \\
\leq \varrho(x_i, x_j) + \eta.
\end{equation}

Now, suppose that all minimal geodesics joining $\pi(y_i)$ to $\pi(y_j)$ are contained in $U$. If
\begin{equation}
(15) \quad d_M(y_i, y_j) < \min_{z \in \partial U}\{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\}
\end{equation}
then all minimal geodesic curves joining $y_i$ to $y_j$ in $M_{f_n}$ are totally embedded in $\pi^{-1}(U)$. Indeed, suppose by contradiction that there exists a minimal geodesic curve $\gamma_0 : [0, \delta] \to M_{f_n}$ joining $y_i$ with $y_j$ which is not totally embedded in $\pi^{-1}(U)$. So there exist $x_0 \in \pi^{-1}(\partial U)$ and $t_0 \in (0, \delta)$ such $\gamma_0(t_0) = x_0$. We then have
\begin{equation}
(16) \quad d_{M_{f_n}}(y_i, y_j) = l(\gamma_0) = \int_0^\delta \|\dot{\gamma}_0(t)\|_{M_{f_n}} dt \\
= \int_0^{t_0} \|\dot{\gamma}_0(t)\|_{M_{f_n}} dt + \int_{t_0}^{\delta} \|\dot{\gamma}_0(t)\|_{M_{f_n}} dt \\
\geq \int_0^{t_0} \|\pi_*(\dot{\gamma}_0(t))\|_B dt + \int_{t_0}^{\delta} \|\pi_*(\dot{\gamma}_0(t))\|_B dt \\
\geq d_B(\pi(y_i), \pi(x_0)) + d_B(\pi(x_0), \pi(y_j)) \\
\geq \min_{z \in \partial U}\{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\}.
\end{equation}

But
\[d_M(y_i, y_j) \geq d_{Mf_n}(y_i, y_j).\]
So we get 
\[ d_M(y_i, y_j) \geq \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\}, \]
which contradicts (15).

Let \( \gamma : [0, \delta] \to M_{f_n} \) be a minimal geodesic curve joining \( y_i \) to \( y_j \). Because all geodesic curves joining \( y_i \) to \( y_j \) are totally embedded in \( \pi^{-1}(U) \),
\[ d_{M_{f_n}}(y_i, y_j) = \int_0^\delta \|\dot{\gamma}(t)\|_{M_{f_n}} \, dt = \int_0^\delta \|\dot{\gamma}(t)\|_M \, dt = d_M(y_i, y_j) = \varrho(x_i, x_j). \]
Hence
\[ (17) \quad |d_{M_{f_n}}(y_i, y_j) - \varrho(x_i, x_j)| < \eta. \]

Now, suppose that \( d_M(y_i, y_i) \geq \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\} \). Let \( z_0 \in \partial U \) be a point at which \( \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\} \) is achieved, and let
1. \( \gamma_1 : [0, \delta_1] \to B \) be a minimal geodesic curve joining \( \pi(y_i) \) to \( z_0 \) and \( \tilde{\gamma}_1 \)
   its horizontal lift such \( \tilde{\gamma}_1(0) = y_i \),
2. \( \gamma_2 : [0, \delta_2] \to B \) be a minimal geodesic curve joining \( \pi(y_j) \) to \( z_0 \) and \( \tilde{\gamma}_2 \)
   its horizontal lift such \( \tilde{\gamma}_2(0) = y_j \),
3. \( \gamma_3 : [0, \delta_3] \to \pi^{-1}(z_0) \) be a minimal geodesic curve joining \( \tilde{\gamma}_1(\delta_1) \) to \( \tilde{\gamma}_2(\delta_2) \).

Let \( \gamma : [0, \tilde{\delta}] \to M_{f_n}, \tilde{\delta} = \delta_1 + \delta_2 + \delta_3, \) be given by \( \gamma = \gamma_2^{-1} \ast \gamma_3 \ast \gamma_1 \). Then
\[ (18) \quad d_{M_{f_n}}(y_i, y_j) \leq l(\gamma) = \sum_{i=1}^{3} l(\gamma_i) \leq \varrho(x_i, x_j) + \eta. \]

On the other hand, if \( \gamma : [0, \delta] \to M_{f_n} \) is a minimal geodesic curve from \( y_i \) to \( y_j \) then
1. if \( \gamma([0, \delta]) \subset \pi^{-1}(U) \) then
\[ (19) \quad \eta + d_{M_{f_n}}(y_i, y_j) \geq \varrho(x_i, x_j) \geq \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\} \geq d_M(y_i, y_j); \]
2. if \( \gamma([0, \delta]) \not\subset \pi^{-1}(U) \) then as in (16),
\[ (20) \quad \eta + d_{M_{f_n}}(y_i, y_j) \geq \min_{z \in \partial U} \{d_B(\pi(y_i^l), z) + d_B(z, \pi(y_j^h))\} \geq d_M(y_i, y_j). \]

Hence by (11)–(14) and (17)–(20) we get (10).

Since \( E^n \) and \( E \) are \( 2\eta \)-nets on \( M_{f_n} \) and \( X \) respectively and for any \( i, j \in \{1, \ldots, l\} \), we have
\[ |\varrho(x_i, x_j) - d_{M_{f_n}}(y_i, y_j)| < 2\eta, \]
Lemma 2 implies that \( d_{GH}(M_{f_n}, X) < 6\eta \). This yields our statement.
References


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