Normality and value sharing
with a linear differential polynomial

by INDRAJIT LAHIRI (Kalyani) and SHYAMALI DEWAN (Kolkata)

Abstract. We prove some normality criteria for a family of meromorphic functions and as an application we prove a value distribution theorem for a differential polynomial.

1. Introduction, definitions and results. Let \( \mathbb{C} \) be the open complex plane and \( \mathcal{D} \subset \mathbb{C} \) be a domain. A family \( \mathcal{F} \) of meromorphic functions defined in \( \mathcal{D} \) is said to be normal, in the sense of Montel, if for every sequence \( \{ f_n \} \subset \mathcal{F} \) there exists a subsequence \( \{ f_{n_j} \} \) such that \( \{ f_{n_j} \} \) converges spherically and uniformly on compact subsets of \( \mathcal{D} \) to a meromorphic function or \( \infty \).

\( \mathcal{F} \) is said to be normal at a point \( z_0 \in \mathcal{D} \) if there exists a neighbourhood of \( z_0 \) in which \( \mathcal{F} \) is normal. It is well known that \( \mathcal{F} \) is normal in \( \mathcal{D} \) if and only if it is normal at every point of \( \mathcal{D} \).

Let \( f \) and \( g \) be two meromorphic functions defined in \( \mathcal{D} \). For \( a \in \mathbb{C} \cup \{ \infty \} \) we say that \( f \) and \( g \) share the value \( a \) IM (ignoring multiplicity) if the \( a \)-points of \( f \) and \( g \) coincide in locations only, not necessarily in multiplicities.

For a meromorphic function \( f \) we denote by \( f^\# \) the spherical derivative of \( f \), given by

\[
    f^\#(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.
\]

Also, by \( \Delta \) we denote the unit disc \(|z| < 1\).

In 1992 W. Schwick [15] first established a connection between the normality and value sharing. He proved the following theorem.

**Theorem A ([15]).** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( \mathcal{D} \subset \mathbb{C} \) and \( a_1, a_2, a_3 \) be distinct complex numbers. If for every \( f \in \mathcal{F} \), \( f \) and \( f' \) share \( a_1, a_2, a_3 \) IM in \( \mathcal{D} \) then \( \mathcal{F} \) is normal in \( \mathcal{D} \).

2000 Mathematics Subject Classification: 30D45, 30D35.

Key words and phrases: meromorphic function, differential polynomial, normality.
After the work of Schwick [15] it has become a popular problem to investigate the relation between normality and sharing values.

In 1999 Y. Xu [16] proved the following result.

**Theorem B** ([16]). Let $\mathcal{F}$ be a family of holomorphic functions in a domain $\mathcal{D} \subset \mathbb{C}$ and $b$ be a nonzero complex number. If $f$ and $f'$ share $0, b$ IM in $\mathcal{D}$ for every $f \in \mathcal{F}$ then $\mathcal{F}$ is normal in $\mathcal{D}$.

In 2000 X. Pang and L. Zalcman [12] proved the following result, which improves Theorems A and B.

**Theorem C** ([12]). Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D} \subset \mathbb{C}$ and $a_1, a_2$ be distinct complex numbers. If for every $f \in \mathcal{F}$, $f$ and $f'$ share $a_1, a_2$ IM in $\mathcal{D}$ then $\mathcal{F}$ is normal in $\mathcal{D}$.

At this stage two natural questions may be asked:

1. What would be if $f$ and $f'$ share a single value?
2. What would be if $f'$ is replaced by $f^{(k)}$?

For Question 1 the following result of W. C. Lin and H. X. Yi [11] may be noted.

**Theorem D** ([11]). Let $\mathcal{F}$ be a family of meromorphic functions in $\Delta$. If there exist complex numbers $a$ and $b$ ($b \neq 0$ and $a/b$ not a positive integer) such that for every $f \in \mathcal{F}$, $f$ and $f'$ share a IM in $\Delta$ and $|f(z) - a| \geq \varepsilon$ whenever $f'(z) = b$, where $\varepsilon$ is a positive number, then $\mathcal{F}$ is normal in $\Delta$.

For Question 2, H. Chen and M. Fang [3] proved the following result.

**Theorem E** ([3]). Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D} \subset \mathbb{C}$, $k \geq 2$ be an integer and $a, b, c$ be complex numbers such that $b \neq a$. If for each $f \in \mathcal{F}$, $f$ and $f^{(k)}$ share $a, b$ IM in $\mathcal{D}$ and zeros of $f - c$ have multiplicity at least $1 + k$ then $\mathcal{F}$ is normal in $\mathcal{D}$.

The following result of M. Fang and L. Zalcman [5] improved Theorem E.

**Theorem F** ([5]). Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D} \subset \mathbb{C}$, $k \geq 2$ be an integer and $a, b, c, d$ be complex numbers such that $b \neq a$. If for each $f \in \mathcal{F}$, $f$ and $f^{(k)}$ share $a, b$ IM in $\mathcal{D}$ and zeros of $f - c$ have multiplicity at least $k$ then $\mathcal{F}$ is normal in $\mathcal{D}$.

Theorem F is a consequence of the following theorem, also due to Fang and Zalcman [5].

**Theorem G** ([5]). Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D} \subset \mathbb{C}$, $k$ be a positive integer and $a, b, c, d$ be complex numbers such that $b \neq a, 0$ and $c \neq 0$. If, for each $f \in \mathcal{F}$, all zeros of $f - d$ have multiplicity at least $k$, $f$ and $f^{(k)} - a$ share $0$ IM and $f(z) = c$ whenever $f^{(k)}(z) = b$, then $\mathcal{F}$ is normal in $\mathcal{D}$ for $k \geq 2$, and for $k = 1$ so long as $a \neq (1 + m)b$, $m = 1, 2, \ldots$. 
In this paper we investigate the situation when the derivative is replaced by a linear differential polynomial with constant coefficients generated by \( f \). Throughout the paper we denote by \( H_k(f) = H_k(f; a_1, \ldots, a_k) \) a linear differential polynomial generated by a meromorphic function \( f \) of the following form:

\[
H_k(f) = H_k(f; a_1, \ldots, a_k) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_1 f^{(1)}
\]

where \( k \) is a positive integer and \( a_1, \ldots, a_k \neq 0 \) are constants.

We now state the main result of the paper.

**Theorem 1.1.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( \mathfrak{D} \subset \mathbb{C} \) and \( a, b, c, d \) be finite complex numbers such that \( c \neq 0 \). If there exists a differential polynomial \( H_k(f) = H_k(f; a_1, \ldots, a_k) \) such that for each \( f \in \mathcal{F} \),

(i) \( f - d \) does not have any zero with multiplicity less than \( k \),

(ii) \( f - a \) and \( H_k(f) - b \) share the value \( 0 \) IM,

(iii) \( |f(z) - a| \geq \varepsilon \) whenever \( H_k(f) = c \), where \( \varepsilon \) is a positive number,

then \( \mathcal{F} \) is normal in \( \mathfrak{D} \) for \( k \geq 2 \), and for \( k = 1 \) so long as \( b/c \neq 1 + m \) for any positive integer \( m \).

The following example shows that condition (i) of Theorem 1.1 is essential.

**Example 1.1.** Let \( f_n(z) = ne^z - ne^{-z} + 1 \) for \( n = 1, 2, \ldots \) and \( \mathfrak{D} = \mathbb{C} \). We choose \( k = 2 \), \( a = 1 \), \( b = 0 \), \( c = 1 \) and \( \varepsilon = 1 \). Then for any given finite complex number \( d \),

\[
f_n(z) - d = \frac{ne^{2z} + (1 - d)e^z - n}{e^z}
\]

has only simple zeros in \( \mathfrak{D} \) (except possibly for only one value of \( n \) for which \( d = 1 \pm 2ni \)). Also \( f_n(z) - a \) and \( f_n^{(2)}(z) - b \) share 0 IM and \( |f_n(z) - a| = 2 > \varepsilon \) whenever \( f_n^{(2)}(z) = c \). Since \( f_n^{(\#)}(0) = n \to \infty \) as \( n \to \infty \), by Marty’s criterion the family \( \{f_n\} \) is not normal in \( \mathfrak{D} \).

The following example shows that condition (ii) of Theorem 1.1 is essential.

**Example 1.2.** Let \( f_n(z) = nz^2 \) for \( n = 1, 2, \ldots \) and \( \mathfrak{D} = \Delta \). We choose \( k = 2 \), \( a = 0 \), \( b = 0 \), \( d = 0 \) and \( c = 1 \). Then \( f_n(z) - d \) has no zero of multiplicity less than \( k \), \( f_n^{(2)}(z) = 2n \) does not assume the value \( c \), so that condition (iii) of Theorem 1.1 is satisfied but \( f_n(z) \) and \( f_n^{(2)}(z) \) do not share the value \( a = b = 0 \). Since \( f_n(0) = 0 \) for \( n = 1, 2, \ldots \) and for \( z \neq 0 \), \( f_n(z) \to \infty \) as \( n \to \infty \), it follows that the family \( \{f_n\} \) is not normal in \( \mathfrak{D} \).

The following example shows that condition (iii) of Theorem 1.1 is essential.
Example 1.3. Let \( f(z) = e^{nz} \) for \( n = 1, 2, \ldots \) and \( \mathcal{D} = \Delta \). We choose \( k = 2, a = 0, b = 0, c = 1 \) and \( d = 0 \). Then conditions (i) and (ii) of Theorem 1.1 are satisfied. Also we see that \( f_n^{(2)}(z) = c \) implies \( |f_n(z) - a| = 1/n^2 \to 0 \) as \( n \to \infty \) so that we cannot find any \( \varepsilon > 0 \) for which condition (iii) is satisfied. Since \( f_n^\#(0) = n/2 \to \infty \) as \( n \to \infty \), by Marty’s criterion the family \( \{f_n\} \) is not normal in \( \mathcal{D} \).

The following example shows that the condition \( c \neq 0 \) cannot be removed from Theorem 1.1.

Example 1.4. Let \( f_n(z) = e^{nz} - a/n + a \) for \( n = 1, 2, \ldots \) and \( \mathcal{D} = \Delta \). Then \( f_n \) and \( f_n^{(1)} \) share the value \( a \) IM. Also \( f_n^{(1)}(z) \neq 0 \) in \( \mathcal{D} \) so that condition (iii) of Theorem 1.1 is satisfied for \( c = 0 \). Since

\[
 f_n^\#(0) = \frac{n}{1 + |a/n + a|} \to \infty \quad \text{as} \quad n \to \infty,
\]

by Marty’s criterion the family \( \{f_n\} \) is not normal in \( \mathcal{D} \).

The following example shows that for \( k = 1 \) the condition “\( b/c \neq 1 + m \) for any positive integer \( m \)” of Theorem 1.1 is essential.

Example 1.5. Let \( b \) and \( c \) be two nonzero numbers such that \( b = (1 + m)c \), where \( m \) is a positive integer. Also let \( \{\alpha_n\} \) be a sequence of numbers converging to 0 and \( |\alpha_n| < 1 \) for \( n = 1, 2, \ldots \). We suppose that \( \mathcal{D} = \Delta \) and, for \( n = 1, 2, \ldots \),

\[
f_n(z) = c(z - \alpha_n) + \frac{A(\alpha_n)^m}{m(z - \alpha_n)^m},
\]

where \( A \) is a nonzero constant. Then

\[
f_n^{(1)}(z) = c - \frac{A(\alpha_n)^m}{(z - \alpha_n)^{m+1}}
\]

so that \( f_n^{(1)}(z) \) does not assume the value \( c \) and so condition (iii) of Theorem 1.1 is satisfied. Also

\[
f_n(z) = \frac{mc(z - \alpha_n)^{m+1} + A(\alpha_n)^m}{m(z - \alpha_n)^m},
\]

\[
f_n^{(1)}(z) - b = -\frac{mc(z - \alpha_n)^{m+1} + A(\alpha_n)^m}{m(z - \alpha_n)^{m+1}}
\]

so that \( f_n \) and \( f_n^{(1)} \) share 0 IM. Again

\[
f_n^\#(0) = \frac{|c + (-1)^{m+2}/\alpha_n|}{1 + |c\alpha_n + (-1)^m A/m|^2}
\]

\[
\geq \frac{1/|\alpha_n| - |c|}{1 + \{|c|/\alpha_n + |A|/m|^2}} \to \infty \quad \text{as} \quad n \to \infty.
\]

Hence by Marty’s criterion the family \( \{f_n\} \) is not normal in \( \mathcal{D} \).
The following corollary not only extends Theorem G to a linear differential polynomial but also removes the hypothesis \( a \neq b \).

**Corollary 1.1.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( \mathcal{D} \subset \mathbb{C} \) and \( a, b, c, d, \alpha \) be finite complex numbers such that \( b \neq 0 \) and \( c \neq \alpha \). If there exists a differential polynomial \( H_k(f) = H_k(f; a_1, \ldots, a_k) \) such that for each \( f \in \mathcal{F} \),

(i) \( f - d \) does not have any zero of multiplicity less than \( k \),
(ii) \( f - \alpha \) and \( H_k(f) - a \) share the value 0 IM,
(iii) \( f(z) = c \) whenever \( H_k(f) = b \),

then \( \mathcal{F} \) is normal in \( \mathcal{D} \) for \( k \geq 2 \), and for \( k = 1 \) so long as \( a/b \neq 1 + m \) for any positive integer \( m \).

**Remark 1.1.** If we choose \( a = b \) then from conditions (ii) and (iii) of Corollary 1.1 it is obvious that \( \alpha \) and \( a \) are lacunary values of \( f \in \mathcal{F} \) and \( H_k(f) \) respectively.

The following example shows that in Corollary 1.1 the condition \( b \neq 0 \) is essential.

**Example 1.6.** Let \( f_n(z) = e^{nz} \) for \( n = 1, 2, \ldots \) and \( \mathcal{D} = \Delta \). We choose \( \alpha = a = b = d = 0 \). Then \( f_n(z) - d \) does not have any zero and for any positive integer \( k \), \( f_n(z) \) and \( f_n^{(k)}(z) - a \) share the value 0 IM. Since \( f_n^{(k)}(z) \neq b \), it follows that condition (iii) of Corollary 1.1 is satisfied for any complex number \( c \). Since \( f_n^\#(0) = n/2 \to \infty \) as \( n \to \infty \), by Marty’s criterion the family \( \{f_n\} \) is not normal in \( \mathcal{D} \).

The following corollary improves Theorems C and F.

**Corollary 1.2.** Let \( \mathcal{F} \) be a family of meromorphic functions in a domain \( \mathcal{D} \subset \mathbb{C} \) and \( a, b, c \) be finite numbers such that \( a \neq b \). If there exists a differential polynomial \( H_k(f) = H_k(f; a_1, \ldots, a_k) \) such that for each \( f \in \mathcal{F} \),

(i) \( f - c \) does not have any zero of multiplicity less than \( k \),
(ii) \( f \) and \( H_k(f) \) share the values \( a \) and \( b \) IM,

then \( \mathcal{F} \) is normal in \( \mathcal{D} \).

For standard definitions and notations we refer to [7] and [14].

2. **Lemmas.** In this section we present some necessary lemmas.

**Lemma 2.1 ([13]).** Let \( \mathcal{F} \) be a family of meromorphic functions in \( \Delta \) having no zero of multiplicity less than \( k \). Suppose there exists a number \( A \geq 1 \) such that \( |f^{(k)}(z)| \leq A \) whenever \( f(z) = 0 \). If \( \mathcal{F} \) is not normal in \( \Delta \) then there exist, for each \( \alpha \) \( (0 \leq \alpha \leq k) \),

(i) a number \( r \), \( 0 < r < 1 \),
(ii) points $z_n$, $|z_n| < r$,
(iii) functions $f_n \in \mathfrak{F}$ and
(iv) positive numbers $\rho_n$, $\rho_n \to 0$,
such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$ spherically and locally uniformly
to a nonconstant meromorphic function $g$ in $\mathbb{C}$, all of whose zeros have mul-
tiplicity at least $k$ and $g^\#(\xi) \leq g^\#(0) = kA + 1$. Moreover the order of $g$ is
at most 2.

Lemma 2.2 ([5]). Let $f$ be a meromorphic function of finite order and
$a, b \neq 0$ be distinct complex numbers and $k \geq 2$ be an integer. If $f$ has no zero
of multiplicity less than $k$, $f$ and $f^{(k)} - a$ share the value 0 IM and $f^{(k)}$
does not assume the value $b$, then $f$ is a constant.

Lemma 2.3 ([5, 8, 11]). Let $f$ be a nonconstant meromorphic function
of finite order and let $a, b \neq 0$ be distinct complex numbers. If $f$ and $f^{(1)} - a$
share the value 0 IM and $f^{(1)}$ does not assume the value $b$ in $\mathbb{C}$ then
\[
f(z) = b(z - d) + \frac{A}{m(z - d)^m} \quad \text{and} \quad a = (1 + m)b
\]
for some $d \in \mathbb{C}$ and some positive integer $m$.

Lemma 2.4 ([9]). Let $f$ be a nonconstant rational function, and $k$ and
$\lambda \geq 2$ be positive integers such that

(i) $f$ has no zero of multiplicity less than $\lambda$ and the number of zeros of $f$
(counted with multiplicities), if there are any, is not less than $1 + k$,
(ii) if $f$ has any pole then the number of poles of $f$ (counted with multi-
plicities) is greater than $k/(\lambda - 1)$.

Then for every complex number $a \neq 0, \infty$, the function $f^{(k)} + a$ has at least
one zero.

Lemma 2.5. Let $f$ be a nonconstant rational function having no zero
and $k$ be a positive integer. Then for every complex number $a \neq 0, \infty$, the
function $f^{(k)} + a$ has at least one zero.

Proof. Since $f$ has no zero, choosing $\lambda = k + 2$ in Lemma 2.4 we obtain
the result. ■

Lemma 2.6 ([7, p. 60]). Suppose that $f$ is meromorphic and transcendental in $\mathbb{C}$. Then for any positive integer $k$,
\[
T(r, f) \leq (2 + 1/k)N(r, 0; f) + (2 + 2/k)\bar{N}(r, a; f^{(k)}) + S(r, f),
\]
where $a \neq 0, \infty$ is a complex number.

Lemma 2.7 ([2]). Let $f$ be a meromorphic function of finite order. If $f$
has only finitely many critical values then it has only finitely many asymptotic
values.
Lemma 2.8 ([11]). Let $f$ be a transcendental meromorphic function such that $f(0) \neq \infty$ and let the set of finite critical and asymptotic values of $f$ be bounded. Then there exists $R > 0$ such that

$$|f'(z)| \geq \frac{|f(z)|}{2\pi |z|} \log \frac{|f(z)|}{R}$$

for all $z \in \mathbb{C} \setminus \{0\}$ which are not poles of $f$.

Lemma 2.9 ([6, 10]). Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$ and $k \geq 2$ be an integer. If $f$ and $f^{(k)}$ do not assume the value 0 in $\mathbb{C}$ then either $f(z) = e^{Az+B}$ or $f(z) = (Az+B)^{-m}$, where $A \neq 0$ and $B$ are constants and $m$ is a positive integer.

Lemma 2.10 ([4]). Let $f$ be a meromorphic function in $\mathbb{C}$. If there exists a constant $M > 0$ such that $f^{\#}(z) \leq M$ in $\mathbb{C}$ then the order of $f$ is at most 2.

Lemma 2.11 ([7, p. 57]). Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$ and $H_k(f)$ be nonconstant. Then for any complex number $a \neq 0, \infty$,

$$T(r, f) \leq N(r, \infty; f) + N(r, 0; f) + \widetilde{N}(r, a; H_k(f)) + S(r, f).$$

3. Proof of the theorem and corollaries

Proof of Theorem 1.1. Since normality is a local property, without loss of generality we may assume that $f = \Delta$. Also since $H_k(f - a) = H_k(f)$, we may additionally suppose that $a = 0$. First we suppose that $a_k = 1$. We now consider the following cases.

Case I. Let $k \geq 2$ and $d = 0$. Suppose that $\mathfrak{F}$ is not normal in $\Delta$. Then by Lemma 2.1 for $\alpha = k$ we can find a sequence $\{z_n\}$ of points with $|z_n| < r$ $(0 < r < 1)$, a sequence of positive numbers $\varrho_n \rightarrow 0$ and a sequence $\{f_n\} \subset \mathfrak{F}$ of functions such that

$$g_n(\xi) = \varrho_n^{-k} f_n(z_n + \varrho_n \xi) \rightarrow g(\xi)$$

spherically and locally uniformly, where $g$ is a nonconstant meromorphic function in $\mathbb{C}$ and $g$ has no zero of multiplicity less than $k$. Also $g^\#(\xi) \leq g^\#(0) = k(A + 1) + 1$ and $g$ is of order at most 2, where $A = \max\{|b|, |c|\}$.

We now verify that (I) $g$ and $g^{(k)} - b$ share the value 0 IM, and that (II) $g^{(k)}$ does not assume the value $c$ in $\mathbb{C}$.

Let $g(\xi_0) = 0$. Then by Hurwitz's theorem there exists a sequence $\xi_n \rightarrow \xi_0$ such that $g_n(\xi_n) = 0$ for all sufficiently large values of $n$. So for all sufficiently large values of $n$ we get $f_n(z_n + \varrho_n \xi_n) = 0$, and so for all sufficiently large values of $n$, $H_k(f_n(z_n + \varrho_n \xi_n)) = b$. Hence

$$g_n^{(k)}(\xi_n) + a_{k-1} \varrho_n g_n^{(k-1)}(\xi_n) + \cdots + a_1 \varrho_n^{k-1} g_n^{(1)}(\xi_n) = b.$$ 

Letting $n \rightarrow \infty$ we obtain $g^{(k)}(\xi_0) = b$. 


Next let \( g^{(k)}(\eta_0) = b \). First we verify that \( g^{(k)}(\xi) \neq b \). If \( g^{(k)}(\xi) \equiv b \) then \( g \) becomes a polynomial of degree at most \( k \). Since \( g \) has no zero of multiplicity less than \( k \) and \( g \) is nonconstant, it follows that \( g \) is a polynomial of degree \( k \) and so it has a single zero of multiplicity \( k \). Hence we can write

\[
(3.1) \quad g(\xi) = \frac{b(\xi - \xi_1)^k}{k!}.
\]

By a simple calculation we deduce from (3.1) that \( g^#(0) \leq k/2 \) if \( |\xi_1| \geq 1 \) and \( g^#(0) \leq |b| \) if \( |\xi_1| < 1 \). Therefore \( g^#(0) < k(|b| + 1) + 1 \), which is a contradiction.

Since \( g^{(k)}(\eta_0) = b \) and \( g^{(k)}(\eta) + a_{k-1}e_\eta g^{(k-1)}(\eta) + \cdots + a_1g^{k-1}_\eta g^{(1)}(\eta) \) converges uniformly to \( g^{(k)}(\eta) \) in some neighbourhood of \( \eta_0 \), by Hurwitz’s theorem there exists a sequence \( \eta_n \to \eta_0 \) such that for all large values of \( n \),

\[
g^{(k)}(\eta_n) + a_{k-1}e_\eta g^{(k-1)}(\eta_n) + \cdots + a_1g^{k-1}_\eta g^{(1)}(\eta_n) = b
\]

and so \( H_k(f_n(z_n + \varrho_n\eta_n)) = b \). Therefore for all sufficiently large values of \( n \) we get \( f_n(z_n + \varrho_n\eta_n) = 0 \) and so \( g_n(\eta_n) = 0 \). Letting \( n \to \infty \) we obtain \( g(\eta_0) = 0 \). Therefore (I) is verified.

Let \( g^{(k)}(\zeta_0) = c \). Then as above we can show that \( g^{(k)}(\zeta) \neq c \). Since \( g^{(k)}(\zeta) + a_{k-1}e_\zeta g^{(k-1)}(\zeta) + \cdots + a_1g^{k-1}_\zeta g^{(1)}(\zeta) \) converges uniformly to \( g^{(k)}(\zeta) \) in some neighbourhood of \( \zeta_0 \), by Hurwitz’s theorem there exists a sequence \( \zeta_n \to \zeta_0 \) such that for all large values of \( n \),

\[
g^{(k)}(\zeta_n) + a_{k-1}e_\zeta g^{(k-1)}(\zeta_n) + \cdots + a_1g^{k-1}_\zeta g^{(1)}(\zeta_n) = c
\]

and so \( H_k(f_n(z_n + \varrho_n\zeta_n)) = c \). Therefore \( |f_n(z_n + \varrho_n\zeta_n)| \geq \varepsilon \) and so \( |g_n(\zeta_n)| \geq \varepsilon/g_n^{(1)} \) for all large values of \( n \). This shows that \( g(\zeta_0) = \infty \), which is a contradiction. So (II) is verified.

If \( b \neq c \), by Lemma 2.2, \( g \) becomes a constant, which is impossible. Let \( b = c \). Then from (I) and (II) we see that \( g \) does not assume the value 0 and \( g^{(k)} \) does not assume the value \( c \neq 0 \). If \( g \) is transcendental, by Lemma 2.6 we get \( T(r, g) = S(r, g) \), which is a contradiction. If \( g \) is rational, by Lemma 2.5, \( g \) becomes a constant, which is impossible. Therefore the family \( \mathfrak{F} \) is normal.

**Case II.** Let \( k \geq 2 \) and \( d \neq 0 \). Suppose that \( \mathfrak{F}_1 = \{f - d : f \in \mathfrak{F}\} \). If \( \mathfrak{F}_1 \) is not normal in \( \Delta \), by Lemma 2.1 for \( \alpha = 0 \) we can find a sequence \( \{z_n\} \) of points with \( |z_n| < r \) \((0 < r < 1)\), a sequence of positive numbers \( \varrho_n \to 0 \) and a sequence \( \{f_n - d\} \subset \mathfrak{F}_1 \) of functions such that

\[
g_n(\xi) = f_n(z_n + \varrho_n\xi) - d \to g(\xi)
\]

spherically and locally uniformly, where \( g \) is a nonconstant meromorphic function in \( \mathbb{C} \) and \( g \) has no zero of multiplicity less then \( k \). Further \( g \) is of order at most 2.
We now verify that (III) \( g^{(k)} \) does not assume the value 0 in \( \mathbb{C} \), and that (IV) \( g + d \) does not assume the value 0 in \( \mathbb{C} \).

Let \( g^{(k)}(\xi_0) = 0 \) for some \( \xi_0 \in \mathbb{C} \). Also we see that \( g^{(k)}(\xi) \neq 0 \), for otherwise \( g \) becomes a polynomial of degree less than \( k \), which is impossible because \( g \) is nonconstant and does not have any zero of multiplicity less than \( k \).

Since in a neighbourhood of \( \xi_0 \),

\[
g_n^{(k)}(\xi) + a_{k-1}g_n^{(k-1)}(\xi) + \cdots + a_1g_n^{(1)}(\xi) - c_n^kb_n
\]

converges uniformly to \( g^{(k)}(\xi) \), by Hurwitz’s theorem there exists a sequence \( \xi_n \to \xi_0 \) such that for all large values of \( n \),

\[
g_n^{(k)}(\xi_n) + a_{k-1}g_n^{(k-1)}(\xi_n) + \cdots + a_1g_n^{(1)}(\xi_n) - c_n^kb_n = 0,
\]

and so for all large values of \( n \) we get \( H_k(f_n(z_n + \varrho_n\xi_n)) = b \). Therefore for all large values of \( n \) we obtain \( f_n(z_n + \varrho_n\xi_n) = 0 \) and so \( g_n(\xi_n) + d = 0 \). Letting \( n \to \infty \) we get

(3.2) \quad \quad g(\xi_0) + d = 0.

Again since in a neighbourhood of \( \xi_0 \),

\[
g_n^{(k)}(\xi) + a_{k-1}g_n^{(k-1)}(\xi) + \cdots + a_1g_n^{(1)}(\xi) - c_n^kc
\]

converges uniformly to \( g^{(k)}(\xi) \), by Hurwitz’s theorem there exists a sequence \( \chi_n \to \xi_0 \) such that

\[
g_n^{(k)}(\chi_n) + a_{k-1}g_n^{(k-1)}(\chi_n) + \cdots + a_1g_n^{(1)}(\chi_n) - c_n^kc = 0
\]

for all large values of \( n \). Hence for all large values of \( n \) we deduce that \( H_k(f_n(z_n + \varrho_n\chi_n)) = c \). So for all large values of \( n \),

\[
|f_n(z_n + \varrho_n\chi_n)| \geq \varepsilon, \quad \text{i.e.,} \quad |g_n(\chi_n) + d| \geq \varepsilon.
\]

Letting \( n \to \infty \) we obtain \( |g(\xi_0) + d| \geq \varepsilon \), which contradicts (3.2). Therefore (III) is verified.

Next let \( g(\beta_0) + d = 0 \). Then by Hurwitz’s theorem there exists a sequence \( \beta_n \to \beta_0 \) such that for all large values of \( n \), \( f_n(z_n + \varrho_n\beta_n) - d = g_n(\beta_n) = -d \) and so \( f_n(z_n + \varrho_n\beta_n) = 0 \). Hence for all large values of \( n \) we deduce that \( H_k(f_n(z_n + \varrho_n\beta_n)) = b \) and so

\[
g_n^{(k)}(\beta_n) + a_{k-1}g_n^{(k-1)}(\beta_n) + \cdots + a_1g_n^{(1)}(\beta_n) = b \varrho_n^k.
\]

Letting \( n \to \infty \) we get \( g^{(k)}(\beta_0) = 0 \), which contradicts (III). Therefore (IV) is verified.

Now by Lemma 2.9 we see that either \( g(\xi) = -d + e^{Az+B} \) or \( g(\xi) = -d + 1/(Az+B)^m \). Since \( d \neq 0 \), it follows that \( g \) has only simple zeros, which is impossible. Therefore \( \mathcal{F}_1 \) and so \( \mathcal{F} \) is normal.

Case III. Let \( k = 1 \). In this case condition (i) of the theorem is immaterial and so the proof does not depend on \( d \). If \( \mathcal{F} \) is not normal in \( \Delta \),
proceeding as Case I we can show that there exists a nonconstant meromorphic function $g$ of finite order such that $g$ and $g^{(1)} - b$ share the value $0$ IM and $g^{(1)}$ does not assume the value $c$ in $\mathbb{C}$.

If $b \neq c$ then by Lemma 2.3 we get $b = (1 + m)c$ for some positive integer $m$, which is impossible. Let $b = c$. Then $g$ does not assume the value $0$ and $g^{(1)}$ does not assume the value $c$. If $g$ is rational, by Lemma 2.5, $g$ becomes a constant, which is impossible. If $g$ is transcendental, by Lemma 2.6 we get $T(r, g) = S(r, g)$, which is a contradiction. Therefore the family $\mathfrak{F}$ is normal.

Finally, suppose that $a_k \neq 1$. We now put $G_k(f) = (1/a_k)H_k(f)$, $b_1 = b/a_k$ and $c_1 = c/a_k$. Then the leading coefficient of $G_k(f)$ is 1 and $b_1/c_1 = b/c$. Also the following hold:

(i) $f - d$ has no zero of multiplicity less than $k$,
(ii) $f - a$ and $G_k(f) - b_1$ share the value 0 IM,
(iii) $|f(z) - a| \geq \varepsilon$ whenever $G_k(f) = c_1$.

Therefore the family $\mathfrak{F}$ is normal in this case as well by the result for $a_k = 1$. This proves the theorem. 

**Proof of Corollary 1.1.** Since $c \neq \alpha$, we choose an $\varepsilon$ such that $0 < \varepsilon < |c - \alpha|$. Then from condition (iii) we see that if $H_k(f) = b$ then $|f(z) - \alpha| = |c - \alpha| > \varepsilon$, which is condition (iii) of Theorem 1.1. Hence the corollary follows from Theorem 1.1. 

**Proof of Corollary 1.2.** Interchanging $a$ and $b$ if necessary, we may choose $|a| \leq |b|$. Since $a \neq b$, it follows that $b \neq 0$ and $a/b$ is not a positive integer. We now choose an $\varepsilon$ such that $0 < \varepsilon < |b - a|$. So we see that if $H_k(f) = b$ then $|f(z) - a| = |b - a| > \varepsilon$. Hence the corollary follows from Theorem 1.1. 

**4. Application.** In this section we prove a value distribution theorem for a differential polynomial which follows from Theorem 1.1.

**Theorem 4.1.** Let $f$ be a transcendental meromorphic function and $a_1, \ldots, a_k \neq 0$ be constants such that $H_k(f^p) = H_k(f^p; a_1, \ldots, a_k)$ is also transcendental, where $p \geq 2$ is an integer. Let $a$ be a finite complex number such that

(i) $f$ has no zero of multiplicity less than $k/p$,
(ii) $f$ and $H_k(f^p) - a$ share the value 0 IM.

Then for every complex number $b \neq 0, \infty$, the function $H_k(f^p) - b$ has infinitely many zeros.

**Proof.** We consider the following cases.

**Case I.** Let $f$ be of infinite order. Then by Lemma 2.10 there exists a sequence $z_n \to \infty$ such that $f^\#(z_n) \to \infty$ as $n \to \infty$. Let $f_n(z) = f(z_n + z)$
for \( n = 1, 2, \ldots \). Then \( f_n^\#(0) = f^\#(z_n) \to \infty \) as \( n \to \infty \). So by Marty’s criterion no subfamily of \( \{ f_n \} \) is normal in \( \Delta \). Suppose that \( H_k(f^p) - b \) has a finite number of zeros. Since \( z_n \to \infty \) as \( n \to \infty \), there exists a positive integer \( N \) such that for \( n \geq N \), \( H_k(f^p_n) - b \) has no zero in \( \Delta \). So by Theorem 1.1 the family \( \{ f_n : n \geq N \} \) is normal in \( \Delta \), which is a contradiction. Therefore \( H_k(f^p) - b \) has infinitely many zeros.

**Case II.** Let \( f \) be of finite order. If \( f \) has only finitely many zeros, by Lemma 2.11 we get

\[
T(r, f^p) \leq \mathcal{N}(r, \infty; f^p) + \mathcal{N}(r, b; H_k(f^p)) + S(r, f^p)
\]

and so

\[
(p - 1)T(r, f) \leq \mathcal{N}(r, b; H_k(f^p)) + S(r, f),
\]

which shows that \( H_k(f^p) - b \) has infinitely many zeros.

Let \( f \) have infinitely many zeros, say \( w_1, w_2, \ldots \). We put \( g(z) = a_k h^{(k-1)}(z) + a_{k-1} h^{(k-2)}(z) + \cdots + a_1 h(z) - bz \), where \( h(z) = \{ f(z) \}^p \). Let \( g'(z) = H_k(f^p) - b \) have only finitely many zeros. So \( g \) has only finitely many critical values and so, by Lemma 2.7, \( g \) has only finitely many asymptotic values. We assume, without loss of generality, that \( g(0) \neq \infty \). Then by Lemma 2.8 there exists \( R > 0 \) such that for \( n = 1, 2, \ldots \),

\[
\left| \frac{w_n g'(w_n)}{g(w_n)} \right| \geq \frac{1}{2\pi} \log \frac{|g(w_n)|}{R} = \frac{1}{2\pi} \log \frac{|bw_n|}{R},
\]

so that

\[
\left| \frac{w_n g'(w_n)}{g(w_n)} \right| \to \infty \quad \text{as} \quad n \to \infty.
\]

On the other hand, for \( n = 1, 2, \ldots \) we get

\[
\left| \frac{w_n g'(w_n)}{g(w_n)} \right| = \frac{|a - b|}{|b|},
\]

which is a contradiction. Therefore \( H_k(f^p) - b \) has infinitely many zeros. This proves the theorem. \( \blacksquare \)

**Acknowledgements.** The authors are thankful to the referee for her/his valuable suggestions.

**References**


Department of Mathematics
University of Kalyani
West Bengal 741235, India
E-mail: indr9431@dataone.in

Department of Mathematics
Bhairab Ganguly College
Kolkata 700056, India
E-mail: shyamalidewan@rediffmail.com

Received 23.2.2006
and in final form 31.5.2006 (1658)