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Collapse of warped submersions

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Abstract. We generalize the concept of warped manifold to Riemannian submersions $\pi : M \to B$ between two compact Riemannian manifolds (M, g_M) and (B, g_B) in the following way. If $f : B \to (0, \infty)$ is a smooth function on B which is extended to a function $\widetilde{f} = f \circ \pi$ constant along the fibres of π then we define a new metric g_f on M by

$$g_f|_{\mathcal{H}\times\mathcal{H}} \equiv g_M|_{\mathcal{H}\times\mathcal{H}}, \quad g_f|_{\mathcal{V}\times T\widetilde{M}} \equiv \widetilde{f}^2 g_M|_{\mathcal{V}\times T\widetilde{M}},$$

where $\mathcal{H}$ and $\mathcal{V}$ denote the bundles of horizontal and vertical vectors. The manifold (M, g_f) obtained that way is called a *warped submersion*. The function f is called a *warping function*.

We show a necessary and sufficient condition for convergence of a sequence of warped submersions to the base B in the Gromov–Hausdorff topology. Finally, we consider an example of a sequence of warped submersions which does not converge to its base.

1. Introduction

1.1. *Riemannian submersion.* Recall that a mapping $\pi : M \to B$ between two Riemannian manifolds (M, g_M) and (B, g_B), $\dim B \le \dim M$, is called a *Riemannian submersion* if it has maximal rank, and $g_M(u, w) = g_B(\pi_* u, \pi_* w)$ for any vectors $u, w \in (\operatorname{Ker} \pi_*)^\perp$. We denote by $\mathcal{V}(x) = \operatorname{Ker} \pi_{*x}$ ($\mathcal{H}(x) = (\operatorname{Ker} \pi_{*x})^\perp$ resp.) the subspace of vertical (horizontal) vectors.

Lemma 1. *Let $\pi : M \to B$ be a Riemannian submersion, where M, B are compact Riemannian manifolds. The function $\widetilde{d} : B \ni x \mapsto \operatorname{diam}^M(\pi^{-1}(x))$ is continuous.*

Proof. Let $\varepsilon > 0$ and $x_0 \in B$. Since π is continuous, there exist points $y_1, y_2 \in \pi^{-1}(x_0)$ such that $d_M(y_1, y_2) = \operatorname{diam}^M(\pi^{-1}(x_0))$.

Let $x \in B(x_0, \varepsilon/2) \subset B$ and let $\gamma : [0, \delta] \to B$, $\delta > 0$, be a geodesic curve such that $\gamma(0) = x$, $\gamma(\delta) = x_0$, $l(\gamma) = d_B(x, x_0)$. Denote by γ_i, $i = 1, 2$, the horizontal lifts of γ such that $\gamma_i(\delta) = y_i$. It is clear that $l(\gamma_i) = l(\gamma) < \varepsilon/2$.

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Hence

$$\begin{aligned}(1)\qquad \operatorname{diam}^M(\pi^{-1}(x_0)) &= d_M(y_1, y_2)\\ &\le l(\gamma_1) + d_M(\gamma_1(0), \gamma_2(0)) + l(\gamma_2)\\ &\le \varepsilon + \operatorname{diam}^M(\pi^{-1}(x)).\end{aligned}$$

In the same way we show that

$$(2)\qquad \operatorname{diam}^M(\pi^{-1}(x)) \le \varepsilon + \operatorname{diam}^M(\pi^{-1}(x_0)).$$

Formulae (1) and (2) imply the continuity. ∎

As a result, in further considerations we can assume that

$$\operatorname{diam}^M(\pi^{-1}(z)) \le 1$$

for any $z \in B$.

1.2. *Gromov–Hausdorff topology.* The *Gromov–Hausdorff distance* between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as

$$(3)\quad d_{\mathrm{GH}}(X, Y) := \inf\{\widetilde{d}_H(X, Y) : \widetilde{d} \text{ is an admissible metric on } X \amalg Y\}.$$

An *admissible metric* on $X \amalg Y$ is a metric that is an extension of d_X and d_Y. Such a metric always exists, e.g.

$$\begin{aligned}&\widetilde{d}|_{X\times X} \equiv d_X, \quad \widetilde{d}|_{Y\times Y} \equiv d_Y,\\ &\widetilde{d}(x, y) = \max\{\operatorname{diam}(X), \operatorname{diam}(Y)\}, \quad x \in X,\ y \in Y.\end{aligned}$$

In [1] it is shown that (3) defines a metric on the set of isometry classes of compact metric spaces. In further considerations we will need the following two facts.

Lemma 2 (Gromov). *If (X, d_X) and (Y, d_Y) are compact metric spaces and*

$$A = \{x_1, \ldots, x_k\} \subset X, \quad B = \{y_1, \ldots, y_k\} \subset Y$$

are ε-nets on X and Y, respectively, and if

$$|d_X(x_i, x_j) - d_Y(y_i, y_j)| \le \varepsilon, \quad 1 \le i, j \le k,$$

then $d_{\mathrm{GH}}(X, Y) \le 3\varepsilon$.

A proof can be found in [3].

Theorem 1. *Let $((X_i, d_{X_i}))_{i\in\mathbb{N}}$, (Y, d_Y) be compact metric spaces. If $X_i \to Y$ in the Gromov–Hausdorff topology then for any $\eta > 0$ and for any η-net $A = \{y_1, \ldots, y_l\}$ on X there exists a sequence of 2η-nets $A^i = \{x_1^i, \ldots, x_l^i\}$ on X_i such that A is a quasi-isometric limit of A^i, i.e. for any $j, k \in \{1, \ldots, l\}$,*

$$|d_Y(y_j, y_k) - d_{X_i}(x_j^i, x_k^i)| \to 0 \quad \text{as } i \to \infty.$$

A proof can be found in [1].

1.3. *Warped submersion.* Let (M, g_M), (B, g_B) be compact Riemannian manifolds, $\pi : M \to B$ a Riemannian submersion, and $f : B \to (0, \infty)$ a C^∞-function on B. Then $\widetilde{f} = f \circ \pi$ is a smooth function on M constant along the fibres of π. Denote by g_f the metric on M given by

$$g_f|_{\mathcal{H}\times\mathcal{H}} \equiv g_M|_{\mathcal{H}\times\mathcal{H}}, \quad g_f|_{\mathcal{V}\times T\widetilde{M}} \equiv \widetilde{f}^2 g_M|_{\mathcal{V}\times T\widetilde{M}}.$$

The manifold M with metric g_f will be called a *warped submersion* and denoted by M_f. The function f will be called a *warping function.*

2. Main results. Let $(f_n : B \to (0, \infty))_{n\in\mathbb{N}}$ be a sequence of smooth warping functions uniformly bounded on B by a constant C. We ask what should be assumed about (f_n) to ensure that the manifold B is the limit of M_{f_n} in the Gromov–Hausdorff topology.

THEOREM 2. *$M_{f_n} \to (B, g_B)$ in the Gromov–Hausdorff topology if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ there exists an ε-net $A^n \subset B$ such that*

$$f_n|_{A^n} < \varepsilon.$$

Proof. $\Leftarrow$ Let $\eta > 0$ and $n > N$. Let $A^n = \{y_1, \ldots, y_k\}$ be an η-net on B such that $f_n|_{A^n} < \eta$. Select points $x_i \in M_{f_n}$, $i \in \{1, \ldots, k\}$, in such a way that $\pi(x_i) = y_i$. Note that the set $\{x_i\}_{i\in\{1,\ldots,k\}}$ is a 2η-net on M_{f_n}. Indeed, let $y \in M_{f_n}$. There exists $j \in \{1, \ldots, k\}$ such that $d_B(\pi(y), y_j) < \eta$. Let $\gamma : [0, \delta] \to B$ be a minimal geodesic curve joining $\pi(y)$ and y_j and $\widetilde{\gamma}$ its horizontal lift such that $\widetilde{\gamma}(0) = y$. We have

$$d_{M_{f_n}}(y, x_j) \le l(\widetilde{\gamma}) + \operatorname{diam}^{M_{f_n}}(\pi^{-1}(y_j)) \le l(\gamma) + \eta < 2\eta.$$

Moreover, for all $i, j \in \{1, \ldots, k\}$,

$$d_B(y_i, y_j) \le d_{M_{f_n}}(x_i, x_j). \tag{4}$$

Furthermore, if $\gamma : [0, \delta] \to B$ is a minimal geodesic curve joining x_i to x_j and $\widetilde{\gamma}$ its horizontal lift such that $\widetilde{\gamma}(0) = x_i$ then

$$d_{M_{f_n}}(x_i, x_j) \le l(\widetilde{\gamma}) + d_{M_{f_n}}(\widetilde{\gamma}(\delta), x_j) \le d_B(y_i, y_j) + \eta. \tag{5}$$

Hence, from (4) and (5), $|d_B(y_i, y_j) - d_{M_{f_n}}(x_i, x_j)| < 2\eta$ for all $i, j \in \{1, \ldots, k\}$. Lemma 2 gives us the statement.

$\Rightarrow$ Suppose that there exists $\varepsilon_0 > 0$ and a sequence $n_k \to \infty$ such that for any $k \in \mathbb{N}$ and any ε_0-net $A \subset B$ there exists $x \in A$ such that $f_{n_k}(x) \ge \varepsilon_0$. It is obvious that there exist $E_0 > 0$ and $y_0 \in B$ such that $f_{n_k}|_{B(y_0,E_0)} \ge \varepsilon_0$ for all $k \in \mathbb{N}$.

Now, suppose that $M_{f_{n_k}} \to B$ in the Gromov–Hausdorff topology. By Theorem 1, for any η-net $A = \{y_1, \ldots, y_l\} \subset B$ there exists a sequence of 2η-nets $A^{n_k} = \{x_1^{n_k}, \ldots, x_l^{n_k}\} \subset M_{f_{n_k}}$ such that A is a quasi-isometric limit of A^{n_k}. Moreover, if A is minimal and η is small enough,

$$\begin{aligned} &l \min \operatorname{vol}^B B(x,\eta/4) \le \operatorname{vol} B, \\ &l \max \operatorname{vol}^{M_{f_{n_k}}} B(x,2\eta) \ge \operatorname{vol} M_{f_{n_k}}. \end{aligned} \tag{6}$$

Recall that for any compact manifold $\widetilde{M}$ there exists $\widetilde{\eta} > 0$ and a constant $\widetilde{C} \ge 1$ such that for all $\eta < \widetilde{\eta}$ and $x \in \widetilde{M}$,

$$\frac{1}{\widetilde{C}} \eta^{\dim \widetilde{M}} \le \operatorname{vol} B(x,\eta) \le \widetilde{C} \eta^{\dim \widetilde{M}}. \tag{7}$$

Hence, by (6) and (7),

$$\begin{aligned} 0 &< \varepsilon_0^{\dim M - \dim B} \cdot \operatorname{vol}^M \pi^{-1}(B(y_0, E_0)) \le \operatorname{vol} M_{f_n} \\ &\le \operatorname{vol} B \, \frac{\max \operatorname{vol}^{M_{f_{n_k}}} B(x,2\eta)}{\min \operatorname{vol}^B B(x,\eta/4)} \le \operatorname{vol} B \, \frac{C_M C_B C^{\dim M - \dim B} \cdot (2\eta)^{\dim M}}{(\eta/4)^{\dim B}}. \end{aligned}$$

Hence $M_{f_{n_k}}$ cannot converge to M. This yields our statement. ∎

3. Examples. Let $U \subset B$ be an open set and let $f : B \to [0,\infty)$ be a function such that $f|_U \equiv 1$ and $f|_{B \setminus U} \equiv 0$. Let $(f_n : B \to (0,\infty))_{n \in \mathbb{N}}$ be a sequence of smooth functions on B such that

$$f_n|_{U \setminus B(\partial U, 1/n)} \equiv 1, \quad f_n|_{B \setminus U} \equiv 1/n, \quad f_n \le 1. \tag{8}$$

It is obvious that $f_n \to f$. Moreover, the condition of Theorem 2 does not hold, so the limit of the sequence M_{f_n} cannot be B. We then ask what the limit of M_{f_n} is (if it exists).

Let $\sim$ be the equivalence relation on M given by

$$x \sim y \iff (\pi(x) = \pi(y) \text{ and } \pi(x) \in B \setminus U) \text{ or } (x = y \text{ and } \pi(x) \in U)$$

Let $\gamma_x^y : [0, \delta_x^y] \to B$ be a minimal geodesic curve joining $x, y \in B$. Let us set $X = M/{\sim}$ and define $\varrho : X \times X \to [0,\infty)$ as follows. If all $\gamma_{\pi(y)}^{\pi(x)}$ are contained in U then

$$\varrho(x,y) = \min\{\min_{z \in \partial U}\{d_B(\pi(x), z) + d_B(z, \pi(y))\}, d_M(x,y)\};$$

if not,

$$\varrho(x,y) = d_B(\pi(x), \pi(y)).$$

It is easy to show that ϱ is a metric on X. This follows immediately from the fact that

$$d_M(x,y) \ge d_B(\pi(x), \pi(y))$$

and d_B and d_M are metrics on B and M respectively.

Now we can prove the following theorem.

THEOREM 3. *$M_{f_n} \to (X, \varrho)$ as $n \to \infty$ in the Gromov–Hausdorff topology.*

Proof. Let $\eta > 0$ and let $E = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_l\}$ be an η-net on X such that

$$\{x_1, \ldots, x_k\} \subset \pi^{-1}(U)/\sim, \quad \{x_{k+1}, \ldots, x_l\} \subset \pi^{-1}(B \setminus U)/\sim.$$

By (8) there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\operatorname{diam}^{M_{f_n}}([x]_\sim) < \eta \quad \text{for } x \in M \setminus \pi^{-1}(U),$$
$$\operatorname{diam}^{M_{f_n}}(\pi^{-1}(\pi(x_j))) = 1 \quad \text{for } j = 1, \ldots, k.$$

Let $n > N$ and let $E^n = \{y_1, \ldots, y_l\}$ be such that

$$[y_i]_\sim = x_i \quad \text{for } i = 1, \ldots, l. \tag{9}$$

The set E^n is a 2η-net on M_{f_n}. Indeed, let $y \in M_{f_n}$. There exists $j \in \{1, \ldots, l\}$ such that $\varrho([y]_\sim, x_j) < \eta$. We consider the following cases:

1. $y \in \pi^{-1}(U)$ and $j \in \{1, \ldots, k\}$,
2. $y \in \pi^{-1}(U)$ and $j \in \{k+1, \ldots, l\}$,
3. $y \in \pi^{-1}(B \setminus U)$ and $j \in \{1, \ldots, k\}$,
4. $y \in \pi^{-1}(B \setminus U)$ and $j \in \{k+1, \ldots, k\}$.

We only handle the first case. The others are similar. Let $y \in \pi^{-1}(U)$, $j \in \{1, \ldots, k\}$. If any minimal geodesic curve $\gamma_{\pi(x_j)}^{\pi(y)} \subset U$ then, since $[y]_\sim = \{y\}$ and (9),

$$\varrho([y]_\sim, x_j) = \min\{\min_{z \in \partial U}\{d_B(\pi(y), z) + d_B(z, \pi(x_j))\}, d_M(y, x_j)\}.$$

If $\varrho([y]_\sim, x_j) = d_M(y, x_j)$ then

$$d_{M_{f_n}}(y, y_j) \le d_M(y, y_j) = \varrho([y]_\sim, x_j) < 2\eta.$$

Else if $\varrho([y]_\sim, x_j) = \min_{z \in \partial U}\{d_B(\pi(y), z) + d_B(z, \pi(x_j))\}$ then

$$\varrho([y]_\sim, x_j) = \min\{\min_{z \in \partial U}\{d_B(\pi([y]_\sim), z) + d_B(z, \pi(x_j))\}$$

and for some $z_0 \in \partial U$,

$$\begin{aligned} d_{M_{f_n}}(y, y_j) &\le d_B(\pi(y), z_0) + d_B(z_0, \pi(y_j)) + \operatorname{diam}(\pi^{-1}(z_0)) \\ &= d_B(\pi([y]_\sim), z_0) + d_B(z_0, \pi(x_j)) + \eta \\ &= \varrho([y]_\sim, x_j) + \eta < 2\eta. \end{aligned}$$

Furthermore, for any $i, j \in \{1, \ldots, l\}$, we have

$$|\varrho(x_i, x_j) - d_{M_{f_n}}(y_i, y_j)| < 2\eta. \tag{10}$$

Indeed, if $k+1 \le i \le l$, $j \in \{1, \ldots, l\}$ then

$$\varrho(x_i, x_j) = d_B(\pi(x_i), \pi(x_j)) \le d_{M_{f_n}}(y_i, y_j) + \eta \tag{11}$$

and as above

$$d_{M_{f_n}}(y_i, y_j) \le d_M(\pi(y_i), \pi(y_j)) + \operatorname{diam}^{M_{f_n}}(\pi^{-1}(\pi(y_j))) \le \varrho(x_i, x_j) + \eta. \tag{12}$$

Let $1 < i, j \le k$. Suppose that there exists a geodesic curve

$$\gamma^{\pi(x_i)}_{\pi(x_j)} : [0, \delta^{\pi(x_i)}_{\pi(x_j)}] \to B$$

not contained in U. Then $\varrho(x_i, x_j) = d_B(\pi(x_i), \pi(x_j))$ and there exists $t_0 \in [0, \delta^{\pi(x_i)}_{\pi(x_j)}]$ such that $\gamma(t_0) \notin U$. Hence $\operatorname{diam}^{M_{f_n}}(\pi^{-1}(\gamma(t_0))) < \eta$. Moreover,

$$\varrho(x_i, x_j) = d_B(y_i, y_j) \le d_{M_{f_n}}(y_i, y_j) \le d_{M_{f_n}}(y_i, y_j) + \eta \tag{13}$$

and

$$d_{M_{f_n}}(y_i, y_j) \le d_B(\pi(y_i), \gamma(t_0)) + d_B(\gamma(t_0), y_j) + \operatorname{diam}^{M_{f_n}} \pi^{-1}(\gamma(t_0)) \le \varrho(x_i, x_j) + \eta. \tag{14}$$

Now, suppose that all minimal geodesics joining $\pi(y_i)$ to $\pi(y_j)$ are contained in U. If

$$d_M(y_i, y_j) < \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\} \tag{15}$$

then all minimal geodesic curves joining y_i to y_j in M_{f_n} are totally embedded in $\pi^{-1}(U)$. Indeed, suppose by contradiction that there exists a minimal geodesic curve $\gamma_0 : [0, \delta] \to M_{f_n}$ joining y_i with y_j which is not totally embedded in $\pi^{-1}(U)$. So there exist $x_0 \in \pi^{-1}(\partial U)$ and $t_0 \in (0, \delta)$ such $\gamma_0(t_0) = x_0$. We then have

$$\begin{aligned} d_{M_{f_n}}(y_i, y_j) &= l(\gamma_0) = \int_0^\delta \|\dot\gamma_0(t)\|_{M_{f_n}}\, dt \\ &= \int_0^{t_0} \|\dot\gamma_0(t)\|_{M_{f_n}}\, dt + \int_{t_0}^{\delta} \|\dot\gamma_0(t)\|_{M_{f_n}}\, dt \\ &\ge \int_0^{t_0} \|(\dot\gamma_0(t))^\perp\|_{M_{f_n}}\, dt + \int_{t_0}^{\delta} \|(\dot\gamma_0(t))^\perp\|_{M_{f_n}}\, dt \\ &\ge \int_0^{t_0} \|\pi_*(\dot\gamma_0(t))^\perp\|_B\, dt + \int_{t_0}^{\delta} \|\pi_*(\dot\gamma_0(t)^\perp)\|_B\, dt \\ &\ge d_B(\pi(y_i), \pi(x_0)) + d_B(\pi(x_0), \pi(y_j) \\ &\ge \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j)\}. \end{aligned} \tag{16}$$

But

$$d_M(y_i, y_j) \ge d_{M_{f_n}}(y_i, y_j).$$

So we get

$$d_M(y_i, y_j) \geq \min_{z \in \partial U} \{d_B(\pi(y_i), z) + d_B(z, \pi(y_j)\},$$

which contradicts (15).

Let $\gamma : [0, \delta] \to M_{f_n}$ be a minimal geodesic curve joining y_i to y_j. Because all geodesic curves joining y_i to y_j are totally embeded in $\pi^{-1}(U)$,

$$d_{M_{f_n}}(y_i, y_j) = \int_0^\delta \|\dot{\gamma}(t)\|_{M_{f_n}}\, dt = \int_0^\delta \|\dot{\gamma}(t)\|_M\, dt = d_M(y_i, y_j) = \varrho(x_i, x_j).$$

Hence

$$|d_{M_{f_n}}(y_i, y_j) - \varrho(x_i, x_j)| < \eta. \tag{17}$$

Now, suppose that $d_M(y_i, y_i) \geq \min_{z \in \partial U}\{d_B(\pi(y_i), z) + d_B(z, \pi(y_j)\}$. Let $z_0 \in \partial U$ be a point at which $\min_{z \in \partial U}\{d_B(\pi(y_i), z) + d_B(z, \pi(y_j)\}$ is achieved, and let

1. $\gamma_1 : [0, \delta_1] \to B$ be a minimal geodesic curve joining $\pi(y_i)$ to z_0 and $\widetilde{\gamma}_1$ its horizontal lift such $\widetilde{\gamma}_1(0) = y_i$,
2. $\gamma_2 : [0, \delta_2] \to B$ be a minimal geodesic curve joining $\pi(y_j)$ to z_0 and $\widetilde{\gamma}_2$ its horizontal lift such $\widetilde{\gamma}_2(0) = y_j$,
3. $\gamma_3 : [0, \delta_3] \to \pi^{-1}(z_0)$ be a minimal geodesic curve joining $\widetilde{\gamma}_1(\delta_1)$ to $\widetilde{\gamma}_2(\delta_2)$.

Let $\gamma : [0, \widetilde{\delta}] \to M_{f_n}$, $\widetilde{\delta} = \delta_1 + \delta_2 + \delta_3$, be given by $\gamma = \gamma_2^{-1} * \gamma_3 * \gamma_1$. Then

$$d_{M_{f_n}}(y_i, y_j) \leq l(\gamma) = \sum_{i=1}^{3} l(\gamma_i) \leq \varrho(x_i, x_j) + \eta. \tag{18}$$

On the other hand, if $\gamma : [0, \delta] \to M_{f_n}$ is a minimal geodesic curve from y_i to y_j then

1. if $\gamma([0, \delta]) \subset \pi^{-1}(U)$ then

$$\begin{aligned} \eta + d_{M_{f_n}}(y_i, y_j) &\geq l(\gamma) = \int_0^\delta \|\dot{\gamma}(t)\|_{M_{f_n}} dt = \int_0^\delta \|\dot{\gamma}(t)\|_M dt \\ &\geq \min_{z \in \partial U}\{d_B(\pi(y_i), z) + d_B(z, \pi(y_j))\} \geq \varrho(x_i, x_j); \end{aligned} \tag{19}$$

2. if $\gamma([0, \delta]) \not\subset \pi^{-1}(U)$ then as in (16),

$$\eta + d_{M_{f_n}}(y_i, y_j) \geq \min_{z \in \partial U}\{d_B(\pi(y_i^n), z) + d_B(z, \pi(y_j^n))\} \geq \varrho(x_i, x_j). \tag{20}$$

Hence by (11)–(14) and (17)–(20) we get (10).

Since E^n and E are 2η-nets on M_{f_n} and X respectively and for any $i, j \in \{1, \ldots, l\}$, we have

$$|\varrho(x_i, x_j) - d_{M_{f_n}}(y_i, y_j)| < 2\eta,$$

Lemma 2 implies that $d_{\mathrm{GH}}(M_{f_n}, X) < 6\eta$. This yields our statement. ■

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