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Normality and value sharing with a linear differential polynomial

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Abstract. We prove some normality criteria for a family of meromorphic functions and as an application we prove a value distribution theorem for a differential polynomial.

1. Introduction, definitions and results. Let $\mathbb{C}$ be the open complex plane and $\mathfrak{D} \subset \mathbb{C}$ be a domain. A family $\mathfrak{F}$ of meromorphic functions defined in $\mathfrak{D}$ is said to be *normal*, in the sense of Montel, if for every sequence $\{f_n\} \subset \mathfrak{F}$ there exists a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ converges spherically and uniformly on compact subsets of $\mathfrak{D}$ to a meromorphic function or ∞.

$\mathfrak{F}$ is said to be *normal at a point* $z_0 \in \mathfrak{D}$ if there exists a neighbourhood of z_0 in which $\mathfrak{F}$ is normal. It is well known that $\mathfrak{F}$ is normal in $\mathfrak{D}$ if and only if it is normal at every point of $\mathfrak{D}$.

Let f and g be two meromorphic functions defined in $\mathfrak{D}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g *share the value a IM* (*ignoring multiplicity*) if the a-points of f and g coincide in locations only, not necessarily in multiplicities.

For a meromorphic function f we denote by $f^{\#}$ the *spherical derivative* of f, given by

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Also, by Δ we denote the unit disc $|z| < 1$.

In 1992 W. Schwick [15] first established a connection between the normality and value sharing. He proved the following theorem.

Theorem A ([15]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and a_1, a_2, a_3 be distinct complex numbers. If for every $f \in \mathfrak{F}$, f and f' share a_1, a_2, a_3 IM in $\mathfrak{D}$ then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

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After the work of Schwick [15] it has become a popular problem to investigate the relation between normality and sharing values.

In 1999 Y. Xu [16] proved the following result.

Theorem B ([16]). *Let $\mathfrak{F}$ be a family of holomorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and b be a nonzero complex number. If f and f' share $0, b$ IM in $\mathfrak{D}$ for every $f \in \mathfrak{F}$ then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

In 2000 X. Pang and L. Zalcman [12] proved the following result, which improves Theorems A and B.

Theorem C ([12]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and a_1, a_2 be distinct complex numbers. If for every $f \in \mathfrak{F}$, f and f' share a_1, a_2 IM in $\mathfrak{D}$ then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

At this stage two natural questions may be asked:

1. What would be if f and f' share a single value?
2. What would be if f' is replaced by $f^{(k)}$?

For Question 1 the following result of W. C. Lin and H. X. Yi [11] may be noted.

Theorem D ([11]). *Let $\mathfrak{F}$ be a family of meromorphic functions in Δ. If there exist complex numbers a and b ($b \neq 0$ and a/b not a positive integer) such that for every $f \in \mathfrak{F}$, f and f' share a IM in Δ and $|f(z) - a| \geq \varepsilon$ whenever $f'(z) = b$, where ε is a positive number, then $\mathfrak{F}$ is normal in Δ.*

For Question 2, H. Chen and M. Fang [3] proved the following result.

Theorem E ([3]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$, $k \geq 2$ be an integer and a, b, c be complex numbers such that $b \neq a$. If for each $f \in \mathfrak{F}$, f and $f^{(k)}$ share a, b IM in $\mathfrak{D}$ and zeros of $f - c$ have multiplicity at least $1 + k$ then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

The following result of M. Fang and L. Zalcman [5] improved Theorem E.

Theorem F ([5]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$, $k \geq 2$ be an integer and a, b, c be complex numbers such that $b \neq a$. If for each $f \in \mathfrak{F}$, f and $f^{(k)}$ share a, b IM in $\mathfrak{D}$ and zeros of $f - c$ have multiplicity at least k then $\mathfrak{F}$ is normal in $\mathfrak{D}$.*

Theorem F is a consequence of the following theorem, also due to Fang and Zalcman [5].

Theorem G ([5]). *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$, k be a positive integer and a, b, c, d be complex numbers such that $b \neq a, 0$ and $c \neq 0$. If, for each $f \in \mathfrak{F}$, all zeros of $f - d$ have multiplicity at least k, f and $f^{(k)} - a$ share 0 IM and $f(z) = c$ whenever $f^{(k)}(z) = b$, then $\mathfrak{F}$ is normal in $\mathfrak{D}$ for $k \geq 2$, and for $k = 1$ so long as $a \neq (1 + m)b$, $m = 1, 2, \ldots$.*

In this paper we investigate the situation when the derivative is replaced by a linear differential polynomial with constant coefficients generated by f. Throughout the paper we denote by $H_k(f) = H_k(f; a_1, \dots, a_k)$ a linear differential polynomial generated by a meromorphic function f of the following form:

$$H_k(f) = H_k(f; a_1, \dots, a_k) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \dots + a_1 f^{(1)}$$

where k is a positive integer and $a_1, \dots, a_k \neq 0$ are constants.

We now state the main result of the paper.

THEOREM 1.1. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and a, b, c, d be finite complex numbers such that $c \neq 0$. If there exists a differential polynomial $H_k(f) = H_k(f; a_1, \dots, a_k)$ such that for each $f \in \mathfrak{F}$,*

(i) *$f - d$ does not have any zero with multiplicity less than k,*
(ii) *$f - a$ and $H_k(f) - b$ share the value 0 IM,*
(iii) *$|f(z) - a| \geq \varepsilon$ whenever $H_k(f) = c$, where ε is a positive number,*

then $\mathfrak{F}$ is normal in $\mathfrak{D}$ for $k \geq 2$, and for $k = 1$ so long as $b/c \neq 1 + m$ for any positive integer m.

The following example shows that condition (i) of Theorem 1.1 is essential.

EXAMPLE 1.1. Let $f_n(z) = ne^z - ne^{-z} + 1$ for $n = 1, 2, \dots$ and $\mathfrak{D} = \mathbb{C}$. We choose $k = 2$, $a = 1$, $b = 0$, $c = 1$ and $\varepsilon = 1$. Then for any given finite complex number d,

$$f_n(z) - d = \frac{ne^{2z} + (1-d)e^z - n}{e^z}$$

has only simple zeros in $\mathfrak{D}$ (except possibly for only one value of n for which $d = 1 \pm 2ni$). Also $f_n(z) - a$ and $f_n^{(2)}(z) - b$ share 0 IM and $|f_n(z) - a| = 2 > \varepsilon$ whenever $f_n^{(2)}(z) = c$. Since $f_n^{\#}(0) = n \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following example shows that condition (ii) of Theorem 1.1 is essential.

EXAMPLE 1.2. Let $f_n(z) = nz^2$ for $n = 1, 2, \dots$ and $\mathfrak{D} = \Delta$. We choose $k = 2$, $a = 0$, $b = 0$, $d = 0$ and $c = 1$. Then $f_n(z) - d$ has no zero of multiplicity less than k, $f_n^{(2)}(z) = 2n$ does not assume the value c, so that condition (iii) of Theorem 1.1 is satisfied but $f_n(z)$ and $f_n^{(2)}(z)$ do not share the value $a = b = 0$. Since $f_n(0) = 0$ for $n = 1, 2, \dots$ and for $z \neq 0$, $f_n(z) \to \infty$ as $n \to \infty$, it follows that the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following example shows that condition (iii) of Theorem 1.1 is essential.

EXAMPLE 1.3. Let $f(z) = e^{nz}$ for $n = 1, 2, \dots$ and $\mathfrak{D} = \Delta$. We choose $k = 2$, $a = 0$, $b = 0$, $c = 1$ and $d = 0$. Then conditions (i) and (ii) of Theorem 1.1 are satisfied. Also we see that $f_n^{(2)}(z) = c$ implies $|f_n(z) - a| = 1/n^2 \to 0$ as $n \to \infty$ so that we cannot find any $\varepsilon > 0$ for which condition (iii) is satisfied. Since $f_n^{\#}(0) = n/2 \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following example shows that the condition $c \neq 0$ cannot be removed from Theorem 1.1.

EXAMPLE 1.4. Let $f_n(z) = e^{nz} - a/n + a$ for $n = 1, 2, \dots$ and $\mathfrak{D} = \Delta$. Then f_n and $f_n^{(1)}$ share the value a IM. Also $f_n^{(1)}(z) \neq 0$ in $\mathfrak{D}$ so that condition (iii) of Theorem 1.1 is satisfied for $c = 0$. Since

$$f_n^{\#}(0) = \frac{n}{1 + |a/n + a|} \to \infty \quad \text{as } n \to \infty,$$

by Marty's criterion the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following example shows that for $k = 1$ the condition "$b/c \neq 1 + m$ for any positive integer m" of Theorem 1.1 is essential.

EXAMPLE 1.5. Let b and c be two nonzero numbers such that $b = (1 + m)c$, where m is a positive integer. Also let $\{\alpha_n\}$ be a sequence of numbers converging to 0 and $|\alpha_n| < 1$ for $n = 1, 2, \dots$. We suppose that $\mathfrak{D} = \Delta$ and, for $n = 1, 2, \dots$,

$$f_n(z) = c(z - \alpha_n) + \frac{A(\alpha_n)^m}{m(z - \alpha_n)^m},$$

where A is a nonzero constant. Then

$$f_n^{(1)}(z) = c - \frac{A(\alpha_n)^m}{(z - \alpha_n)^{m+1}}$$

so that $f_n^{(1)}(z)$ does not assume the value c and so condition (iii) of Theorem 1.1 is satisfied. Also

$$f_n(z) = \frac{mc(z - \alpha_n)^{m+1} + A(\alpha_n)^m}{m(z - \alpha_n)^m},$$

$$f_n^{(1)}(z) - b = -\frac{mc(z - \alpha_n)^{m+1} + A(\alpha_n)^m}{m(z - \alpha_n)^{m+1}}$$

so that f_n and $f_n^{(1)}$ share 0 IM. Again

$$\begin{aligned} f_n^{\#}(0) &= \frac{|c + (-1)^{m+2}/\alpha_n|}{1 + |-c\alpha_n + (-1)^m A/m|^2} \\ &\geq \frac{1/|\alpha_n| - |c|}{1 + \{|c|\,|\alpha_n| + |A|/m\}^2} \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Hence by Marty's criterion the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following corollary not only extends Theorem G to a linear differential polynomial but also removes the hypothesis $a \neq b$.

COROLLARY 1.1. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and a, b, c, d, α be finite complex numbers such that $b \neq 0$ and $c \neq \alpha$. If there exists a differential polynomial $H_k(f) = H_k(f; a_1, \ldots, a_k)$ such that for each $f \in \mathfrak{F}$,*

(i) *$f - d$ does not have any zero of multiplicity less than k,*
(ii) *$f - \alpha$ and $H_k(f) - a$ share the value 0 IM,*
(iii) *$f(z) = c$ whenever $H_k(f) = b$,*

then $\mathfrak{F}$ is normal in $\mathfrak{D}$ for $k \geq 2$, and for $k = 1$ so long as $a/b \neq 1 + m$ for any positive integer m.

REMARK 1.1. If we choose $a = b$ then from conditions (ii) and (iii) of Corollary 1.1 it is obvious that α and a are lacunary values of $f \in \mathfrak{F}$ and $H_k(f)$ respectively.

The following example shows that in Corollary 1.1 the condition $b \neq 0$ is essential.

EXAMPLE 1.6. Let $f_n(z) = e^{nz}$ for $n = 1, 2, \ldots$ and $\mathfrak{D} = \Delta$. We choose $\alpha = a = b = d = 0$. Then $f_n(z) - d$ does not have any zero and for any positive integer k, $f_n(z)$ and $f_n^{(k)}(z) - a$ share the value 0 IM. Since $f_n^{(k)}(z) \neq b$, it follows that condition (iii) of Corollary 1.1 is satisfied for any complex number c. Since $f_n^{\#}(0) = n/2 \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathfrak{D}$.

The following corollary improves Theorems C and F.

COROLLARY 1.2. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D} \subset \mathbb{C}$ and a, b, c be finite numbers such that $a \neq b$. If there exists a differential polynomial $H_k(f) = H_k(f; a_1, \ldots, a_k)$ such that for each $f \in \mathfrak{F}$,*

(i) *$f - c$ does not have any zero of multiplicity less than k,*
(ii) *f and $H_k(f)$ share the values a and b IM,*

then $\mathfrak{F}$ is normal in $\mathfrak{D}$.

For standard definitions and notations we refer to [7] and [14].

2. Lemmas. In this section we present some necessary lemmas.

LEMMA 2.1 ([13]). *Let $\mathfrak{F}$ be a family of meromorphic functions in Δ having no zero of multiplicity less than k. Suppose there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If $\mathfrak{F}$ is not normal in Δ then there exist, for each α $(0 \leq \alpha \leq k)$,*

(i) *a number r, $0 < r < 1$,*

(ii) *points* z_n, $|z_n| < r$,
(iii) *functions* $f_n \in \mathfrak{F}$ *and*
(iv) *positive numbers* ϱ_n, $\varrho_n \to 0$,

such that $g_n(\xi) = \varrho_n^{-\alpha} f_n(z_n + \varrho_n \xi) \to g(\xi)$ *spherically and locally uniformly to a nonconstant meromorphic function* g *in* $\mathbb{C}$, *all of whose zeros have multiplicity at least* k *and* $g^{\#}(\xi) \le g^{\#}(0) = kA + 1$. *Moreover the order of* g *is at most* 2.

LEMMA 2.2 ([5]). *Let* f *be a meromorphic function of finite order and* $a, b \ne 0$ *be distinct complex numbers and* $k \ge 2$ *be an integer. If* f *has no zero of multiplicity less than* k, f *and* $f^{(k)} - a$ *share the value* 0 *IM and* $f^{(k)}$ *does not assume the value* b, *then* f *is a constant.*

LEMMA 2.3 ([5, 8, 11]). *Let* f *be a nonconstant meromorphic function of finite order and let* $a, b \ne 0$ *be distinct complex numbers. If* f *and* $f^{(1)} - a$ *share the value* 0 *IM and* $f^{(1)}$ *does not assume the value* b *in* $\mathbb{C}$ *then*

$$f(z) = b(z-d) + \frac{A}{m(z-d)^m} \quad \text{and} \quad a = (1+m)b$$

for some $d \in \mathbb{C}$ *and some positive integer* m.

LEMMA 2.4 ([9]). *Let* f *be a nonconstant rational function, and* k *and* $\lambda \ge 2$ *be positive integers such that*

(i) f *has no zero of multiplicity less than* λ *and the number of zeros of* f (*counted with multiplicities*), *if there are any, is not less than* $1 + k$,
(ii) *if* f *has any pole then the number of poles of* f (*counted with multiplicities*) *is greater than* $k/(\lambda - 1)$.

Then for every complex number $a \ne 0, \infty$, *the function* $f^{(k)} + a$ *has at least one zero.*

LEMMA 2.5. *Let* f *be a nonconstant rational function having no zero and* k *be a positive integer. Then for every complex number* $a \ne 0, \infty$, *the function* $f^{(k)} + a$ *has at least one zero.*

Proof. Since f has no zero, choosing $\lambda = k + 2$ in Lemma 2.4 we obtain the result. ■

LEMMA 2.6 ([7, p. 60]). *Suppose that* f *is meromorphic and transcendental in* $\mathbb{C}$. *Then for any positive integer* k,

$$T(r,f) \le (2 + 1/k)N(r,0;f) + (2 + 2/k)\overline{N}(r,a;f^{(k)}) + S(r,f),$$

where $a \ne 0, \infty$ *is a complex number.*

LEMMA 2.7 ([2]). *Let* f *be a meromorphic function of finite order. If* f *has only finitely many critical values then it has only finitely many asymptotic values.*

Lemma 2.8 ([1]). *Let f be a transcendental meromorphic function such that $f(0) \neq \infty$ and let the set of finite critical and asymptotic values of f be bounded. Then there exists $R > 0$ such that*

$$|f'(z)| \geq \frac{|f(z)|}{2\pi|z|} \log \frac{|f(z)|}{R}$$

for all $z \in \mathbb{C} \setminus \{0\}$ which are not poles of f.

Lemma 2.9 ([6, 10]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and $k \geq 2$ be an integer. If f and $f^{(k)}$ do not assume the value 0 in $\mathbb{C}$ then either $f(z) = e^{Az+B}$ or $f(z) = (Az+B)^{-m}$, where $A \neq 0$ and B are constants and m is a positive integer.*

Lemma 2.10 ([4]). *Let f be a meromorphic function in $\mathbb{C}$. If there exists a constant $M > 0$ such that $f^{\#}(z) \leq M$ in $\mathbb{C}$ then the order of f is at most 2.*

Lemma 2.11 ([7, p. 57]). *Let f be a nonconstant meromorphic function in $\mathbb{C}$ and $H_k(f)$ be nonconstant. Then for any complex number $a \neq 0, \infty$,*

$$T(r,f) \leq \overline{N}(r,\infty;f) + N(r,0;f) + \overline{N}(r,a;H_k(f)) + S(r,f).$$

3. Proof of the theorem and corollaries

Proof of Theorem 1.1. Since normality is a local property, without loss of generality we may assume that $\mathfrak{D} = \Delta$. Also since $H_k(f-a) = H_k(f)$, we may additionally suppose that $a = 0$. First we suppose that $a_k = 1$. We now consider the following cases.

Case I. Let $k \geq 2$ and $d = 0$. Suppose that $\mathfrak{F}$ is not normal in Δ. Then by Lemma 2.1 for $\alpha = k$ we can find a sequence $\{z_n\}$ of points with $|z_n| < r$ $(0 < r < 1)$, a sequence of positive numbers $\varrho_n \to 0$ and a sequence $\{f_n\} \subset \mathfrak{F}$ of functions such that

$$g_n(\xi) = \varrho_n^{-k} f_n(z_n + \varrho_n \xi) \to g(\xi)$$

spherically and locally uniformly, where g is a nonconstant meromorphic function in $\mathbb{C}$ and g has no zero of multiplicity less than k. Also $g^{\#}(\xi) \leq g^{\#}(0) = k(A+1)+1$ and g is of order at most 2, where $A = \max\{|b|, |c|\}$.

We now verify that (I) g and $g^{(k)} - b$ share the value 0 IM, and that (II) $g^{(k)}$ does not assume the value c in $\mathbb{C}$.

Let $g(\xi_0) = 0$. Then by Hurwitz's theorem there exists a sequence $\xi_n \to \xi_0$ such that $g_n(\xi_n) = 0$ for all sufficiently large values of n. So for all sufficiently large values of n we get $f_n(z_n + \varrho_n \xi_n) = 0$, and so for all sufficiently large values of n, $H_k(f_n(z_n + \varrho_n \xi_n)) = b$. Hence

$$g_n^{(k)}(\xi_n) + a_{k-1}\varrho_n g_n^{(k-1)}(\xi_n) + \cdots + a_1 \varrho_n^{k-1} g_n^{(1)}(\xi_n) = b.$$

Letting $n \to \infty$ we obtain $g^{(k)}(\xi_0) = b$.

Next let $g^{(k)}(\eta_0)=b$. First we verify that $g^{(k)}(\xi)\not\equiv b$. If $g^{(k)}(\xi)\equiv b$ then g becomes a polynomial of degree at most k. Since g has no zero of multiplicity less than k and g is nonconstant, it follows that g is a polynomial of degree k and so it has a single zero of multiplicity k. Hence we can write

$$g(\xi)=\frac{b(\xi-\xi_1)^k}{k!}. \tag{3.1}$$

By a simple calculation we deduce from (3.1) that $g^{\#}(0)\le k/2$ if $|\xi_1|\ge 1$ and $g^{\#}(0)\le |b|$ if $|\xi_1|<1$. Therefore $g^{\#}(0)<k(|b|+1)+1$, which is a contradiction.

Since $g^{(k)}(\eta_0)=b$ and $g_n^{(k)}(\eta)+a_{k-1}\varrho_n g_n^{(k-1)}(\eta)+\cdots+a_1\varrho_n^{k-1}g_n^{(1)}(\eta)$ converges uniformly to $g^{(k)}(\eta)$ in some neighbourhood of η_0, by Hurwitz's theorem there exists a sequence $\eta_n\to\eta_0$ such that for all large values of n,

$$g_n^{(k)}(\eta_n)+a_{k-1}\varrho_n g_n^{(k-1)}(\eta_n)+\cdots+a_1\varrho_n^{k-1}g_n^{(1)}(\eta_n)=b$$

and so $H_k(f_n(z_n+\varrho_n\eta_n))=b$. Therefore for all sufficiently large values of n we get $f_n(z_n+\varrho_n\eta_n)=0$ and so $g_n(\eta_n)=0$. Letting $n\to\infty$ we obtain $g(\eta_0)=0$. Therefore (I) is verified.

Let $g^{(k)}(\zeta_0)=c$. Then as above we can show that $g^{(k)}(\zeta)\not\equiv c$. Since $g_n^{(k)}(\zeta)+a_{k-1}\varrho_n g_n^{(k-1)}(\zeta)+\cdots+a_1\varrho_n^{k-1}g_n^{(1)}(\zeta)$ converges uniformly to $g^{(k)}(\zeta)$ in some neighbourhood of ζ_0, by Hurwitz's theorem there exists a sequence $\zeta_n\to\zeta_0$ such that for all large values of n,

$$g_n^{(k)}(\zeta_n)+a_{k-1}\varrho_n g_n^{(k-1)}(\zeta_n)+\cdots+a_1\varrho_n^{k-1}g_n^{(1)}(\zeta_n)=c$$

and so $H_k(f_n(z_n+\varrho_n\zeta_n))=c$. Therefore $|f_n(z_n+\varrho_n\zeta_n)|\ge\varepsilon$ and so $|g_n(\zeta_n)|\ge\varepsilon/\varrho_n^k$ for all large values of n. This shows that $g(\zeta_0)=\infty$, which is a contradiction. So (II) is verified.

If $b\ne c$, by Lemma 2.2, g becomes a constant, which is impossible. Let $b=c$. Then from (I) and (II) we see that g does not assume the value 0 and $g^{(k)}$ does not assume the value $c\ne 0$. If g is transcendental, by Lemma 2.6 we get $T(r,g)=S(r,g)$, which is a contradiction. If g is rational, by Lemma 2.5, g becomes a constant, which is impossible. Therefore the family $\mathfrak{F}$ is normal.

Case II. Let $k\ge 2$ and $d\ne 0$. Suppose that $\mathfrak{F}_1=\{f-d : f\in\mathfrak{F}\}$. If $\mathfrak{F}_1$ is not normal in Δ, by Lemma 2.1 for $\alpha=0$ we can find a sequence $\{z_n\}$ of points with $|z_n|<r$ $(0<r<1)$, a sequence of positive numbers $\varrho_n\to 0$ and a sequence $\{f_n-d\}\subset\mathfrak{F}_1$ of functions such that

$$g_n(\xi)=f_n(z_n+\varrho_n\xi)-d\to g(\xi)$$

spherically and locally uniformly, where g is a nonconstant meromorphic function in $\mathbb{C}$ and g has no zero of multiplicity less then k. Further g is of order at most 2.

We now verify that (III) $g^{(k)}$ does not assume the value 0 in $\mathbb{C}$, and that (IV) $g+d$ does not assume the value 0 in $\mathbb{C}$.

Let $g^{(k)}(\xi_0) = 0$ for some $\xi_0 \in \mathbb{C}$. Also we see that $g^{(k)}(\xi) \not\equiv 0$, for otherwise g becomes a polynomial of degree less than k, which is impossible because g is nonconstant and does not have any zero of multiplicity less than k.

Since in a neighbourhood of ξ_0,

$$g_n^{(k)}(\xi) + a_{k-1}\varrho_n g_n^{(k-1)}(\xi) + \cdots + a_1\varrho_n^{k-1} g_n^{(1)}(\xi) - \varrho_n^k b$$

converges uniformly to $g^{(k)}(\xi)$, by Hurwitz's theorem there exists a sequence $\xi_n \to \xi_0$ such that for all large values of n,

$$g_n^{(k)}(\xi_n) + a_{k-1}\varrho_n g_n^{(k-1)}(\xi_n) + \cdots + a_1\varrho_n^{k-1} g_n^{(1)}(\xi_n) - \varrho_n^k b = 0,$$

and so for all large values of n we get $H_k(f_n(z_n + \varrho_n\xi_n)) = b$. Therefore for all large values of n we obtain $f_n(z_n + \varrho_n\xi_n) = 0$ and so $g_n(\xi_n) + d = 0$. Letting $n \to \infty$ we get

$$g(\xi_0) + d = 0. \tag{3.2}$$

Again since in a neighbourhood of ξ_0,

$$g_n^{(k)}(\xi) + a_{k-1}\varrho_n g_n^{(k-1)}(\xi) + \cdots + a_1\varrho_n^{k-1} g_n^{(1)}(\xi) - \varrho_n^k c$$

converges uniformly to $g^{(k)}(\xi)$, by Hurwitz's theorem there exists a sequence $\chi_n \to \xi_0$ such that

$$g_n^{(k)}(\chi_n) + a_{k-1}\varrho_n g_n^{(k-1)}(\chi_n) + \cdots + a_1\varrho_n^{k-1} g_n^{(1)}(\chi_n) - \varrho_n^k c = 0$$

for all large values of n. Hence for all large values of n we deduce that $H_k(f_n(z_n + \varrho_n\chi_n)) = c$. So for all large values of n,

$$|f_n(z_n + \varrho_n\chi_n)| \geq \varepsilon, \quad \text{i.e.,} \quad |g_n(\chi_n) + d| \geq \varepsilon.$$

Letting $n \to \infty$ we obtain $|g(\xi_0) + d| \geq \varepsilon$, which contradicts (3.2). Therefore (III) is verified.

Next let $g(\beta_0)+d = 0$. Then by Hurwitz's theorem there exists a sequence $\beta_n \to \beta_0$ such that for all large values of n, $f_n(z_n + \varrho_n\beta_n) - d = g_n(\beta_n) = -d$ and so $f_n(z_n + \varrho_n\beta_n) = 0$. Hence for all large values of n we deduce that $H_k(f_n(z_n + \varrho_n\beta_n)) = b$ and so

$$g_n^{(k)}(\beta_n) + a_{k-1}\varrho_n g_n^{(k-1)}(\beta_n) + \cdots + a_1\varrho_n^{k-1} g_n^{(1)}(\beta_n) = b\varrho_n^k.$$

Letting $n \to \infty$ we get $g^{(k)}(\beta_0) = 0$, which contradicts (III). Therefore (IV) is verified.

Now by Lemma 2.9 we see that either $g(\xi) = -d + e^{Az+B}$ or $g(\xi) = -d + 1/(Az+B)^m$. Since $d \neq 0$, it follows that g has only simple zeros, which is impossible. Therefore $\mathfrak{F}_1$ and so $\mathfrak{F}$ is normal.

Case III. Let $k = 1$. In this case condition (i) of the theorem is immaterial and so the proof does not depend on d. If $\mathfrak{F}$ is not normal in Δ, proceeding as Case I we can show that there exists a nonconstant meromorphic function g of finite order such that g and $g^{(1)} - b$ share the value 0 IM and $g^{(1)}$ does not assume the value c in $\mathbb{C}$.

If $b \neq c$ then by Lemma 2.3 we get $b = (1+m)c$ for some positive integer m, which is impossible. Let $b = c$. Then g does not assume the value 0 and $g^{(1)}$ does not assume the value c. If g is rational, by Lemma 2.5, g becomes a constant, which is impossible. If g is transcendental, by Lemma 2.6 we get $T(r,g) = S(r,g)$, which is a contradiction. Therefore the family $\mathfrak{F}$ is normal.

Finally, suppose that $a_k \neq 1$. We now put $G_k(f) = (1/a_k)H_k(f)$, $b_1 = b/a_k$ and $c_1 = c/a_k$. Then the leading coefficient of $G_k(f)$ is 1 and $b_1/c_1 = b/c$. Also the following hold:

(i) $f - d$ has no zero of multiplicity less than k,
(ii) $f - a$ and $G_k(f) - b_1$ share the value 0 IM,
(iii) $|f(z) - a| \geq \varepsilon$ whenever $G_k(f) = c_1$.

Therefore the family $\mathfrak{F}$ is normal in this case as well by the result for $a_k = 1$. This proves the theorem. ∎

Proof of Corollary 1.1. Since $c \neq \alpha$, we choose an ε such that $0 < \varepsilon < |c - \alpha|$. Then from condition (iii) we see that if $H_k(f) = b$ then $|f(z) - \alpha| = |c - \alpha| > \varepsilon$, which is condition (iii) of Theorem 1.1. Hence the corollary follows from Theorem 1.1. ∎

Proof of Corollary 1.2. Interchanging a and b if necessary, we may choose $|a| \leq |b|$. Since $a \neq b$, it follows that $b \neq 0$ and a/b is not a positive integer. We now choose an ε such that $0 < \varepsilon < |b - a|$. So we see that if $H_k(f) = b$ then $|f(z) - a| = |b - a| > \varepsilon$. Hence the corollary follows from Theorem 1.1. ∎

4. Application. In this section we prove a value distribution theorem for a differential polynomial which follows from Theorem 1.1.

THEOREM 4.1. *Let f be a transcendental meromorphic function and $a_1, \ldots, a_k \neq 0$ be constants such that $H_k(f^p) = H_k(f^p; a_1, \ldots, a_k)$ is also transcendental, where $p \geq 2$ is an integer. Let a be a finite complex number such that*

(i) *f has no zero of multiplicity less than k/p,*
(ii) *f and $H_k(f^p) - a$ share the value 0 IM.*

Then for every complex number $b \neq 0, \infty$, the function $H_k(f^p) - b$ has infinitely many zeros.

Proof. We consider the following cases.

CASE I. Let f be of infinite order. Then by Lemma 2.10 there exists a sequence $z_n \to \infty$ such that $f^{\#}(z_n) \to \infty$ as $n \to \infty$. Let $f_n(z) = f(z_n + z)$

for $n = 1, 2, \ldots$. Then $f_n^{\#}(0) = f^{\#}(z_n) \to \infty$ as $n \to \infty$. So by Marty's criterion no subfamily of $\{f_n\}$ is normal in Δ. Suppose that $H_k(f^p) - b$ has a finite number of zeros. Since $z_n \to \infty$ as $n \to \infty$, there exists a positive integer N such that for $n \geq N$, $H_k(f_n^p) - b$ has no zero in Δ. So by Theorem 1.1 the family $\{f_n : n \geq N\}$ is normal in Δ, which is a contradiction. Therefore $H_k(f^p) - b$ has infinitely many zeros.

Case II. Let f be of finite order. If f has only finitely many zeros, by Lemma 2.11 we get

$$T(r, f^p) \leq \overline{N}(r, \infty; f^p) + \overline{N}(r, b; H_k(f^p)) + S(r, f^p)$$

and so

$$(p-1)T(r, f) \leq \overline{N}(r, b; H_k(f^p)) + S(r, f),$$

which shows that $H_k(f^p) - b$ has infinitely many zeros.

Let f have infinitely many zeros, say $w_1, w_2, \ldots$. We put $g(z) = a_k h^{(k-1)}(z) + a_{k-1}h^{(k-2)}(z) + \cdots + a_1 h(z) - bz$, where $h(z) = \{f(z)\}^p$. Let $g'(z) = H_k(f^p) - b$ have only finitely many zeros. So g has only finitely many critical values and so, by Lemma 2.7, g has only finitely many asymptotic values. We assume, without loss of generality, that $g(0) \neq \infty$. Then by Lemma 2.8 there exists $R > 0$ such that for $n = 1, 2, \ldots$,

$$\left|\frac{w_n g'(w_n)}{g(w_n)}\right| \geq \frac{1}{2\pi} \log \frac{|g(w_n)|}{R} = \frac{1}{2\pi} \log \frac{|bw_n|}{R},$$

so that

$$\left|\frac{w_n g'(w_n)}{g(w_n)}\right| \to \infty \quad \text{as } n \to \infty.$$

On the other hand, for $n = 1, 2, \ldots$ we get

$$\left|\frac{w_n g'(w_n)}{g(w_n)}\right| = \frac{|a-b|}{|b|},$$

which is a contradiction. Therefore $H_k(f^p) - b$ has infinitely many zeros. This proves the theorem. ∎

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