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A comparative analysis of Bernstein type estimates for the derivative of multivariate polynomials

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Abstract. We compare the yields of two methods to obtain Bernstein type pointwise estimates for the derivative of a multivariate polynomial in a domain where the polynomial is assumed to have sup norm at most 1. One method, due to Sarantopoulos, relies on inscribing ellipses in a convex domain K. The other, pluripotential-theoretic approach, mainly due to Baran, works for even more general sets, and uses the pluricomplex Green function (the Zaharjuta–Siciak extremal function). When the inscribed ellipse method is applied on nonsymmetric convex domains, a key role is played by the generalized Minkowski functional $\alpha(K,x)$. With the aid of this functional, our current knowledge of the best constant in the multivariate Berstein inequality is precise within a constant $\sqrt{2}$ factor. Recently L. Milev and the author derived the exact yield of the inscribed ellipse method in the case of the simplex, and a number of numerical improvements were obtained compared to the general estimates known. Here we compare the yields of this real, geometric method and the results of the complex, pluripotential-theoretical approach in the case of the simplex. We observe a few remarkable facts, comment on the existing conjectures, and formulate a number of new hypotheses.

1. Introduction. If p is a univariate algebraic polynomial of degree at most n, then by the classical Bernstein–Szegő inequality ([41], [13], [9]) we have

$$|p'(x)| \le \frac{n\sqrt{\|p\|^2_{C[a,b]} - p^2(x)}}{\sqrt{(b-x)(x-a)}} \quad (a < x < b). \tag{1}$$

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This inequality is sharp for every n and every $x \in (a,b)$, as

$$\sup\left\{\frac{|p'(x)|}{\sqrt{\|p\|^2_{C[a,b]}-p^2(x)}} : \deg p \le n,\ |p(x)| < \|p\|_{C[a,b]}\right\} = \frac{n}{\sqrt{(b-x)(x-a)}}.$$

We may say that the upper estimate (1) is exact, and the right hand side is just the “true Bernstein factor” of the problem.

Polynomials and continuous polynomials are also defined on topological vector spaces X (see e.g. [14]). The set of continuous polynomials over X will be denoted by $\mathcal{P} = \mathcal{P}(X)$, and the polynomials in $\mathcal{P}$ with degree not exceeding n by $\mathcal{P}_n = \mathcal{P}_n(X)$.

In the multivariate setting a number of extensions were proved for the classical result (1). However, due to the geometric variety of possible convex sets replacing intervals of $\mathbb{R}$, our present knowledge is still not final. The exact Bernstein inequality is known only for symmetric convex bodies, and we are within a bound of some constant factor in the general, nonsymmetric case.

We may define formally, for any topological vector space X, a subset $K \subset X$, and a point $x \in K$, the nth *Bernstein factor* as

$$B_n(K,x) := \frac{1}{n}\sup\left\{\frac{\|Dp(x)\|}{\sqrt{\|p\|^2_{C(K)}-p^2(x)}} : \deg p \le n,\ |p(x)| < \|p\|_{C(K)}\right\}, \tag{2}$$

where $Dp(x)$ is the derivative of p at x, and for any unit vector $y \in X$,

$$B_n(K,x,y) := \frac{1}{n}\sup\left\{\frac{\langle Dp(x),y\rangle}{\sqrt{\|p\|^2_{C(K)}-p^2(x)}} : \deg p \le n,\ |p(x)| < \|p\|_{C(K)}\right\}, \tag{3}$$

where $\langle Dp(x),y\rangle$ is the directional derivative in direction y (which equals the value attained by the gradient, as a linear functional, at y).

Our aim is to investigate these and related quantities, and to analyze methods of estimating them.

2. The inscribed ellipse method of Sarantopoulos. Recall that a set $K \subset X$ is called a *convex body* in a normed space (or a topological vector space) X if it is a bounded, closed convex set with nonempty interior. The convex body K is *symmetric* if there exists a center of symmetry x so that reflection of K at x leaves the set invariant, that is, $K = -(K-x)+x = -K+2x$. We will call K *centrally symmetric* if it is symmetric with respect to the origin, i.e. $K = -K$. This occurs iff K can be considered the unit ball with respect to a norm $\|\cdot\|_{(K)}$, which is then equivalent to the original norm $\|\cdot\|$ of X in view of $B_{X,\|\cdot\|}(0,r) \subset K \subset B_{X,\|\cdot\|}(0,R)$.

The *maximal chord of* K *in direction* $v \neq 0$ is

$$(4)\quad \begin{aligned} \tau(K,v) &:= \sup\{\lambda \geq 0 : \exists y,\, z \in K \text{ such that } z = y + \lambda v\} \\ &= \sup\{\lambda \geq 0 : K \cap (K + \lambda v) \neq \emptyset\} \\ &= \sup\{\lambda \geq 0 : \lambda v \in K - K\} \\ &= 2\sup\{\lambda \geq 0 : \lambda v \in C\} \quad \text{where} \quad C := C(K) := \tfrac{1}{2}(K - K). \end{aligned}$$

Usually $\tau(K,v)$ is not a "maximal" chord length, but only a supremum. Nevertheless, we shall use the familiar finite-dimensional terminology (see for example [42]).

The *support function* to K, where K can be an arbitrary set, is defined for all $v^* \in X^*$ (sometimes only for $v^* \in S^* := \{v^* \in X^* : \|v^*\| = 1\}$) as

$$(5)\qquad h(K,v^*) := \sup_K v^* = \sup\{\langle v^*, x\rangle : x \in K\},$$

and the *width of* K *in direction* $v^* \in X^*$ (or $v^* \in S^*$) is

$$(6)\quad \begin{aligned} w(K,v^*) &:= h(K,v^*) + h(K,-v^*) = \sup_K v^* + \sup_K(-v^*) \\ &= \sup\{\langle v^*, x - y\rangle : x,y \in K\} = 2h(C,v^*) = w(C,v^*). \end{aligned}$$

Then the *minimal width* of K is $w(K) := \inf_{S^*} w(K,v^*)$ and the sharp inequalities

$$(7)\qquad w(K) \leq \tau(K,v) \leq \operatorname{diam} K, \quad w(K) \leq w(K,v^*) \leq \operatorname{diam} K$$

always hold, even in infinite-dimensional spaces (cf. [36, §2]).

In $\mathbb{R}$ the position of a point $x \in \mathbb{R}$ with respect to the "convex body" I can be expressed simply by $|x|$ (as $\pm x$ occupy symmetric positions). In the multivariate case the most frequent tool is the Minkowski functional. For any $x \in X$ the *Minkowski functional* or (*Minkowski*) *distance function* [16, p. 57] or *gauge* [33, p. 28] or *Minkowski gauge functional* [31, §1.1(d)] is defined as

$$(8)\qquad \varphi_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}.$$

Clearly (8) is a norm on X if and only if the convex body K is centrally symmetric with respect to the origin. In that case the norm $\|\cdot\|_{(K)} := \varphi_K$ can be used in approximation-theoretic questions as well. As said above, for $\|\cdot\|_{(K)}$ the unit ball of X will be K itself. In case K is nonsymmetric, the so-called *generalized Minkowski functional* $\alpha(K,x)$ emerged in the problem of quantitative description of the position of a point $x \in \mathbb{R}^d$ with respect to the convex body K. This notion also goes back to Minkowski [25] and Radon [32] (see also [15], [36]). There are several ways to introduce it; perhaps the shortest is the following. First let

$$\gamma(K,x) := \inf\left\{2\,\frac{\sqrt{\|x-a\|\,\|x-b\|}}{\|a-b\|} : a,b\in\partial K,\ x\in[a,b]\right\}. \tag{9}$$

Then we can set

$$\alpha(K,x) := \sqrt{1-\gamma^2(K,x)}. \tag{10}$$

In fact, the wide applicability of (10) stems from the fact that this geometric quantity incorporates quite nicely the geometric aspects of the configuration of x with respect to K, which is mirrored by about a dozen (!), sometimes strikingly different-looking, equivalent defnitions of $\alpha(K,x)$. For the above and many other equivalent formulations with full proofs, further geometric properties and some notes on the applications in approximation theory, see [36] and the references therein; for the first appearance of it in approximation-theoretic questions, see [37].

The *method of inscribed ellipses* was introduced by Y. Sarantopoulos [38]. It works for arbitrary interior points of any, possibly nonsymmetric convex body. The crux of the method is the following

Lemma 1 (Inscribed Ellipse Lemma, Sarantopoulos, 1991). *Let K be any subset in a vector space X. Suppose that $x\in K$ and the ellipse*

$$\mathbf{r}(t) = a\cos t + by\sin t + x - a \qquad (t\in[-\pi,\pi)). \tag{11}$$

lies inside K. Then for any polynomial p of degree at most n we have the Bernstein type inequality

$$|\langle Dp(x),y\rangle| \le \frac{n}{b}\sqrt{\|p\|^2_{C(K)} - p^2(x)}. \tag{12}$$

Theorem 1 (Sarantopoulos, 1991). *Let p be any polynomial of degree at most n over the normed space X. Then for any unit vector $y\in X$ we have the Bernstein type inequality*

$$|\langle Dp(x),y\rangle| \le \frac{n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{\sqrt{1-\|x\|^2_{(K)}}}. \tag{13}$$

Theorem 2 (Sarantopoulos, 1991). *Let K be a symmetric convex body and y a unit vector in the normed space X. Let p be any polynomial of degree at most n. Then*

$$|\langle Dp(x),y\rangle| \le \frac{2n\sqrt{\|p_n\|^2_{C(K)} - p^2(x)}}{\tau(K,y)\sqrt{1-\varphi^2(K,x)}}.$$

In particular,

$$\|Dp(x)\| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\varphi^2(K,x)}},$$

where $w(K)$ stands for the width of K.

The above solves the problem for the case of a symmetric convex body K. However, in the general, nonsymmetric case it can be rather difficult to determine or even estimate the b-parameter of the "best ellipse", which can be inscribed in a convex body K through $x \in K$ and be tangential to direction y. Still, we can formalize what we want to find.

DEFINITION 1 (Milev–Révész, 2003). For any $K \subset X$ and $x, y \in K$, the *best ellipse constant* is the extremal quantity

$$E(K,x,y) := \sup\{b : \mathbf{r} \subset K \text{ with } \mathbf{r} \text{ as given in (11)}\}. \tag{14}$$

Also, in [23] we defined

$$E(K,x) := \inf\{E(K,x,y) : y \in X,\ \|y\| = 1\}. \tag{15}$$

Clearly, the inscribed ellipse method yields Bernstein type estimates whenever we can derive some estimate of the ellipse constants. In the case of symmetric convex bodies, Sarantopoulos's Theorems 1 and 2 are sharp; for the nonsymmetric case we only know the following result.

THEOREM 3 (Kroó–Révész [20], 1998). *Let K be an arbitrary convex body in a normed space X, and let $x \in \operatorname{int} K$ and $\|y\| = 1$. Then*

$$|\langle Dp(x), y\rangle| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{\tau(K,y)\sqrt{1-\alpha(K,x)}} \tag{16}$$

for any polynomial p of degree at most n. Moreover,

$$\|Dp(x)\| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha(K,x)}} \le \frac{2\sqrt{2}n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}}. \tag{17}$$

Note that in [20] the best ellipse is not found; for most cases, the construction there only gives a good estimate, but not an exact value of (14) or (15). (In fact, here we have quoted [20] in a strengthened form: the original paper contains a somewhat weaker formulation.)

It is worth recalling here that geometrically the proof of (16) follows the following idea. To construct an ellipse through x, parallel to y there, and inscribed in K, it suffices to find the best such ellipse (i.e., of maximal possible b-parameter), which is inscribed in the quadrangle formed by the vertices of a *maximal chord in direction* y (or, in infinite dimensions, some chord ε-almost maximal in that direction), and the vertices of the *parallel chord through* x. That ellipse is precisely calculated, and its b-parameter is estimated independently of the location of these chords (even if they degenerate into one line, in which case the ellipse becomes a line segment). (In general the best b-parameter cannot be calculated, though.) We will recall this geometrical construction later.

One of the most intriguing questions in this area is the following conjecture, formulated first in [36].

Conjecture A (Révész–Sarantopoulos, 2001). *Let X be a topological vector space, and K be a convex body in X. For every $x \in \operatorname{int} K$ and every (bounded) polynomial p of degree at most n over X we have*

$$\|Dp(x)\| \le \frac{2n\sqrt{\|p\|^2_{C(K)} - p^2(x)}}{w(K)\sqrt{1-\alpha^2(K,x)}},$$

where $w(K)$ stands for the width of K.

3. Some results on the simplex. We denote by $|x|_2 := (\sum_{i=1}^d x_i^2)^{1/2}$ the Euclidean norm of $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let

$$\Delta := \Delta_d := \Big\{(x_1, \dots, x_d) : x_i \ge 0,\ i = 1, \dots, d,\ \sum_{i=1}^d x_i \le 1\Big\}$$

be the standard simplex in $\mathbb{R}^d$. For fixed $x \in \operatorname{int}\Delta$, and $y = (y_1, \dots, y_d)$, $|y|_2 = 1$, the best ellipse constant of Δ is, by Definition 1, $E(\Delta, x, y)$. By a tedious calculation via the Kuhn–Tucker theorem and some geometry, the following was obtained in [23].

Theorem 4 (Milev–Révész, 2003). *Let $p \in \mathcal{P}_n^d$. Then for every $x \in \operatorname{int}\Delta$ and $y \in \mathbb{S}^{d-1}$ we have*

$$|D_y p(x)| \le \frac{n\sqrt{\|p\|^2_{C(\Delta)} - p^2(x)}}{E(\Delta, x, y)}, \tag{18}$$

where

$$E(\Delta, x, y) = \left\{\frac{y_1^2}{x_1} + \dots + \frac{y_d^2}{x_d} + \frac{(y_1 + \dots + y_d)^2}{1 - x_1 - \dots - x_d}\right\}^{-1/2}. \tag{19}$$

Note that

$$\frac{1}{E(\Delta, x, y)} \le \frac{2}{\tau(\Delta, y)\sqrt{1-\alpha(\Delta, x)}} \tag{20}$$

for every $x \in \operatorname{int}\Delta$ and $y \in \mathbb{S}^1$, which is *not* accidental: the general estimate (16) must also be valid for Δ, and the precise value, calculated for Δ, can only be better. But equality occurs for some directions; we will return to this point soon.

From now on let us restrict ourselves to the case $d = 2$. We denote the vertices of Δ by $O = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$ and the centroid (i.e. mass point) of Δ by $M = (1/3, 1/3)$. It is calculated in [23] that

$$\alpha(\Delta, x) = 1 - 2r(x) \tag{21}$$

with

$$r := r(x) = \min\{x_1, x_2, 1 - x_1 - x_2\} = \begin{cases} x_1, & x \in \triangle OMB, \\ x_2, & x \in \triangle OMA, \\ 1 - x_1 - x_2, & x \in \triangle AMB, \end{cases}$$

and if $y = (\cos\varphi, \sin\varphi)$ $(0 \le \varphi \le \pi)$ then

$$\tau(\Delta, y) = \begin{cases} 1/(y_1 + y_2), & \varphi \in [0, \pi/2], \\ 1/y_2, & \varphi \in (\pi/2, 3\pi/4], \\ -1/y_1, & \varphi \in (3\pi/4, \pi]. \end{cases} \tag{22}$$

Then it can be calculated that we have equality in (20) exactly for the directions $y = (\cos\varphi, \sin\varphi)$ with $\varphi = 0, \pi/2, 3\pi/4 + \pi\mathbb{Z}$ and for some values of x.

Why is that so? For these and only these vectors, can we have a coincidence of the above geometrical figure, the quadrangle in the proof of (16), and the exact domain in which we must really inscribe the ellipse through x and parallel to y there; for all other directions the maximal chord in direction y lies strictly inside Δ, and another ellipse, slightly stretched behind that chord, can also be inscribed. Therefore, it is geometrically natural that nothing better can be obtained (than the ellipse calculated in Theorem 3) only for these directions, while for other directions precise calculation of the best ellipse must always yield a better ellipse constant.

Denote by $|Dp(x)|_2$ the Euclidean length of the gradient vector of p at x, also equal to the operator norm $\|Dp(x)\|$ with respect to the Euclidean norm. In [23] the following estimates were deduced from Theorem 4.

PROPOSITION 5 (Milev–Révész, 2003). *Let $p \in \mathcal{P}_n^2$. Then for every $x \in \operatorname{int} \Delta$ we have*

$$|Dp(x)|_2 \le \frac{n\sqrt{\|p\|^2_{C(\Delta)} - p^2(x)}}{E(\Delta, x)}, \tag{23}$$

where

$$E(\Delta, x) = \sqrt{\frac{2x_1x_2(1 - x_1 - x_2)}{x_1(1 - x_1) + x_2(1 - x_2) + D(x)}} \tag{24}$$

with

$$\begin{aligned} D(x) &:= \sqrt{[x_1(1 - x_1) + x_2(1 - x_2)]^2 - 4x_1x_2(1 - x_1 - x_2)} \\ &= \sqrt{[x_1(1 - x_1) - x_2(1 - x_2)]^2 + 4x_1^2x_2^2} > 0 \qquad (\forall x \in \operatorname{int} \Delta). \end{aligned} \tag{25}$$

From this the following improvements of Theorem 3 were achieved for the special case of $K = \Delta$.

Proposition 6 (Milev-Révész, 2003). *Let $p \in \mathcal{P}_n^2$ and $\|p\|_{C(\Delta)} = 1$. Then for every $x \in \operatorname{int} \Delta$ we have*

$$|Dp(x)|_2 \le \frac{\sqrt{3}\, n \sqrt{\|p\|^2_{C(\Delta)} - p^2(x)}}{w(\Delta)\sqrt{1-\alpha(\Delta,x)}}. \tag{26}$$

Furthermore, using the quantity $\sqrt{1-\alpha^2(\Delta,x)}$ on the right, we even have

$$|Dp(x)|_2 \le \frac{\sqrt{3+\sqrt{5}}\, n \sqrt{\|p\|^2_{C(\Delta)} - p^2(x)}}{w(\Delta)\sqrt{1-\alpha^2(\Delta,x)}}. \tag{27}$$

The result (27) improves the constant in Theorem 3 but falls short of yielding Conjecture A, since $2\sqrt{2} = 2.8284\ldots > \sqrt{3+\sqrt{5}} = 2.2882\ldots > 2$. On the way of proving these, it was noted that no better constants follow from the inscribed ellipse method, interpreted so that $E(K,x)$ is considered the yield of the ellipse method. We shall return to this subject later on.

4. Baran's pluripotential-theoretic method. Another method of considerable success in proving Bernstein and Markov type inequalities is the pluripotential-theoretic approach. Classically, all that was considered only in the finite-dimensional case, but nowadays even the normed spaces setting is cultivated. To explain the method, one needs an understanding of *complexifications of real normed spaces* (see e.g. [28, 6]), as well as the *Zaharjuta–Siciak extremal function* $V(z)$. We start with a formulation which is perhaps easier to digest. It is very much like the Chebyshev problem (cf. [36, §8]), except that we consider it all over the complexification $Y := X + iX$ of X, take logarithms, and after normalization by the degree, merge the information derived from all polynomials of any degree into one clustered quantity. Namely, for any bounded $E \subset Y$, V_E vanishes on E, while outside E we have the definition

$$V_E(z) := \sup\left\{\frac{1}{n}\log|p(z)| : 0 \ne p \in \mathcal{P}_n(Y),\ \|p\|_E \le 1,\ n \in \mathbb{N}\right\} \quad (z \notin E). \tag{28}$$

For $E \subset X$ one can easily restrict even to $p \in \mathcal{P}(X)$.

Note that $\log|p(z)|$ is a *plurisubharmonic* function (PSH, for short), as its one (complex) dimensional restrictions are just logarithms of univariate polynomials over $\mathbb{C}$. After normalization by the degree, $(1/n)\log|p(z)|$ has very regular growth towards infinity: it is at most $\log_+|z| + O(1)$. So it is reasonable to consider the Lelong class of *all* such functions:

$$\mathcal{L}(E) := \{u \in \mathrm{PSH} : u|_E \le 0,\ u(z) \le \log|z| + O(1)\ (|z| \to \infty)\} \tag{29}$$

and to define

$$U_E(z) := \sup\{u(z) : u \in \mathcal{L}(E)\}. \tag{30}$$

This function may be named the pluricomplex Green function. The Zaharjuta–Siciak theorem says that (30) and (28) are equal, at least as long as $E \subset \mathbb{C}^d$ is compact, which we now assume together with E being a *nonpluripolar* set. (A set $E \subset \mathbb{C}^d$ is pluripolar if there exists a PSH function vanishing on E; otherwise, the set is called nonpluripolar.) Then, being suprema of PSH functions (subharmonic functions on all complex "lines"), they are, modulo upper semicontinuous regularization, PSH themselves. They play a central role in the theory.

An extension of the Laplace and Poisson equations is the so-called complex Monge–Ampère equation, using the operator

$$(\partial\overline{\partial}u)^d := d!4^d \det\left[\frac{\partial^2 u}{\partial z_j \overline{\partial} z_k}(z)\right] dV(z), \tag{31}$$

where $dV(z) = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_d \wedge dy_d$ is just the usual volume element in $\mathbb{C}^d$. At first, the complex Monge–Ampère operator is applied only to smooth functions, $u \in \mathrm{PSH} \cap C^2$ say, but due to the work of Bedford and Taylor [7], the operator extends, in the appropriate sense, to the whole set of locally bounded PSH functions (which covers the case of the upper semicontinuous regularization V_E^* for any nonpluripolar E, see e.g. [19]). Therefore, it makes sense to consider

$$(\partial\overline{\partial}V_E^*)^d, \tag{32}$$

which is then a compactly supported measure λ_E and is called the *complex equilibrium measure* of the set E. It is shown [7] that in fact the support lies in the polynomial convex hull $\widehat{E}$ of E; in case E is convex, $\widehat{E} = E$ and $V_E^* = V_E$; moreover, this measure is normalized in a certain sense, as $\lambda|_E(\mathbb{C}^d) = \lambda|_E(\widehat{E}) = (2\pi)^d$.

For the theory of plurisubharmonic functions and some recent developments concerning Bernstein and Markov type inequalities for convex bodies or even more general sets, we refer to [1–8, 10, 19, 21, 22, 26, 30].

There are further yields of the theory of PSH functions, when applied to the Bernstein problem: here we present a few results of Mirosław Baran. For more precise notation we now introduce (interpreting $0/0$ as 0 here)

Definition 2.

$$G(E,x) := \left\{\frac{\operatorname{grad} p(x)}{n\sqrt{\|p\|^2 - p(x)^2}} : \mathbf{0} \neq p \in \mathcal{P}_n,\ n \in \mathbb{N}\right\}, \tag{33}$$

and following Baran we also consider

$$\widetilde{G}(E,x) := \operatorname{con} G(E,x). \tag{34}$$

Clearly $\sup_{n\in\mathbb{N}} B_n(E,x) = \sup_{u\in G(E,x)} \|u\|$ for any compact $E \subset \mathbb{R}^d$.

THEOREM 7 (Baran, 1995). *Let E be a compact subset of $\mathbb{R}^d$ with nonempty interior. Then the equilibrium measure $\lambda|_E$ is absolutely continuous in the interior of E with respect to the Lebesgue measure of $\mathbb{R}^d$. Denote its density function by $\lambda(x)$ for all $x \in \operatorname{int} E$. Then $(1/d!)\lambda(x) \geq \operatorname{vol} \widetilde{G}(E,x)$ for a.a. $x \in \operatorname{int} E$. Moreover, if E is a symmetric convex domain of $\mathbb{R}^d$, then $(1/d!)\lambda(x) = \operatorname{vol} \widetilde{G}(E,x)$ for a.a. $x \in \operatorname{int} E$.*

CONJECTURE B (Baran, 1995). *We have $(1/d!)\lambda(x) = \operatorname{vol} \widetilde{G}(E,x)$ even if E is a nonsymmetric convex body in $\mathbb{R}^d$.*

Now consider $E = K \subset X$, where K is now a convex body. Our more precise results in [35] (see also [36, §8]) yield

$$V_K(x) = \log(\alpha(K,x) + \sqrt{\alpha(K,x)^2 - 1}).$$

However, in the Bernstein problem the values of V_K are much more of interest for *complex* points $z = x + iy$, in particular for $x \in K$ and y small and nonzero. More precisely, the important quantity is the normal (sub)derivative

$$D_y^+ V_E(x) := \liminf_{\varepsilon\to 0} \frac{V_E(x+i\varepsilon y)}{\varepsilon}, \tag{35}$$

as this quantity occurs in the following estimate of the directional derivative and thus also in the gradient.

THEOREM 8 (Baran, 1994 & 2004). *Let $E \subset X$ be any bounded, closed set, $x \in \operatorname{int} E$ and $0 \neq y \in X$. Then for all $p \in \mathcal{P}_n(X)$ we have*

$$|\langle Dp(x), y\rangle| \leq n D_y^+ V_E(x) \sqrt{\|p\|_E^2 - p(x)^2}. \tag{36}$$

Proof. For $\mathbb{R}^d$ and partial derivatives this is contained in [3]; the case of infinite-dimensional spaces is considered in [6], but only for symmetric convex bodies. The same estimate occurs, without proof but with reference to Baran, in the recent publication [11]. For arbitrary directions $y \in \mathbb{R}^d$ one can consider a rotation $A : \mathbb{R}^d \to \mathbb{R}^d$. ∎

It is not obvious how such theoretical estimates can be applied to concrete cases. First, one has to find the value of V_E precisely enough to be able to compute even its derivative. Only then do we really have something. However, even that is addressed by considering the Bedford–Taylor theory of the Monge–Ampère equation and the equilibrium measure [7], as the density of the equilibrium measure gives the extremal function. In some concrete applications all that may be calculated, a particular example (see [5, Example 4.8]) being the following.

PROPOSITION 9 (Baran, 1995). *The extremal function of the standard simplex in* $\mathbb{R}^d$ *is*

$$V_\Delta(z) = \log|h(|z_1| + \cdots + |z_n| + |1 - (z_1 + \cdots + z_n)|)|.$$

Here $h(z) := z + \sqrt{z^2 - 1}$ *is inverse to the Joukowski mapping* $\zeta \mapsto (1/2)(\zeta + 1/\zeta)$, *with the choice of the square root that is positive for positive* z *exceeding* 1, *so that* h *maps to the exterior of the unit disk.*

From this and the calculation with the rotated directions above, we can deduce (1)

PROPOSITION 10. *For the standard simplex* Δ *of* $\mathbb{R}^d$, *any unit vector* $y = (y_1, \ldots, y_n)$ *and any* $x = (x_1, \ldots, x_n) \in \operatorname{int} \Delta$ *we have the formula*

$$D_y^+ V_\Delta(x) = \sqrt{\frac{y_1^2}{x_1} + \cdots + \frac{y_n^2}{x_n} + \frac{(y_1 + \cdots + y_n)^2}{1 - (x_1 + \cdots + x_n)}}. \tag{37}$$

Hence we are led to the following surprising corollary.

COROLLARY 11. *The pluripotential-theoretic estimate* (36) *of Baran*, *calculated for the standard simplex of* $\mathbb{R}^d$ *in* (37), *gives the result exactly identical to* (18), *obtained from the inscribed ellipse method.*

Much remains to be explained in this striking coincidence, the first thing being

HYPOTHESIS A. *Let* $K \subset X$ *be a convex body. Then for all points* $x \in \operatorname{int} K$ *the inscribed ellipse method and the pluripotential-theoretic method of Baran results in exactly the same estimate*, *i.e. for all* $y \in S^*$ *we have*

$$D_y^+ V_K(x) = \frac{1}{E(K, x, y)}. \tag{38}$$

5. Further geometric calculations. At this point it seems worth formulating a few naturally occurring assumptions.

HYPOTHESIS B. *Let* $K \subset X$ *be a convex body. Then for all* $x \in \operatorname{int} K$ *the exact Bernstein factor is just what results from the pluripotential-theoretic method of Baran*:

$$B_n(K, x) = \sup_{y \in S^+} D_y^+ V_K(x). \tag{39}$$

HYPOTHESIS C. *Let* $K \subset X$ *be a convex body. Then for all* $x \in \operatorname{int} K$ *the exact Bernstein factor is just what results from the inscribed ellipse method of Sarantopoulos*:

$$B_n(K, x) = \frac{1}{E(K, x)}. \tag{40}$$

(1) The same formula is mentioned in [11, p. 145].

These hypotheses are certainly not true for the directional derivatives in *all* directions $y \in S^*$, where both methods can be improved upon for some y, as is seen below. Care has to be exercised in formulating conjectures and hypotheses in these matters: the situation is more complex than one might like to have, and the simple heuristics of extending the results of the symmetric case sometimes fails. In this respect see [12, 21, 22] and [11], where another case of deviation from symmetric case extension is observed for the so-called "Baran metric" on the simplex.

There is an important and immediate observation we have not utilized yet. Namely, we have exhibited methods (actually, two equivalently strong ones) to estimate $D_y p(x)$. However, if we are looking for the total derivative $\operatorname{grad} p(x)$, then the estimate we used was only the trivial $\|\operatorname{grad} p(x)\| \le \sup_{y\in S^*} |D_y p(x)|$. Can we do any better? Yes, we can, depending on the estimating functions we have for $D_y p(x)$.

Consider e.g. the estimates from Theorem 3, which was obtained also for the simplex and thus the triangle Δ. For the triangle we have an explicit computation of the maximal chords $\tau(\Delta, x)$ (cf. (22)), and also of the generalized Minkowski functional $\alpha(\Delta, x)$ (see (21)), so everything is explicit and we can compute the estimating functions. As an example, consider e.g. the point $M := (1/3, 1/3)$ and compute all quantities involved in the normalization of the directional derivative estimates. As a result, we can exactly determine the arising domain $H(\Delta, M)$, where in general we write

$$H := H(K,x) := \{v = ty : y = (y_1, \dots, y_d),\ |t| \le r(y)\} \tag{41}$$

with $r(y)$ being the available normalized estimate for the directional derivative in direction y.

It turns out that the domain $H(\Delta, M)$ described by the general estimates of Theorem 3 is a fleecy-cloud like domain which is symmetric with respect to the origin, and its upper half is (the part above the x-axis of) the union of three disks: $D((\sqrt{3/2}, \sqrt{3/2}), \sqrt{3}) \cup D((0, \sqrt{3/2}), \sqrt{3/2}) \cup D((-\sqrt{3/2}, 0), \sqrt{3/2})$. (Here the reader may wish to draw a figure for better visualization.) An immediate observation is that the domain is *not convex*, and so this is certainly not an exact description of all possible directional derivatives of the gradient.

We can conclude that if some domain (41) is given with $r(y)$ being some normalized estimate for the directional derivative in direction y, then to bound $G(K,x)$ an additional process of restricting to the "kernel" part

$$\widetilde{H} := \widetilde{H}(K,x) := \bigcap_{y\in S^*} \{v : |\langle v, y\rangle| \le r(y)\} \tag{42}$$

is available. That is, we always have $\widetilde{G}(K,x) \subset \widetilde{H}$. Note that $\widetilde{H}$ is a convex, symmetric domain for any point set H.

In order to illustrate this "kernel technique", let us come back to the above case of estimates from Theorem 3 for the triangle at point M. After some standard considerations with Thales circles we find that $\widetilde{H}$ is the hexagonal domain

$$\widetilde{H}(\Delta, M) \\ = \mathrm{con}\{(\sqrt{6},0),(\sqrt{6},\sqrt{6}),(0,\sqrt{6}),(-\sqrt{6},0),(-\sqrt{6},-\sqrt{6}),(0,-\sqrt{6})\}.$$

Observe that the area of the possible stretch of G is considerably reduced from the "fleecy-cloud" domain to the derived hexagonal domain as

$$\operatorname{area} H(\Delta, M) = 9 + \frac{9}{2\pi} = 23.137\ldots$$

while area $\widetilde{H}(\Delta, M) = 18$. For comparison recall that Baran's Conjecture B would say that the area should be $\frac{1}{2}\lambda_\Delta(M) = \pi/\sqrt{3^{-3}} = 16.324\ldots$.

Let us calculate the "kernel set" $\widetilde{H}(\Delta, x)$ from the exact estimates (18), (36), (37) which we obtain from the ellipse (and hence also from Baran's) method. We obtain the following ([2]).

Proposition 12. *With the above notations*, $\widetilde{H}(\Delta, x)$ *is an ellipse domain. Moreover*, *its major axis* $\mu := \mu(x)$ *and minor axis* $\nu := \nu(x)$ *are given by*

$$\begin{aligned} \mu &= \sqrt{\frac{2}{x_1(1-x_1)+x_2(1-x_2)+D(x)}}, \\ \nu &= \sqrt{\frac{2}{x_1(1-x_1)+x_2(1-x_2)-D(x)}}, \end{aligned} \tag{43}$$

where $D(x)$ *is the quantity defined in* (25).

Proof. For fixed $x \in \Delta$ we are to describe the solution set (42) for $K = \Delta$, with $r(y)$ being the quantity (19). That is, we determine all those vectors $u = (u_1, u_2) \in \mathbb{R}^2$ which satisfy $|\langle u, y\rangle| \le 1/E(\Delta, x, y)$ for all $y = (\cos\varphi, \sin\varphi)$. Using (19) and squaring, we see that the defining inequalities describe the set

$$\Big\{u :\ (u_1\cos\varphi + u_2\sin\varphi)^2 \le \frac{\cos^2\varphi}{x_1} + \frac{\sin^2\varphi}{x_2} \\ + \frac{(\cos\varphi+\sin\varphi)^2}{1-x_1-x_2}\ (\forall \varphi \in \mathbb{R})\Big\}. \tag{44}$$

([2]) These computations were executed jointly with Nikola Naidenov from the University of Sofia during the author's stay in Sofia in October 2004. The author regrets that in spite of his undoubted contribution [29] to this work, Nikola Naidenov chose not to be named as a coauthor.

Putting $x_3 := 1 - x_1 - x_2$, the case of $\cos^2\varphi > 0$ yields

$$(u_1 + u_2 t)^2 \le \frac{1}{x_1} + \frac{t^2}{x_2} + \frac{(1+t)^2}{x_3} \quad (\forall t := \tan\varphi \in \mathbb{R}), \tag{45}$$

which is a second degree inequality in t. Solving it we arrive at

$$a u_1^2 + b u_2^2 - c u_1 u_2 \le 1, \tag{46}$$

where the coefficients are all strictly positive and have the form

$$\begin{aligned} &a := a(x) := x_1(1-x_1), \quad b := b(x) := x_2(1-x_2), \\ &c := c(x) := 2x_1x_2. \end{aligned} \tag{47}$$

Thus (46) determines an ellipse domain, and calculation of its axes leads to the result. ∎

So we are led to the following result.

THEOREM 13. *With the above notations, we have*

$$\operatorname{area} \widetilde{H}(\Delta, x) = \frac{\pi}{\sqrt{x_1 x_2 (1 - x_1 - x_2)}}.$$

Proof. As is well known, the area of an ellipse domain with axes μ and ν is $\pi\mu\nu$, hence Proposition 12 leads to the asserted value. ∎

COROLLARY 14. *We have* $G(x) \subseteq \operatorname{con} G(x) \subseteq \widetilde{H}(x)$ *with* $\operatorname{area} \widetilde{H}(x) = \frac{1}{2}\lambda(x)$. *Hence either* $\operatorname{con} G(x) = \widetilde{H}(x)$ *for all* $x \in \Delta$, *or Baran's Conjecture* B *fails.*

Proof. One must compute the density function $\lambda(x)$ of the equilibrium measure. This has already been done by Baran, [5, Example 4.8]: we have $\lambda(x) = 2\pi/\sqrt{x_1 x_2 (1 - x_1 - x_2)}$. On comparing to Theorem 13 we find the asserted identity. Since $\widetilde{H}$ is an ellipse domain and also $\operatorname{con} G$ is a convex domain, the inclusion $\operatorname{con} G(x) \subset \widetilde{H}(x)$ and equality of their areas entails that $\operatorname{con} G(x) = \widetilde{H}(x)$. On the other hand, if at some point $x \in \Delta$ the respective areas differ, then $\operatorname{area} \operatorname{con} G(x) < \operatorname{area} \widetilde{H}(x) = \frac{1}{2}\lambda(x)$, hence the conjectured identity of Baran fails. ∎

REMARK 1. While using the information on the support functional from $H(\Delta, x)$ improves upon the known area estimates, it does not improve the maximal gradient norm estimate of [23].

Indeed, as $\widetilde{H}(\Delta, x)$ is an ellipse domain, we have to consider its major axis. It turns out that in the case of the standard triangle, this calculation yields $\max_{v \in \widetilde{H}} \|v\| = \max_{v \in H} \|v\| = 1/E(\Delta, x)$.

Note that $\max_{v \in V} \|v\| = \max_{v \in \operatorname{con} V} \|v\|$ for any set V, hence regarding the maximal gradient norm estimate it makes no difference whether we consider $\operatorname{con} G(x)$ or $G(x)$ only. Also note that starting from a set $H \supset G$

and considering the "kernel" $\widetilde{H}$, we necessarily obtain a convex set, so from $G \subset \widetilde{H}$ it follows that even taking the convex hull we still have $\operatorname{con} G \subset \widetilde{H}$.

COROLLARY 15. *Conjectures* A *and* B *cannot hold simultaneously.*

Proof. According to Corollary 14, Baran's Conjecture B holds if only there can be no improvement on the estimates of the ellipse (or Baran's) method on the simplex. But then Conjecture A fails. Conversely, if Conjecture A holds, then there is an improvement at least at certain points and in certain directions compared to the estimates of the ellipse (or Baran's) method, hence the estimates of Corollary 14 strictly exceed the right value and Baran's Conjecture B fails. ■

6. Concluding remarks. Also, another real, geometric method of obtaining Bernstein type inequalities, due to Skalyga [39, 40], should be mentioned here; the difficulty with it is that to the best of our knowledge, no one has ever been able to compute, neither for the seemingly least complicated case of the standard triangle of $\mathbb{R}^2$, nor in any other particular nonsymmetric case, the yield of that abstract method. Hence in spite of some remarks that the method is sharp in some sense, it is unclear how close these estimates are to the right answer and of what use they can be in any concrete cases.

Given the above findings, it seems plausible that Conjecture A, if not true, can be disproved by some explicit example. To construct a polynomial with large gradient, as compared to the norm, means to construct a highly oscillating polynomial. For that, various natural and more intricate ideas were tried by Nikola Naidenov [29] in Sofia during the Fall of 2004. We hope he will report on his experiences in the near future.

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