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Generalized shadowing for discrete semidynamical systems

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Abstract. We provide a unified approach to different types of shadowing. This enables us to generalize some known shadowing result.

1. Introduction. Shadowing is one of the crucial stability concepts in dynamical systems [2, 3]. Roughly speaking it concerns the question if (and when) an approximate orbit can be approximated by an exact one. This is often important because using numerical methods we can only obtain approximate trajectories.

However, to precisely formulate shadowing one needs to know what is understood by an approximate trajectory. This causes problems, as various definitions are applied in different settings. The most commonly used is the notion of pseudoorbit, which in a sense corresponds to stability in the supremum norm. However, there also exists limit shadowing [3, p. 64], L^p-shadowing [3, p. 68] and weighted shadowing [3, p. 71].

The aim of this paper is to provide a unified approach in which all the above mentioned shadowings can be studied. To do this we introduce a generalization of a norm on a complete sequence space which we call onorm. In our opinion it gives the right setting to formulate general shadowing results.

To illustrate this, using the method from [1] we prove some general shadowing results for semiflows with discrete time.

2. Generalized norm. In this section we introduce the notion of a complete onorm which will be crucial in our considerations. A complete onorm can be understood as a generalization of a norm on a complete sequence space.

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}_+ = [0, \infty)$ and $\overline{\mathbb{R}}_+ = [0, \infty]$. We assume that $0 \cdot \infty = 0$.

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In the set $\overline{\mathbb{R}}_+^{\mathbb{N}}$ of all sequences $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ we introduce the relation $\preccurlyeq$ in the following way:

$$(x_k)_{k\in\mathbb{N}} \preccurlyeq (y_k)_{k\in\mathbb{N}} \quad \text{if and only if} \quad x_k \le y_k \text{ for every } k \in \mathbb{N}.$$

Definition 1. By an *onorm* we mean a function $\rfloor\cdot\lfloor\, : \overline{\mathbb{R}}_+^{\mathbb{N}} \to \overline{\mathbb{R}}_+$ such that for $\mathbf{x}, \mathbf{y} \in \overline{\mathbb{R}}_+^{\mathbb{N}}$ and $\alpha \in \overline{\mathbb{R}}_+$ the following conditions are satisfied:

(i) $\rfloor\mathbf{x}\lfloor = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;
(ii) $\rfloor\mathbf{x}+\mathbf{y}\lfloor \le \rfloor\mathbf{x}\lfloor + \rfloor\mathbf{y}\lfloor$;
(iii) $\rfloor\alpha\mathbf{x}\lfloor = \alpha\rfloor\mathbf{x}\lfloor$;
(iv) $\mathbf{x} \preccurlyeq \mathbf{y} \Rightarrow \rfloor\mathbf{x}\lfloor \le \rfloor\mathbf{y}\lfloor$.

If $(\mathbf{x}^n) = (x_k^n)_{k\in\mathbb{N}}$ is a sequence in $\overline{\mathbb{R}}_+^{\mathbb{N}}$ and instead of (ii) the condition

$$\text{(ii)}' \quad \Big\rfloor\sum_{n=1}^{\infty}\mathbf{x}^n\Big\lfloor \le \sum_{n=1}^{\infty}\rfloor\mathbf{x}^n\lfloor, \quad \text{where} \quad \sum_{n=1}^{\infty}\mathbf{x}^n := \Big(\sum_{n=1}^{\infty}x_k^n\Big)_{k\in\mathbb{N}},$$

is satisfied, then we say that $\rfloor\cdot\lfloor$ is a *complete onorm.*

We use the term onorm as the abbreviation of "ordered norm". It turns out that every "reasonable" norm $\|\cdot\|$ on a (complete) sequence space X defines a (complete) onorm. By a *sequence space* we understand any vector subspace of $\mathbb{R}^{\mathbb{N}}$.

Proposition 1. *Let X be a sequence space with a norm satisfying the condition*

$$\mathbf{0} \preccurlyeq \mathbf{x} \preccurlyeq \mathbf{y},\ \mathbf{x} \in \overline{\mathbb{R}}_+^{\mathbb{N}},\ \mathbf{y} \in X \ \Rightarrow\ \mathbf{x} \in X,\ \|\mathbf{x}\| \le \|\mathbf{y}\|. \tag{1}$$

Then the function $\rfloor\cdot\lfloor_X : \overline{\mathbb{R}}_+^{\mathbb{N}} \to \overline{\mathbb{R}}_+$ defined by the formula

$$\rfloor\mathbf{x}\lfloor_X := \begin{cases} \|\mathbf{x}\| & \text{if } \mathbf{x} \in X, \\ \infty & \text{otherwise}, \end{cases}$$

is an onorm on $\overline{\mathbb{R}}_+^{\mathbb{N}}$.

Moreover, *if X is complete and the following condition holds*:

$$\mathbf{x}^n = (x_k^n)_{k\in\mathbb{N}} \in X \text{ for } n \in \mathbb{N},\ \lim_{n\to\infty}\mathbf{x}^n = \mathbf{0} \ \Rightarrow\ \lim_{n\to\infty}x_k^n = 0 \text{ for } k \in \mathbb{N}, \tag{2}$$

then $\rfloor\cdot\lfloor_X$ is a complete onorm.

Proof. It is obvious that under condition (1), $\rfloor\cdot\lfloor$ is an onorm.

We show that (2) together with the completeness of X implies that $\rfloor\cdot\lfloor_X$ is complete.

Let $(\mathbf{x}^n) = (x_k^n)_{k\in\mathbb{N}}$ be a sequence in $\overline{\mathbb{R}}_+^{\mathbb{N}}$. If $\sum_{n=1}^{\infty}\rfloor\mathbf{x}^n\lfloor = \infty$ then (ii)$'$ holds trivially. So assume $\sum_{n=1}^{\infty}\rfloor\mathbf{x}^n\lfloor < \infty$. This means that $\mathbf{x}^n \in X$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\|\mathbf{x}^n\| < \infty$. By the completeness of X the series $\sum_{n=1}^{\infty}\mathbf{x}^n$ is convergent (in the norm topology of X) to some $\mathbf{z} = (z_k)_{k\in\mathbb{N}} \in X$ and

$\|\mathbf{z}\| \le \sum_{n=1}^{\infty} \|\mathbf{x}^n\|$. The proof of (ii)$'$ will be complete if we show that

$$z_k = \sum_{n=1}^{\infty} x_k^n \quad \text{for every } k \in \mathbb{N}.$$

But $\lim_{N\to\infty}(\mathbf{z} - \sum_{n=1}^{N} \mathbf{x}^n) = 0$, which by (2) means that for every $k \in \mathbb{N}$ we have $\lim_{N\to\infty}(z_k - \sum_{n=1}^{N} x_k^n) = 0$. Consequently, $\sum_{n=1}^{\infty} x_k^n = z_k$ for each $k \in \mathbb{N}$. ■

Making use of Proposition 1 we give a few examples of complete onorms.

If $X = l^p$, $p \in [1,\infty]$, then the onorm defined in Proposition 1 takes the form

$$\rfloor (x_k) \lfloor_{l^p} := \Big(\sum_k x_k^p\Big)^{1/p} \quad \text{if } p \in [1,\infty), \quad \rfloor (x_k) \lfloor_{l^\infty} := \sup_{k\in\mathbb{N}} x_k.$$

More generally, if X is a weighted l^p-space with $p \in [1,\infty]$ then the corresponding onorm has the form

$$\rfloor (x_k) \lfloor_{r,l^p} := \Big(\sum_k (r^k x_k)^p\Big)^{1/p} \quad \text{if } p \in [1,\infty), \quad \rfloor (x_k) \lfloor_{r,l^\infty} := \sup_{k\in\mathbb{N}} r^k x_k,$$

where $r > 0$ is fixed.

If $X = c_0$ then

$$\rfloor (x_k) \lfloor_{c_0} := \begin{cases} \sup_{k\in\mathbb{N}} x_k & \text{if } \lim_{k\to\infty} x_k = 0, \\ \infty & \text{otherwise.} \end{cases}$$

We have proved in Proposition 1 that a (reasonable) norm on a complete sequence space produces a complete onorm. Now we prove that a complete onorm can be used to define a complete metric on some set of sequences.

Proposition 2. *Assume that* $\rfloor\cdot\lfloor$ *is a complete onorm. Let* (X,d) *be a complete metric space and let* $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ *be a fixed sequence in* X. *Let*

$$\mathcal{X} := \{(z_k)_{k\in\mathbb{N}} : \rfloor (d(x_k, z_k))_{k\in\mathbb{N}} \lfloor < \infty\}.$$

Then the function $\mathbf{d} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ *defined by*

$$\mathbf{d}(\mathbf{y},\mathbf{z}) = \rfloor (d(y_k, z_k))_{k\in\mathbb{N}} \lfloor$$

is a complete metric on $\mathcal{X}$.

Proof. Clearly $\mathbf{d}$ is a metric on $\mathcal{X}$. We show that $(\mathcal{X},\mathbf{d})$ is complete.

Let $\mathbf{e}_i = (\delta_k^i)_{k\in\mathbb{N}}$, where δ_k^i is the Kronecker delta. Clearly $\rfloor \mathbf{e}_i \lfloor > 0$ for $i \in \mathbb{N}$.

Consider an arbitrary Cauchy sequence $(\mathbf{z}^n)_{n\in\mathbb{N}} = ((z_k^n)_{k\in\mathbb{N}})_{n\in\mathbb{N}}$ in $\mathcal{X}$. To show that $(\mathbf{z}^n)$ is componentwise convergent, fix $k \in \mathbb{N}$. For $n, m \in \mathbb{N}$ we have

$$d(z_k^n, z_k^m)\mathbf{e}_k \preccurlyeq (d(z_l^n, z_l^m))_{l\in\mathbb{N}},$$

and consequently

$$d(z_k^n, z_k^m) \cdot \rfloor \mathbf{e}_k \lfloor = \rfloor d(z_k^n, z_k^m) \cdot \mathbf{e}_k \lfloor \leq \rfloor (d(z_l^n, z_l^m))_{l \in \mathbb{N}} \lfloor = \mathbf{d}(\mathbf{z}^n, \mathbf{z}^m).$$

Since $(\mathbf{z}^n)_{n \in \mathbb{N}}$ is a Cauchy sequence and $\rfloor \mathbf{e}_k \lfloor > 0$ we infer that $(z_k^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X, and hence it is convergent.

Let $z_k := \lim_{n \to \infty} z_k^n$ and $\mathbf{z} := (z_k)_{k \in \mathbb{N}}$. We will prove that $\mathbf{z} \in \mathcal{X}$ and $\lim_{n \to \infty} \mathbf{d}(\mathbf{z}^n, \mathbf{z}) = 0$. Since $(\mathbf{z}^n)_{n \in \mathbb{N}}$ is a Cauchy sequence it has a subsequence $(\mathbf{z}^{n_l})_{l \in \mathbb{N}}$ such that

$$\mathbf{d}(\mathbf{z}^{n_l}, \mathbf{z}^{n_{l+1}}) = \rfloor (d(z_k^{n_l}, z_k^{n_{l+1}}))_{k \in \mathbb{N}} \lfloor < 1/2^l \quad \text{for } l \in \mathbb{N}.$$

But $\lim_{l \to \infty} z_k^{n_l} = z_k$, so

$$d(z_k^{n_j}, z_k) \leq \sum_{i=j}^{\infty} d(z_k^{n_i}, z_k^{n_{i+1}}) \quad \text{for every } k, j \in \mathbb{N}.$$

Thus

$$(d(z_k^{n_j}, z_k))_{k \in \mathbb{N}} \preccurlyeq \Big(\sum_{i=j}^{\infty} d(z_k^{n_i}, z_k^{n_{i+1}}) \Big)_{k \in \mathbb{N}} \quad \text{for } j \in \mathbb{N},$$

and hence

$$\begin{aligned} \rfloor (d(z_k^{n_j}, z_k))_{k \in \mathbb{N}} \lfloor &\leq \Big\rfloor \Big(\sum_{i=j}^{\infty} d(z_k^{n_i}, z_k^{n_{i+1}}) \Big)_{k \in \mathbb{N}} \Big\lfloor \\ &= \Big\rfloor \sum_{i=j}^{\infty} (d(z_k^{n_i}, z_k^{n_{i+1}}))_{k \in \mathbb{N}} \Big\lfloor \leq \sum_{i=j}^{\infty} \rfloor (d(z_k^{n_i}, z_k^{n_{i+1}}))_{k \in \mathbb{N}} \lfloor \\ &\leq \sum_{i=j}^{\infty} \frac{1}{2^i} = \frac{1}{2^{j-1}} \quad \text{for } j \in \mathbb{N}, \end{aligned}$$

so we have

$$\rfloor (d(z_k^{n_j}, z_k))_{k \in \mathbb{N}} \lfloor \leq 1/2^{j-1} \quad \text{for } j \in \mathbb{N}. \tag{3}$$

Making use of this inequality with a fixed $j \in \mathbb{N}$ we obtain

$$\rfloor (d(x_k, z_k))_{k \in \mathbb{N}} \lfloor \leq \rfloor (d(x_k, z_k^{n_j}))_{k \in \mathbb{N}} \lfloor + \rfloor (d(z_k^{n_j}, z_k))_{k \in \mathbb{N}} \lfloor < \infty,$$

which means that $\mathbf{x} \in \mathcal{X}$.

In virtue of (3) we have

$$\lim_{j \to \infty} \mathbf{d}(\mathbf{z}^{n_j}, \mathbf{z}) = \lim_{j \to \infty} \rfloor (d(z_k^{n_j}, z_k))_{k \in \mathbb{N}} \lfloor = 0.$$

Since $(\mathbf{z}^n)_{n \in \mathbb{N}}$ is a Cauchy sequence, this implies that $\lim_{n \to \infty} \mathbf{d}(\mathbf{z}^n, \mathbf{z}) = 0$. ■

Definition 2. We call a complete onorm $\rfloor \cdot \lfloor$ *K-contractive* if

$$\rfloor (x_{k+1})_{k \in \mathbb{N}} \lfloor \leq K \rfloor (x_k)_{k \in \mathbb{N}} \lfloor \quad \text{for every } (x_k)_{k \in \mathbb{N}} \in \overline{\mathbb{R}}_+^{\mathbb{N}}.$$

Example 1. As one can see, the onorms $\rfloor \cdot \lfloor_{l^p}$ and $\rfloor \cdot \lfloor_{c_0}$ are 1-contractive, and $\rfloor \cdot \lfloor_{l^p_r}$ is $1/r$-contractive.

3. Orbits and generalized shadowing. In this section we generalize the classical definition of shadowing with the use of a complete onorm. One can easily see that specifying a complete onorm we can obtain all the previously mentioned notions of shadowing.

If not specified otherwise, $\rfloor\cdot\lfloor$ denotes a fixed complete onorm. Let (X,d) be a metric space and let $\phi : X \to X$ be given.

DEFINITION 3. We say that a sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is an *orbit for* ϕ (or a ϕ-orbit) provided $x_{k+1} = \phi(x_k)$ for every $k \in \mathbb{N}$.

DEFINITION 4. Let $\delta > 0$ be given. A sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is called a $(\delta, \rfloor\cdot\lfloor)$-*pseudoorbit for* ϕ provided

$$\rfloor(d(x_{k+1}, \phi(x_k)))_{k\in\mathbb{N}}\lfloor \le \delta.$$

A sequence $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is called a *pseudoorbit for* ϕ provided

$$\rfloor(d(x_{k+1}, \phi(x_k)))_{k\in\mathbb{N}}\lfloor < \infty.$$

DEFINITION 5. We say that a $\rfloor\cdot\lfloor$-pseudoorbit $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ for ϕ is $(\varepsilon, \rfloor\cdot\lfloor)$-*shadowed* (*or traced*) by a ϕ-orbit $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ if

$$\rfloor(d(x_k, y_k))_{k\in\mathbb{N}}\lfloor \le \varepsilon.$$

REMARK 1. The case most often considered (and simplest) is $\rfloor\cdot\lfloor := \rfloor\cdot\lfloor_{l^\infty}$. Then we speak simply of pseudoorbits. Thus $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is a δ-pseudoorbit for ϕ if

$$d(x_{k+1}, \phi(x_k)) \le \delta \quad \text{for } k \in \mathbb{N}.$$

Analogously, a pseudoorbit $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is ε-shadowed by a ϕ-orbit $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ if

$$d(x_k, y_k) \le \varepsilon \quad \text{for } k \in \mathbb{N}.$$

Any $(\delta, \rfloor\cdot\lfloor)$-pseudoorbit can be considered as a result of a numerical computation of a trajectory, where δ represents the error.

DEFINITION 6. We say that a discrete semidynamical system ϕ has the $\rfloor\cdot\lfloor$-*shadowing property* if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $(\delta, \rfloor\cdot\lfloor)$-pseudoorbit $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ there exists an orbit $\mathbf{y}$ such that $\mathbf{x}$ is $(\varepsilon, \rfloor\cdot\lfloor)$-shadowed by $\mathbf{y}$.

We mention that usually when one talks about shadowing one has in mind shadowing of the whole orbit, and not only of the positive trajectory. Thus what we here call shadowing is sometimes called *shadowing*$^+$ (see [1]). However, since we do not consider the general shadowing at all, for simplicity we speak of shadowing and not of shadowing$^+$.

REMARK 2. Using onorms we are able to unify some results concerning several classical shadowings. For example, for the onorm $\rfloor\cdot\lfloor_{l^\infty}$ we obtain the standard shadowing; $\rfloor\cdot\lfloor_{c_0}$ defines limits shadowing (cf. [3, p. 63]); the onorm

$\rfloor\cdot\lfloor_{l^p}$ gives us $\mathcal{L}_p$-shadowing (cf. [3, p. 68]); and for the onorm $\rfloor\cdot\lfloor_{r,l^p}$ we get weighted $\mathcal{L}_{\bar{r},p}$ shadowing with the sequence $\bar{r} = (r^k)_{k\in\mathbb{N}}$ (cf. [3, p. 71]).

4. Shadowing. Now we are ready to formulate the main result of this paper. It is a generalization of the result from [1]. We show that the notion of an onorm can be useful in proving classical shadowing results in a more general setting.

THEOREM 1. *Let (X, d) be a complete metric space. Let $\phi : X \to X$ be a bijection, let $l = \mathrm{lip}(\phi^{-1})$ and let $\rfloor\cdot\lfloor$ be a K-contractive complete onorm on $\overline{\mathbb{R}}_+^{\mathbb{N}}$. Assume that $Kl < 1$. If $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is an $\rfloor\cdot\lfloor$-pseudoorbit then there exists a unique orbit $\mathbf{y} = (y_k)_{k\in\mathbb{N}}$ which $\rfloor\cdot\lfloor$-shadows $\mathbf{x}$. Moreover, if $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$ is an $(\varepsilon, \rfloor\cdot\lfloor)$-pseudoorbit for some $\varepsilon > 0$, then*

$$\rfloor(d(x_k, y_k))_{k\in\mathbb{N}}\lfloor \le \frac{l\varepsilon}{1 - Kl}.$$

REMARK 3. We underline that in Theorem 1 the continuity of ϕ is not assumed.

Proof of Theorem 1. Fix an ε-pseudoorbit $\mathbf{x} = (x_k)_{k\in\mathbb{N}}$. Define

$$\mathcal{X} := \{\mathbf{z} = (z_k)_{k\in\mathbb{N}} : \rfloor(d(x_k, z_k))_{k\in\mathbb{N}}\lfloor < \infty\}.$$

By Proposition 2 we know that

$$\mathbf{d}(\mathbf{y}, \mathbf{z}) = \rfloor(d(y_k, z_k))_{k\in\mathbb{N}}\lfloor$$

defines a complete metric on $\mathcal{X}$. Define a mapping $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{P}((z_k)_{k\in\mathbb{N}}) := (\phi^{-1}(z_{k+1}))_{k\in\mathbb{N}}.$$

We will show now that $\mathcal{P}$ is a well defined contraction. Since ϕ^{-1} is Lipschitz with Lipschitz constant l, we have

$$\begin{aligned}\rfloor(d(x_k, \phi^{-1}(x_{k+1})))_{k\in\mathbb{N}}\lfloor &= \rfloor(d(\phi^{-1}(\phi(x_k)), \phi^{-1}(x_{k+1})))_{k\in\mathbb{N}}\lfloor\\ &\le \rfloor l(d(\phi(x_k), x_{k+1}))_{k\in\mathbb{N}}\lfloor = l\rfloor(d(\phi(x_k), x_{k+1}))_{k\in\mathbb{N}}\lfloor \le l\varepsilon.\end{aligned}$$

This proves that $\mathcal{P}(\mathbf{x}) \in \mathcal{X}$ and

$$\mathbf{d}(\mathbf{x}, \mathcal{P}(\mathbf{x})) \le l\varepsilon. \tag{4}$$

Consider an arbitrary $\mathbf{z} = (z_k)_{k\in\mathbb{N}} \in \mathcal{X}$. Then we have

$$\begin{aligned}&\rfloor(d(x_k, \phi^{-1}(z_{k+1})))_{k\in\mathbb{N}}\lfloor\\ &\quad\le \rfloor(d(x_k, \phi^{-1}(x_{k+1})) + d(\phi^{-1}(x_{k+1}), \phi^{-1}(z_{k+1})))_{k\in\mathbb{N}}\lfloor\\ &\quad= \rfloor(d(x_k, \phi^{-1}(x_{k+1})))_{k\in\mathbb{N}} + (d(\phi^{-1}(x_{k+1}), \phi^{-1}(z_{k+1})))_{k\in\mathbb{N}}\lfloor\\ &\quad\le \rfloor(d(x_k, \phi^{-1}(x_{k+1})))_{k\in\mathbb{N}}\lfloor + \rfloor(d(\phi^{-1}(x_{k+1}), \phi^{-1}(z_{k+1})))_{k\in\mathbb{N}}\lfloor\\ &\quad\le \mathbf{d}(\mathbf{x}, \mathcal{P}(\mathbf{x})) + l\rfloor(d(x_{k+1}, z_{k+1}))_{k\in\mathbb{N}}\lfloor \le l\varepsilon + lK\rfloor(d(x_k, z_k))_{k\in\mathbb{N}}\lfloor < \infty.\end{aligned}$$

Thus $\mathcal{P}(\mathbf{z}) \in \mathcal{X}$.

Now we show that $\mathcal{P}$ is a contraction. Fix $\mathbf{w} = (w_k)_{k\in\mathbb{N}}, \mathbf{z} = (z_k)_{k\in\mathbb{N}} \in \mathcal{X}$. Then

$$\begin{aligned}\mathbf{d}(\mathcal{P}(\mathbf{w}), \mathcal{P}(\mathbf{z})) &= \rfloor(d(\phi^{-1}(w_{k+1}), \phi^{-1}(z_{k+1})))_{k\in\mathbb{N}}\lfloor \\ &\le \rfloor l(d(w_{k+1}, z_{k+1}))_{k\in\mathbb{N}}\lfloor = l\rfloor(d(w_{k+1}, z_{k+1}))_{k\in\mathbb{N}}\lfloor \\ &\le lK\rfloor(d(w_k, z_k))_{k\in\mathbb{N}}\lfloor = Kl\mathbf{d}(\mathbf{w}, \mathbf{z}).\end{aligned}$$

Hence $\mathcal{P}$ is a contraction with constant $Kl < 1$. Thus using the Banach Contraction Principle we derive that $\mathcal{P}$ has a unique fixed point $\mathbf{y} \in \mathcal{X}$, and on account of (4) we get

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) \le \frac{\mathbf{d}(\mathbf{x}, \mathcal{P}(\mathbf{x}))}{1 - Kl} \le \frac{l\varepsilon}{1 - Kl}.$$

Since $\mathcal{P}(\mathbf{y}) = \mathbf{y}$, we get $(\phi^{-1}(y_{k+1}))_{k\in\mathbb{N}} = (y_k)_{k\in\mathbb{N}}$, which means that $\mathbf{y}$ is an orbit for ϕ. To prove uniqueness of $\mathbf{y}$ suppose that $\widetilde{\mathbf{y}} = (\widetilde{y}_k)_{k\in\mathbb{N}}$ is a ϕ-orbit which shadows $\mathbf{x}$. Then $\widetilde{\mathbf{y}} \in \mathcal{X}$ and

$$\mathcal{P}((\widetilde{y}_k)_{k\in\mathbb{N}}) = (\phi^{-1}(\widetilde{y}_{k+1}))_{k\in\mathbb{N}} = (\widetilde{y}_k)_{k\in\mathbb{N}},$$

which means that $\widetilde{\mathbf{y}}$ is a fixed point for $\mathcal{P}$. Hence $\widetilde{\mathbf{y}} = \mathbf{y}$. ■

As a corollary we obtain a classical shadowing result [1, Proposition 3] in a slightly more general setting.

COROLLARY 1. *Let $\phi : X \to X$ be a bijection such that ϕ^{-1} is a contraction. Then ϕ has the shadowing property and ϕ has the limit shadowing property.*

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