ANNALES
POLONICI MATHEMATICI
88.3 (2006)

The comparison principle and Dirichlet problem in the class $\mathcal{E}_p(f)$, $p > 0$

by Pham Hoang Hiep (Hanoi)

Abstract. We establish the comparison principle in the class $\mathcal{E}_p(f)$. The result obtained is applied to the Dirichlet problem in $\mathcal{E}_p(f)$.

1. Introduction. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. We denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic (psh) functions on Ω. In [BT1,2] the authors established and used the comparison principle to study the Dirichlet problem in $\mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$. Recently, Cegrell introduced a general class $\mathcal{E}$ of psh functions on which the complex Monge–Ampère operator $(dd^c\cdot)^n$ can be defined. He obtained many important results of pluripotential theory in the class $\mathcal{E}$, for example, the comparison principle and solvability of the Dirichlet problem (see [Ce1–3]). In [H], the author proved the comparison principle in the class $\mathcal{F}$.

The aim of the present paper is to continue the study of the class $\mathcal{E}_p(f)$. In Section 3 we prove a comparison principle of the Xing type in the class $\mathcal{E}_p(f)$, $p > 0$. This is aplied to the Dirichlet problem in $\mathcal{E}_p(f)$. In particular, in Section 4, we prove that for a positive measure μ on Ω the equation $(dd^c u)^n = \mu$ has a solution in $\mathcal{E}_p(f)$ if and only if $\mathcal{E}_p(\Omega) \subset L_p(\Omega, \mu)$.

Acknowledgments. The author is grateful to Professor Nguyen Van Khue for suggesting the problem and for many helpful discussions during the preparation of this work. The author is also indebted to the referee for his useful comments that helped to improve the paper.

2. Preliminaries. First we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [BT1,2], [Ce1–3].

2000 *Mathematics Subject Classification*: Primary 32W20; Secondary 32U05.
Key words and phrases: complex Monge–Ampère operator, plurisubharmonic function.
This work was supported by the National Research Program for Natural Sciences, Vietnam.

2.1. Unless otherwise specified, Ω will be a bounded hyperconvex domain in $\mathbb{C}^n$, meaning that there exists a negative exhaustive psh function for Ω.

2.2. Let Ω be a bounded domain in $\mathbb{C}^n$. The C_n-*capacity* in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E, \Omega) = \sup\Big\{ \int_E (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0 \Big\}$$

for every Borel set E in Ω. It is known [BT2] that

$$C_n(E) = \int_\Omega (dd^c h^*_{E,\Omega})^n$$

where $h^*_{E,\Omega}$ is the relative extremal psh function for E (relative to Ω) defined as the smallest upper semicontinuous majorant of $h_{E,\Omega}$,

$$h_{E,\Omega}(z) = \sup\{u(z) : u \in \mathrm{PSH}(\Omega),\ -1 \le u \le 0,\ u \le -1 \text{ on } E\}.$$

The following definition was introduced in [Xi]: A sequence $u_j \in \mathrm{PSH}^-(\Omega)$ *converges* to u *in* C_n-*capacity* if

$$C_n(K \cap \{|u_j - u| > \delta\}) \to 0, \quad j \to \infty, \quad \forall K \subset\subset \Omega,\ \delta > 0.$$

2.3. The following classes of psh functions were introduced by Cegrell in [Ce1,2]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega) : \lim_{z \to \partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c \varphi)^n < \infty\Big\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \forall z_0 \in \Omega\ \exists \text{ a neighbourhood } \omega \ni z_0,$$
$$\exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi \text{ on } \omega,\ \sup_{j \ge 1} \int_\Omega (dd^c \varphi_j)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j \ge 1} \int_\Omega (dd^c \varphi_j)^n < \infty\Big\},$$

$$\mathcal{E}_p = \mathcal{E}_p(\Omega) = \Big\{\varphi \in \mathrm{PSH}(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j \ge 1} \int_\Omega (-\varphi_j)^p (dd^c \varphi_j)^n < \infty\Big\},$$

$$\mathcal{F}_p = \mathcal{F}_p(\Omega) = \Big\{\varphi \in \mathcal{E}_p(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j \ge 1} \int_\Omega (dd^c \varphi_j)^n < \infty\Big\}.$$

2.4. Let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. Recall that the *Perron–Bremermann envelope* of f is defined by

$$U(0, f)(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}(\Omega),\ \overline{\lim_{w \to \xi}}\, \varphi(w) \le f(\xi)\ \forall \xi \in \partial\Omega\}.$$

A plurisubharmonic function u defined on Ω belongs to the class $\mathcal{E}_p(f)$ if there exists a function $\varphi \in \mathcal{E}_p$ such that

$$\varphi + U(0, f) \le u \le U(0, f).$$

Next we introduce some results needed for our paper:

2.5. PROPOSITION. *Let $u_j \in \mathrm{PSH}^-(\Omega)$ be such that u_j is increasing a.e. with respect to the Lebesgue measure to some $u \in \mathrm{PSH}^-(\Omega)$. Then $u_j \to u$ in C_n-capacity as $j \to \infty$.*

Proof. Let $K \subset\subset \Omega$ and $\delta, \varepsilon > 0$. By [BT1,2] we can choose $t > 0$ such that

$$C_n(K \cap \{u_1 < -t\}) < \varepsilon.$$

By Proposition 2.5 in [Cz] there exists j_0 such that

$$C_n(K \cap \{|\max(u_j, -t) - \max(u, -t)| > \delta\}) < \varepsilon, \quad \forall j \ge j_0.$$

For each $j \ge j_0$, we have

$$\begin{aligned} C_n(K \cap \{|u_j - u| > \delta\}) &\le C_n(K \cap \{|\max(u_j, -t) - \max(u, -t)| > \delta\}) \\ &\quad + C_n(K \cap \{u_j < -t\}) + C_n(K \cap \{u < -t\}) \\ &\le C_n(K \cap \{|\max(u_j, -t) - \max(u, -t)| > \delta\}) \\ &\quad + 2C_n(K \cap \{u_1 < -t\}) \\ &\le 3\varepsilon. \end{aligned}$$

2.6. PROPOSITION. *Let $u_j \in \mathcal{E}$ be such that u_j is increasing a.e. with respect to the Lebesgue measure to some $u \in \mathcal{E}$. Then $(dd^c u_j)^n \to (dd^c u)^n$ weakly as $j \to \infty$.*

Proof. Let $D \subset\subset \Omega$. By the remark after Definition 4.6 in [Ce2] we can find $v \in \mathcal{F}$ such that $v|_D = u_1|_D$. We set

$$\widetilde{u}_j = \max(u_j, v), \quad \widetilde{u} = \max(u, v).$$

We have $\mathcal{F} \ni \widetilde{u}_j \nearrow \widetilde{u} \in \mathcal{F}$ and $\widetilde{u}_j|_D = u_j|_D$, $\widetilde{u}|_D = u|_D$. By Proposition 2.5 and Theorem 1.1 in [Ce4] we have $(dd^c \widetilde{u}_j)^n \to (dd^c \widetilde{u})^n$ weakly as $j \to \infty$. Hence $(dd^c u_j)^n \to (dd^c u)^n$ weakly as $j \to \infty$.

2.7. PROPOSITION. *Let $u \in \mathcal{E}$ be such that*

$$s^n C_n(\{u < -s\}) \to 0 \quad \textit{as } s \to \infty.$$

Then $(dd^c u)^n$ is locally absolutely continuous with respect to C_n-capacity.

Proof. Let $D \subset\subset \Omega$. By the remark following Definition 4.6 in [Ce2] we can choose $v \in \mathcal{F}$ such that $v = u$ on D and $v \ge u$ on Ω. We have

$$s^n C_n(\{v < -s\}) \le s^n C_n(\{u < -s\}) \to 0 \quad \text{as } s \to \infty.$$

By Proposition 3.4 in [CKZ], $(dd^c v)^n$ is absolutely continuous with respect to C_n-capacity. Therefore, $(dd^c u)^n$ is locally absolutely continuous with respect to C_n-capacity.

2.8. PROPOSITION. *Let $u \in \mathcal{E}_p(f)$. Then $(dd^c u)^n$ is locally absolutely continuous with respect to C_n-capacity.*

Proof. We can assume that $0 \le f \le 1$. By the definition of $\mathcal{E}_p(f)$, there exists a function $\varphi \in \mathcal{E}_p$ such that

$$\varphi + U(0,f) \le u \le U(0,f).$$

We set $v = u - 1 \in \mathcal{E}$. By Proposition 3.1 in [CKZ] we have

$$s^n C_n(\{v < -s\}) \le s^n C_n(\{\varphi < -s+1\}) \le c_{n,p} e_p(\varphi) \frac{s^n}{(s-1)^{n+p}} \to 0$$

as $s \to \infty$. Using Proposition 2.7 we conclude that $(dd^c u)^n = (dd^c v)^n$ is locally absolutely continuous with respect to C_n-capacity.

2.9. THEOREM. *Let $u, v \in \mathcal{E}_p$ be such that $(dd^c u)^n \le (dd^c v)^n$. Then $u \ge v$.*

Proof. See the proof of Theorem 6.2 in [Ce1] for $p \ge 1$ and Theorem 4.2 in [CHÅ] for $0 < p < 1$.

2.10. THEOREM. *Let $u, v \in \mathcal{E}_p$. Then*

$$\frac{1}{n!} \int_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u<v\}} (r - w_1)(dd^c v)^n \le \int_{\{u<v\}} (r - w_1)(dd^c u)^n$$

for all $w_j \in \mathrm{PSH}(\Omega)$, $0 \le w_j \le 1$, $j = 1, \ldots, n$ and all $r \ge 1$.

Proof. Use Theorem 2.9 and Proposition 4.7 of [KH].

The following theorem was proved by Persson [Per] for $p \ge 1$ and in [CHÅ] for $0 < p < 1$.

2.11. THEOREM. *Let $u_0, u_1, \ldots, u_n$ be functions in $\mathrm{PSH} \cap L^\infty(\Omega)$ such that $\lim_{z\to\partial\Omega} u_j(z) = 0$ for $j = 0, 1, \ldots, n$. Then*

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \le C_{p,n} \Big[\int_\Omega (-u_0)^p (dd^c u_0)^n\Big]^{p/(p+n)} \Big[\int_\Omega (-u_1)^p (dd^c u_1)^n\Big]^{1/(p+n)} \cdots \Big[\int_\Omega (-u_n)^p (dd^c u_n)^n\Big]^{1/(p+n)}.$$

Finally, we need the following theorem on the Dirichlet problem.

2.12. THEOREM. *Let $p > 0$ and μ a positive measure on Ω. Then there exists a unique function $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$ if, and only if, there* *is a constant* $A > 0$ *such that*

$$\int_{\Omega} (-\varphi)^p \, d\mu \le A\Big[\int_{\Omega} (-\varphi)^p (dd^c\varphi)^n\Big]^{p/(p+n)}$$

for every $\varphi \in \mathcal{E}_0$.

Proof. The assumption on μ implies that it vanishes on pluripolar sets and therefore Theorem 5.11 in [Ce2] shows that there exist $\phi \in \mathcal{E}_0$ and $0 \le f \in L^1_{\mathrm{loc}}((dd^c\phi)^n)$ such that $\mu = f(dd^c\phi)^n$. Kołodziej's theorem ([Ko]) implies that there exist $u_j \in \mathcal{E}_0$ such that $(dd^c u_j)^n = \min\{f, j\}(dd^c\phi)^n$. Using the assumption on μ for $\varphi = u_j$, we obtain

$$\int_{\Omega} (-u_j)^p (dd^c u_j)^n \le A^{(n+p)/n}.$$

Thus $u_j \searrow u \in \mathcal{E}_p$ and $(dd^c u)^n = d\mu$. Uniqueness follows from Theorem 2.9. For the converse, let $p > 0$ and assume that there exists $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$. By Theorem 2.1 in [Ce2] there exist $u_j \in \mathcal{E}_0$ such that $u_j \searrow u$. We have

$$B = \sup_{j\ge 1} \int_{\Omega} (-u_j)^p (dd^c u_j)^n < \infty.$$

Theorem 2.11 yields

$$\begin{aligned}\int_{\Omega} (-\varphi)^p \, d\mu &\le \varliminf_{j\to\infty} \int_{\Omega} (-\varphi)^p (dd^c u_j)^n \\ &\le \varliminf_{j\to\infty} \Big[\int_{\Omega} (-\varphi)^p (dd^c\varphi)^n\Big]^{p/(p+n)} \Big[\int_{\Omega} (-u_j)^p (dd^c u_j)^n\Big]^{n/(p+n)} \\ &\le B^{n/(p+n)} \Big[\int_{\Omega} (-\varphi)^p (dd^c\varphi)^n\Big]^{p/(p+n)}.\end{aligned}$$

3. The comparison principle in $\mathcal{E}_p(f)$. In this section we prove the comparison principle in the class $\mathcal{E}_p(f)$ with $p > 0$. The theorem is proved using the ideas from the proof of Theorem 3.10 in [Ce3].

3.1. THEOREM. *Let* $u \in \mathcal{E}_p(f)$ *and* $v \in \mathcal{E}_p(g)$ *with* $f \in C(\partial\Omega)$ *and* $f \ge g$. *Then*

$$(*) \quad \frac{1}{n!} \int_{\{u<v\}} (v-u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int_{\{u<v\}} (r - w_1)(dd^c v)^n \le \int_{\{u<v\}} (r - w_1)(dd^c u)^n$$

for all $w_j \in \mathrm{PSH}(\Omega)$, $0 \le w_j \le 1$, $j = 1, \ldots, n$ *and all* $r \ge 1$.

We need the following

3.2. Lemma. *Let $\varphi \in \mathcal{E}_p$. There exist $\mathcal{E}_0 \ni \varphi_j \searrow \varphi$ and $\mathcal{E}_p \ni \psi_j \nearrow 0$ a.e. such that $\varphi_j + \psi_j \le \varphi \le \varphi_j, \psi_j$.*

Proof. Let $h \in \mathcal{E}_0$ with $h \not\equiv 0$. For every $j > 0$ by Proposition 4.1 in [KH] we have

$$\begin{aligned}(dd^c\varphi)^n &= 1_{\{\varphi > jh\}}(dd^c\varphi)^n + 1_{\{\varphi \le jh\}}(dd^c\varphi)^n \\ &= 1_{\{\varphi > jh\}}(dd^c \max(\varphi, jh))^n + 1_{\{\varphi \le jh\}}(dd^c\varphi)^n,\end{aligned}$$

where 1_E denotes the characteristic function of $E \subset \Omega$. By Kołodziej's theorem ([Ko]) there exists $\varphi_j \in \mathcal{E}_0$ such that

$$(dd^c\varphi_j)^n = 1_{\{\varphi > jh\}}(dd^c \max(\varphi, jh))^n = 1_{\{\varphi > jh\}}(dd^c\varphi)^n.$$

On the other hand, by Theorem 2.12 there exists $\psi_j \in \mathcal{E}_p$ such that

$$(dd^c\psi_j)^n = 1_{\{\varphi \le jh\}}(dd^c\varphi)^n.$$

Therefore

$$\begin{aligned}\max((dd^c\varphi_j)^n, (dd^c\psi_j)^n) &= \max(1_{\{\varphi > jh\}}(dd^c\varphi)^n, 1_{\{\varphi \le jh\}}(dd^c\varphi)^n) \\ &\le (dd^c\varphi)^n = (dd^c\varphi_j)^n + (dd^c\psi_j)^n \\ &\le (dd^c(\varphi_j + \psi_j))^n.\end{aligned}$$

Using Theorem 2.9 we get

$$\varphi_j + \psi_j \le \varphi \le \varphi_j,\, \psi_j$$

and

$$\varphi_j \searrow \widetilde{\varphi} \ge \varphi \quad \text{and} \quad \psi_j \nearrow \widetilde{\psi} \in \mathcal{E}_p \quad \text{a.e.}$$

Thus by Theorem 4.5 in [Ce2] and Proposition 2.6, we have

$$(dd^c\varphi_j)^n \to (dd^c\widetilde{\varphi})^n, \quad (dd^c\psi_j)^n \to (dd^c\widetilde{\psi})^n \quad \text{as } j \to \infty.$$

On the other hand, we also have

$$(dd^c\varphi_j)^n \to (dd^c\varphi)^n, \quad (dd^c\psi_j)^n \to 0 \quad \text{as } j \to \infty.$$

Indeed, let $\omega \in C_0^\infty(\Omega)$. First note that $1_{\{\varphi > jh\}} \to 1_\Omega$, $1_{\{\varphi \le jh\}} \to 0$ except on a pluripolar set, as $j \to \infty$. Then by Proposition 2.8 and Lebesgue's convergence theorem we have

$$\lim_{j\to\infty} \int_\Omega \omega (dd^c\varphi_j)^n = \lim_{j\to\infty} \int_\Omega \omega 1_{\{\varphi > jh\}}(dd^c\varphi)^n = \int_\Omega \omega (dd^c\varphi)^n$$

and

$$\lim_{j\to\infty} \int_\Omega \omega (dd^c\psi_j)^n = \lim_{j\to\infty} \int_\Omega \omega 1_{\{\varphi \le jh\}}(dd^c\varphi)^n = 0.$$

Thus

$$(dd^c\widetilde{\varphi})^n = (dd^c\varphi)^n \quad \text{and} \quad (dd^c\widetilde{\psi})^n = 0.$$

Hence $\widetilde{\varphi} = \varphi$ and $\widetilde{\psi} = 0$.

Proof of Theorem 3.1. Obviously, we may assume that $f \leq -1$. First consider the case $u, v \in \mathcal{E}_p(f)$. Let $\varphi \in \mathcal{E}_p$ be such that

$$\varphi + U(0, f) \leq u, v \leq U(0, f).$$

Replacing u by $u + \varepsilon$, without loss of generality we may assume that

$$U(0, f + \varepsilon) + \varphi \leq u \leq U(0, f + \varepsilon).$$

Using Lemma 3.2 we can find $\mathcal{E}_0 \ni \varphi_j \searrow \varphi$ and $\mathcal{E}_p \ni \psi_j \nearrow 0$ a.e. such that

$$\varphi_j + \psi_j \leq \varphi \leq \varphi_j, \psi_j.$$

For each $j \geq 1$ take $h_j \in \mathcal{E}_0$ such that $h_j < U(0, f)$ on $\{\varphi_j < -\varepsilon\} \subset\subset \Omega$. We set

$$u_j = \max(u, \varphi + \max(U(0, f), h_j)) \in \mathcal{E}_p,$$
$$v_j = \max(v + \psi_j, 2\varphi + \max(U(0, f), h_j)) \in \mathcal{E}_p.$$

Using Theorem 2.10, we have

$$(1) \qquad \frac{1}{n!} \int\limits_{\{u_j < v_j\}} (v_j - u_j)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int\limits_{\{u_j < v_j\}} (r - w_1)(dd^c v_j)^n \leq \int\limits_{\{u_j < v_j\}} (r - w_1)(dd^c u_j)^n$$

for all $w_j \in \mathrm{PSH}(\Omega)$, $0 \leq w_j \leq 1$, $j = 1, \ldots, n$ and all $r \geq 1$. From the inclusions

$$\{u < v + \psi_j\} \subset \{\varphi + U(0, f + \varepsilon) < \psi_j + U(0, f)\}$$
$$\subset \{\varphi_j + \psi_j + U(0, f + \varepsilon) < \psi_j + U(0, f)\} \subset \{\varphi_j < -\varepsilon\}.$$

we have

$$\{u_j < v_j\} \subset \{\varphi_j < -\varepsilon\}.$$

Moreover, $u_j = u$ and $v_j = v + \psi_j$ on $\{\varphi_j < -\varepsilon\}$ because $h_j < U(0, f)$ on $\{\varphi_j < -\varepsilon\}$. It follows from (1) that

$$\frac{1}{n!} \int\limits_{\{u < v + \psi_j\}} (v + \psi_j - u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int\limits_{\{u < v + \psi_j\}} (r - w_1)(dd^c (v + \psi_j))^n \leq \int\limits_{\{u < v + \psi_j\}} (r - w_1)(dd^c u)^n.$$

We get

$$(2) \qquad \frac{1}{n!} \int\limits_{\Omega} 1_{\{u < v + \psi_j\}} (v + \psi_j - u)^n dd^c w_1 \wedge \cdots \wedge dd^c w_n + \int\limits_{\Omega} 1_{\{u < v + \psi_j\}} (r - w_1)(dd^c v)^n \leq \int\limits_{\{u < v\}} (r - w_1)(dd^c u)^n.$$

From $\sup_{j\geq 1}\psi_j = (\sup_{j\geq 1}\psi_j)^* = 0$ except on a pluripolar set, it follows that $1_{\{u<v+\psi_j\}} \nearrow 1_{\{u<v\}}$ and $1_{\{u<v+\psi_j\}}(v+\psi_j-u)^n \nearrow 1_{\{u<v\}}(v-u)^n$ except on a pluripolar set. On the other hand, from the locally absolute continuity of $dd^cw_1\wedge\cdots\wedge dd^cw_n$ and $(dd^cv)^n$ with respect to C_n-capacity (see Proposition 2.8) it follows that $1_{\{u<v+\psi_j\}} \nearrow 1_{\{u<v\}}$ and $1_{\{u<v+\psi_j\}}(v+\psi_j-u)^n \nearrow 1_{\{u<v\}}(v-u)^n$ a.e. with respect to these measures. Thus applying Lebesgue's monotone convergence theorem to (2) we obtain $(*)$ in Theorem 3.1.

Now assume that $u\in\mathcal{E}_p(f)$ and $v\in\mathcal{E}_p(g)$. Then $v_1=\max(u,v)\in\mathcal{E}_p(f)$ and thus $(*)$ holds for u and v_1. Thus using Proposition 4.1 of [KH] and the inclusion $\{u<v\}=\{u<v_1\}$ it follows that $(*)$ holds for u and v. The theorem is proved.

3.3. Theorem. *Let $u\in\mathcal{E}_p(f)$ and $v\in\mathcal{E}(g)$ be such that $f,g\in C(\partial\Omega)$ and $f\geq g$. If $(dd^cu)^n\leq(dd^cv)^n$ then $u\geq v$.*

Proof. Obviously, we may assume that $f\leq -1$. First consider the case $u,v\in\mathcal{E}_p(f)$. Let $\varphi\in\mathcal{E}_p$ be such that

$$\varphi+U(0,f)\leq u,v\leq U(0,f).$$

Using Lemma 3.2 we can find $\mathcal{E}_0\ni\varphi_j\searrow\varphi$ and $\mathcal{E}_p\ni\psi_j\nearrow 0$ a.e. such that

$$\varphi_j+\psi_j\leq\varphi\leq\varphi_j,\psi_j.$$

Theorem 3.1 yields

$$\begin{aligned}(3)\qquad \frac{1}{n!}\int\limits_{\{u+\varepsilon<v+\psi_j\}}&(v+\psi_j-u-\varepsilon)^n dd^cw_1\wedge\cdots\wedge dd^cw_n\\ &+\int\limits_{\{u+\varepsilon<v+\psi_j\}}(r-w_1)(dd^c(v+\psi_j))^n\\ &\leq\int\limits_{\{u+\varepsilon<v+\psi_j\}}(r-w_1)(dd^cu)^n\end{aligned}$$

for all $w_j\in\mathrm{PSH}(\Omega),\ 0\leq w_j\leq 1,\ j=1,\dots,n$ and all $r\geq 1$. From the inclusions

$$\begin{aligned}\{u<v+\psi_j\}&\subset\{\varphi+U(0,f)+\varepsilon<\psi_j+U(0,f)\}\\ &\subset\{\varphi_j+\psi_j+U(0,f)+\varepsilon<\psi_j+U(0,f)\}\subset\{\varphi_j<-\varepsilon\}\end{aligned}$$

we have

$$\{u+\varepsilon<v+\psi_j\}\subset\{\varphi_j<-\varepsilon\}\subset\subset\Omega.$$

Moreover $(dd^cu)^n\leq(dd^cv)^n$. It follows from (3) that

$$\int\limits_{\{u+\varepsilon<v+\psi_j\}}(v+\psi_j-u-\varepsilon)^n dd^cw_1\wedge\cdots\wedge dd^cw_n=0$$

for all $w_j \in \mathrm{PSH}(\Omega)$, $0 \le w_j \le 1$, $j = 1, \ldots, n$. Therefore $u + \varepsilon \ge v + \psi_j$. Letting $j \to \infty$ and $\varepsilon \to 0$, we obtain $u \ge v$.

Now assume that $u \in \mathcal{E}_p(f)$ and $v \in \mathcal{E}_p(g)$. Then $v_1 = \max(u, v) \in \mathcal{E}_p(f)$. By Proposition 4.3 in [KH], we have $(dd^c u)^n \le (dd^c v_1)^n$. Hence $u \ge v_1 \ge v$. The theorem is proved.

4. The Dirichlet problem in $\mathcal{E}_p(f)$. In this section, first using Theorem 3.3 by a standard method we prove the following

4.1. Theorem. *Let μ be a positive measure such that $\mu \le (dd^c v)^n$ with $v \in \mathcal{E}_p(f)$. If $\lim_{z\to\xi} U(0, f) = f(\xi)$ for all $\xi \in \partial\Omega$ then there is a unique function $u \in \mathcal{E}_p(f)$ such that $\mu = (dd^c u)^n$.*

Proof. The uniqueness is known from Theorem 3.3. It remains to show the existence of $u \in \mathcal{E}_p(f)$ such that $\mu = (dd^c u)^n$. By Theorem 6.3 in [Ce1] we can find $\psi \in \mathcal{E}_0$ and $0 \le \varphi \in L^1_{\mathrm{loc}}((dd^c\psi)^n)$ such that $\mu = \varphi(dd^c\psi)^n$. We set $\mu_k = \min(\varphi, k)(dd^c\psi)^n$. Then $\mu_k \le (dd^c k^{1/n}\psi)^n$. By Kołodziej's theorem (see [Ko]) there exists $\omega_k \in \mathcal{E}_0$ such that $(dd^c\omega_k)^n = \mu_k$. From the relations

$$\begin{cases} U((dd^c(\omega_k + U(0, f)))^n, f) = \omega_k + U(0, f), \\ (dd^c(\omega_k + U(0, f)))^n \ge \mu_k, \end{cases}$$

and from Theorem 8.1 in [Ce1] it follows that

$$\begin{cases} (dd^c U(\mu_k, f))^n = \mu_k, \\ U(0, f) \ge U(\mu_k, f) \ge \omega_k + U(0, f). \end{cases}$$

Theorem 3.3 implies that $U(\mu_k, f) \searrow u \ge v$. Obviously, we have $u \in \mathcal{E}_p(f)$ and $\mu = (dd^c u)^n$.

4.2. Example. *There exists $0 \le \varphi \in L^1(\Omega)$ such that no function*

$$u \in \bigcup \{\mathcal{E}_p(f) : p > 0,\ f \in C(\partial\Omega)\}$$

satisfies $(dd^c u)^n \ge \varphi d\lambda$, where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$.

Indeed, take an arbitrary subdomain $D \subset\subset \Omega$. Let $z_j \in D$, $s_j \searrow 0$, $p_j \searrow 0$ and $a_j > 0$ be such that $B(z_j, s_j) = \{z \in \mathbb{C}^n : \|z - z_j\| < s_j\} \subset D$ and $\sum_{j=1}^{\infty} a_j < \infty$. Define

$$\varphi = \sum_{j=1}^{\infty} \frac{a_j}{d_n r_j^{2n}} 1_{B(z_j, r_j)} \in L^1(\Omega)$$

where d_n is the volume of the unit ball in $\mathbb{C}^n$ and $0 < r_j < s_j$ are chosen so that

$$\frac{1}{a_j} (C_n(B(z_j, r_j), \Omega))^{p_j/(p_j+n)} \to 0 \quad \text{as } j \to \infty.$$

Assume that $\varphi d\lambda \le (dd^c u)^n$ for some $u \in \mathcal{E}_p(f)$ with $p > 0$ and $f \in C(\partial\Omega)$. Obviously, we may assume that $f < 0$. Take $\psi \in \mathcal{E}_p$ such that $\psi + U(0,f) \le u \le U(0,f)$. Put

$$\omega = \max\Big(U(0,f), -\frac{M}{\sup_{\overline{D}} h_{D,\Omega}} h_{D,\Omega}\Big) \in \mathcal{E}_0$$

where $M > 0$ is such that $-M < \inf_{\partial\Omega} f < 0$. Hence $\omega = U(0,f)$ on D. Let $\widetilde{u} = \max(u, \psi + \omega)$. We have $\psi + \omega \le \widetilde{u} \le 0$ and $\psi + \omega \in \mathcal{E}_p + \mathcal{E}_0 \subset \mathcal{E}_p$. By [Ce1] we have $\widetilde{u} \in \mathcal{E}_p$. Moreover $\widetilde{u} = u$ on D. Thus for $B_j = B(z_j, r_j)$ we have

$$a_j = \int_{B_j} \varphi\, d\lambda \le \int_{B_j} (dd^c u)^n = \int_{B_j} (dd^c \widetilde{u})^n.$$

Let $\mathcal{E}_0 \ni \widetilde{u}_k \searrow \widetilde{u}$ be as in the definition of $\mathcal{E}_p$. Then $(dd^c \widetilde{u}_k)^n \to (dd^c \widetilde{u})^n$ weakly (see [Ce1]). Theorem 2.11 implies the estimates

$$\begin{aligned}
a_j &\le \int_{B_j} (dd^c \widetilde{u})^n \le \varliminf_{k\to\infty} \int_{B_j} (dd^c \widetilde{u}_k)^n \le \varliminf_{k\to\infty} \int_{\Omega} (-h_{B_j,\Omega})^p (dd^c \widetilde{u}_k)^n \\
&\le C_{p,n} \varliminf_{k\to\infty} \Big[\int_{\Omega} (-h_{B_j,\Omega})^p (dd^c h_{B_j,\Omega})^n\Big]^{p/(p+n)} \Big[\int_{\Omega} (-\widetilde{u}_k)^p (dd^c \widetilde{u}_k)^n\Big]^{n/(p+n)} \\
&\le \alpha \Big[\int_{\Omega} (dd^c h_{B_j})^n\Big]^{p/(p+n)} = \alpha [C_n(B_j, \Omega)]^{p/(p+n)}
\end{aligned}$$

where $C_{p,n}$ is a positive constant and

$$\alpha = C_{p,n} \Big[\sup_{k\ge 1} \int_{\Omega} (-\widetilde{u}_k)^p (dd^c \widetilde{u}_k)^n\Big]^{n/(p+n)} < \infty.$$

This is impossible, because

$$\lim_{j\to\infty} \frac{[C_n(B_j,\Omega)]^{p/(p+n)}}{a_j} \le \lim_{j\to\infty} \frac{[C_n(B_j,\Omega)]^{p_j/(p_j+n)}}{a_j} = 0.$$

4.3. Theorem. *Let $f \in C(\partial\Omega)$ be such that*

$$\lim_{z\to\xi} U(0,f)(z) = f(\xi) \qquad \forall \xi \in \partial\Omega$$

and

$$U(0,f) + U(0,-f) \in \mathcal{E}_p.$$

Assume that μ is a positive measure on Ω. Then the following are equivalent:

(i) *There exists a function $u \in \mathcal{E}_p(f)$ with $(dd^c u)^n = \mu$.*

(ii) *There exists a constant $A > 0$ such that*

$$(**) \qquad \int_{\Omega} (-\varphi)^p\, d\mu \le A\Big[\int_{\Omega} (-\varphi)^p (dd^c \varphi)^n\Big]^{p/(p+n)} \qquad \forall \varphi \in \mathcal{E}_0(\Omega).$$

(iii) *There exists a constant $A > 0$ such that*

$$\int_{\Omega'} (-\varphi)^p \, d\mu \le A\Big[\int_{\Omega'} (-\varphi)^p (dd^c\varphi)^n\Big]^{p/(p+n)} \qquad \forall \varphi \in \mathcal{E}_0(\Omega')$$

for all hyperconvex subdomains $\Omega' \subset\subset \Omega$.

(iv) $\mathcal{E}_p(\Omega) \subset L_p(\Omega, \mu)$.

Proof. (i)⇒(ii). Suppose that $\mu = (dd^c u)^n$ for some $u \in \mathcal{E}_p(f)$. Take $\psi \in \mathcal{E}_p$ with

$$\psi + U(0, f) \le u \le U(0, f).$$

Hence

$$\psi + U(0, f) + U(0, -f) \le u + U(0, -f) \le 0.$$

It follows that $u + U(0, -f) \in \mathcal{E}_p$ because $\psi + U(0, f) + U(0, -f) \in \mathcal{E}_p$. By Theorem 2.12, $(dd^c(u + U(0, -f)))^n$ satisfies $(**)$. Hence so also does $\mu = (dd^c u)^n$.

(ii)⇒(i). Assume that μ satisfies $(**)$. From Theorem 2.12 we find $v \in \mathcal{E}_p$ such that $(dd^c v)^n = \mu$. Since $\mu \le (dd^c(v + U(0, f)))^n$, using Theorem 4.1 we have (ii)⇒(i).

(ii)⇒(iii). Assume that $(**)$ holds for all $\varphi \in \mathcal{E}_0(\Omega)$. Since Theorem 2.12 we can write $\mu = (dd^c u)^n$ for some $u \in \mathcal{E}_p(\Omega)$. By [Åh] we find $v \in \mathcal{F}(\Omega')$ such that $(dd^c v)^n = \mu|_{\Omega'}$. By the comparison principle we have $v \ge u|_{\Omega'}$. Therefore

$$\int_{\Omega'} (-v)^p (dd^c v)^n \le \int_{\Omega} (-u)^p (dd^c u)^n.$$

Theorem 2.11 implies that $(**)$ holds for $\varphi \in \mathcal{E}_0(\Omega')$ with

$$A = C_{p,n}\Big[\int_{\Omega} (-u)^p (dd^c u)^n\Big]^{p/(p+n)},$$

which is independent of Ω'.

(iii)⇒(ii). Take an increasing exhaustion sequence of Ω by relatively compact hyperconvex subdomains Ω_j. Let $\varphi \in \mathcal{E}_0(\Omega)$. By [Åh], there are $\varphi_j \in \mathcal{F} \cap L^\infty(\Omega_j)$ such that $(dd^c\varphi_j)^n = (dd^c\varphi)^n|_{\Omega_j}$. The comparison principle implies that $\varphi_j \searrow \varphi$. We have

$$\int_{\Omega_j} (-\varphi_j)^p \, d\mu \le A\Big[\int_{\Omega_j} (-\varphi_j)^p (dd^c\varphi_j)^n\Big]^{p/(p+n)} = A\Big[\int_{\Omega_j} (-\varphi_j)^p (dd^c\varphi)^n\Big]^{p/(p+n)}$$

for all $j \ge 1$. Letting $j \to \infty$, we have

$$\int_{\Omega} (-\varphi)^p \, d\mu \le A\Big[\int_{\Omega} (-\varphi)^p (dd^c\varphi)^n\Big]^{p/(p+n)}.$$

(ii)⇒(iv) is obvious. In order to prove (iv)⇒(ii) we need

4.4. LEMMA.

(a) *If* $p \geq 1$ *then*

$$\int_\Omega \Big(-\sum_{j=1}^k \alpha_j u_j\Big)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n \leq C_{p,n} \max_{1\leq j\leq k} \int_\Omega (-u_j)^p (dd^c u_j)^n$$

for all $u_1,\dots,u_k \in \mathcal{E}_p$ *and* $0 \leq \alpha_1,\dots,\alpha_k \leq 1$ *with* $\sum_{j=1}^k \alpha_j = 1$.

(b) *If* $0 < p < 1$ *then*

$$\int_\Omega \Big(-\sum_{j=1}^k \alpha_j u_j\Big)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n \leq C_{p,n}\Big(\sum_{j=1}^k \alpha_j^p\Big) \max_{1\leq j\leq k} \int_\Omega (-u_j)^p (dd^c u_j)^n$$

for all $u_1,\dots,u_k \in \mathcal{E}_p$ *and* $0 \leq \alpha_1,\dots,\alpha_k \leq 1$ *with* $\sum_{j=1}^k \alpha_j = 1$, *where* $C_{p,n}$ *is as in Theorem* 2.11.

Proof. Set

$$e_p(u) = \int_\Omega (-u)^p (dd^c u)^n, \quad u \in \mathcal{E}_p, \quad M = \max_{1\leq j\leq k} \int_\Omega (-u_j)^p (dd^c u_j)^n.$$

(a) By Theorem 2.11 we have

$$\begin{aligned}
\Big(e_p\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^{1/p} &= \Big[\int_\Omega \Big(-\sum_{j=1}^k \alpha_j u_j\Big)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n\Big]^{1/p} \\
&\leq \sum_{j=1}^k \Big[\int_\Omega (-\alpha_j u_j)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n\Big]^{1/p} \\
&= \sum_{j=1}^k \alpha_j \Big[\int_\Omega (-u_j)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n\Big]^{1/p} \\
&= \sum_{j=1}^k \alpha_j \Big[\int_\Omega (-u_j)^p \sum_{1\leq i_1,\dots,i_n\leq k} \alpha_{i_1}\cdots\alpha_{i_n} dd^c u_{i_1}\wedge\cdots\wedge dd^c u_{i_n}\Big]^{1/p} \\
&\leq \sum_{j=1}^k \alpha_j \Big[C_{p,n} \sum_{1\leq i_1,\dots,i_n\leq k} \alpha_{i_1}\cdots\alpha_{i_n} e_p(u_j)^{p/(p+n)} e_p(u_{i_1})^{1/(p+n)} \cdots \\
&\qquad\qquad e_p(u_{i_n})^{1/(p+n)}\Big]^{1/p} \\
&\leq \sum_{j=1}^k \alpha_j \Big[C_{p,n} M \sum_{1\leq i_1,\dots,i_n\leq k} \alpha_{i_1}\cdots\alpha_{i_n}\Big]^{1/p} \\
&= (C_{p,n}M)^{1/p} \sum_{j=1}^k \alpha_j [(\alpha_1+\cdots+\alpha_k)^n]^{1/p} = (C_{p,n}M)^{1/p}.
\end{aligned}$$

Hence $e_p(\sum_{j=1}^k \alpha_j u_j) \leq C_{p,n} M$.

(b) By Theorem 2.11 we have

$$
\begin{aligned}
e_p\Big(\sum_{j=1}^k \alpha_j u_j\Big) &= \int_\Omega \Big(-\sum_{j=1}^k \alpha_j u_j\Big)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n \\
&\le \sum_{j=1}^k \int_\Omega (-\alpha_j u_j)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n \\
&= \sum_{j=1}^k \alpha_j^p \int_\Omega (-u_j)^p \Big(dd^c\Big(\sum_{j=1}^k \alpha_j u_j\Big)\Big)^n \\
&= \sum_{j=1}^k \alpha_j^p \Big[\int_\Omega (-u_j)^p \sum_{1\le i_1,\dots,i_n\le k} \alpha_{i_1}\cdots\alpha_{i_n} dd^c u_{i_1}\wedge\cdots\wedge dd^c u_{i_n}\Big] \\
&\le \sum_{j=1}^k \alpha_j^p \Big[C_{p,n} \sum_{1\le i_1,\dots,i_n\le k} \alpha_{i_1}\cdots\alpha_{i_n} e_p(u_j)^{p/(p+n)} e_p(u_{i_1})^{1/(p+n)} \\
&\qquad\qquad \cdots e_p(u_{i_n})^{1/(p+n)}\Big] \\
&\le \sum_{j=1}^k \alpha_j^p \Big[C_{p,n} M \sum_{1\le i_1,\dots,i_n\le k} \alpha_{i_1}\cdots\alpha_{i_n}\Big] \\
&= C_{p,n} M \sum_{j=1}^k \alpha_j^p (\alpha_1+\cdots+\alpha_k)^n = C_{p,n} M \sum_{j=1}^k \alpha_j^p.
\end{aligned}
$$

Now we prove that (iv)⇒(ii). Assume that $(**)$ is not true. Then we can find $\varphi_j \in \mathcal{E}_0(\Omega)$ such that

$$
\int_\Omega (-\varphi_j)^p\, d\mu \ge 4^{jp}\Big[\int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n\Big]^{p/(p+n)}.
$$

Set

$$
\psi_j = \frac{\varphi_j}{e_p(\varphi_j)^{1/(p+n)}}, \qquad j \ge 1.
$$

Obviously, we have $e_p(\psi_j) = 1$ and

$$
\begin{aligned}
\int_\Omega (-\psi_j)^p\, d\mu &= \frac{1}{e_p(\varphi_j)^{p/(p+n)}} \int_\Omega (-\varphi_j)^p\, d\mu \\
&\ge \frac{4^{jp}}{e_p(\varphi_j)^{p/(p+n)}} \Big[\int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n\Big]^{p/(p+n)}.
\end{aligned}
$$

Thus $e_p(\psi_j) = 1$ and $\int_\Omega (-\psi_j)^p\, d\mu \ge 4^{jp}$. Let $\psi = \sum_{j=1}^\infty \psi_j/2^j$. Then

$$
\mathcal{E}_0 \ni \sum_{j=1}^k \frac{\psi_j}{2^j} \searrow \psi \quad \text{as } k\to\infty
$$

and by Lemma 4.4 there exists $D_{p,n} > 0$ such that

$$e_p\bigg(\sum_{j=1}^{k} \frac{\psi_j}{2^j}\bigg) \le D_{p,n} \max(e_p(\psi_1), \ldots, e_p(\psi_k)) \le D_{p,n} \quad \text{for all } j \ge 1.$$

Therefore $\psi \in \mathcal{E}_p(\Omega) \subset L_p(\Omega, \mu)$. Since

$$\int_{\Omega} (-\psi_j)^p \, d\mu = 2^{jp} \int_{\Omega} \bigg(-\frac{\psi_j}{2^j}\bigg)^p \, d\mu \le 2^{jp} \int_{\Omega} (-\psi)^p \, d\mu \quad \text{for all } j \ge 1,$$

it follows that

$$\infty > \int_{\Omega} (-\psi)^p \, d\mu \ge \frac{1}{2^{jp}} \int_{\Omega} (-\psi_j)^p \, d\mu \ge 2^{jp} \quad \text{for all } j \ge 1,$$

which is impossible.

4.5. Corollary. *Let μ be a finite positive measure on Ω such that*

$$\mu(E) \le A(C_n(E, \Omega))^{\alpha}$$

for all Borel sets $E \subset \Omega$, where A and α are positive constants with $\alpha > p/(p+n)$. Then there exists a unique $u \in \mathcal{F}_p$ such that $(dd^c u)^n = \mu$.

Proof. By Theorem 4.3 it suffices to show that $\mathcal{E}_p(\Omega) \subset L_p(\Omega, \mu)$. Given $\varphi \in \mathcal{E}_p(\Omega)$. By using the inequality $C_n(\{\varphi < -s\}) \le C_\varphi / s^{p+n}$ for all $s > 0$ (see Proposition 3.1 in [CKZ]) we have

$$\begin{aligned}
\int_{\Omega} (-\varphi)^p \, d\mu &= \int_{\{\varphi < -1\}} (-\varphi)^p \, d\mu + \int_{\{\varphi \ge -1\}} (-\varphi)^p \, d\mu \\
&\le \int_{\{\varphi < -1\}} (-\varphi)^p \, d\mu + \mu(\Omega) = \int_1^{\infty} p t^{p-1} \mu(\{\varphi < -t\}) \, dt + \mu(\Omega) \\
&\le Ap \int_1^{\infty} t^{p-1} C_n(\{\varphi < -t\})^{\alpha} \, dt + \mu(\Omega) \\
&\le Ap \int_1^{\infty} C_\varphi^{\alpha} \frac{dt}{t^{\alpha(p+n)+1-p}} + \mu(\Omega) < \infty.
\end{aligned}$$

Therefore $\varphi \in L_p(\Omega, \mu)$.

References

[Åh] P. Åhag, *The complex Monge–Ampère operator on bounded hyperconvex domains*, Ph.D. thesis, Umeå Univ., 2002.

[BT1] E. Bedford and B. A. Taylor, *The Dirichlet problem for the complex Monge–Ampère operator*, Invent. Math. 37 (1976), 1–44.

[BT2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[Ce1] U. Cegrell, *Pluricomplex energy*, Acta Math. 1980 (1998), 187–217.

[Ce2] —, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[Ce3] —, *Boundary values of plurisubharmonic functions and a general Dirichlet problem for the complex Monge–Ampère operator*, Dept. of Math., Mid Sweden Univ., Research Report No. 1 (2005).

[Ce4] —, *Convergence in capacity*, preprint, 2005.

[CKZ] U. Cegrell, S. Kołodziej and A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularities*, Math. Z. 250 (2005), 7–22.

[Cz] R. Czyż, *Convergence in capacity of the Perron–Bremermann envelope*, Michigan Math. J. 53 (2005), 497–509.

[CHÅ] R. Czyż, P. Hiep and P. Åhag, *Concerning the energy class* $\mathcal{E}_p$ *for* $0 < p < 1$, IMUJ preprint, 2005.

[H] P. Hiep, *A characterization of bounded plurisubharmonic functions*, Ann. Polon. Math. 85 (2004), 233–238.

[KH] N. Khue and P. Hiep, *Some properties of the complex Monge–Ampère in the Cegrell's classes and applications*, preprint, 2006.

[Kl] M. Klimek, *Pluripotential Theory*, Oxford Univ. Press, New York, 1991.

[Ko] S. Kołodziej, *The range of the complex Monge–Ampère operator, II*, Indiana Univ. Math. J. 44 (1995), 765–782.

[Per] L. Persson, *A Dirichlet principle for the complex Monge–Ampère operator*, Ark. Mat. 37 (1999), 345–356.

[Xi] Y. Xing, *Continuity of the complex Monge–Ampère operator*, Proc. Amer. Math. Soc. 124 (1996), 457–467.

Department of Mathematics
Hanoi University of Education (Dai Hoc Su Pham HaNoi)
Cau Giay, Hanoi, VietNam
E-mail: phamhoanghiep_vn@yahoo.com

Received 25.9.2005
and in final form 27.1.2006 (1611)