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On prolongation of connections

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Abstract. Let $Y \to M$ be a fibred manifold with m-dimensional base and n-dimensional fibres. Let r, m, n be positive integers. We present a construction B^r of rth order holonomic connections $B^r(\Gamma, \nabla) : Y \to J^rY$ on $Y \to M$ from general connections $\Gamma : Y \to J^1Y$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M. Then we prove that any construction B of rth order holonomic connections $B(\Gamma, \nabla) : Y \to J^rY$ on $Y \to M$ from general connections $\Gamma : Y \to J^1Y$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M is equal to B^r. Applying B^r, for any bundle functor $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ on fibred (m, n)-manifolds we present a construction $\mathcal{F}^r_q$ of rth order holonomic connections $\mathcal{F}^r_q(\Theta, \nabla) : FY \to J^r(FY)$ on $FY \to M$ from qth order holonomic connections $\Theta : Y \to J^qY$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M (for $q = r = 1$ we have a well-known classical construction $\mathcal{F}(\Gamma, \nabla) : FY \to J^1(FY)$). Applying B^r we also construct a so-called (Γ, ∇)-lift of a wider class of geometric objects. In Appendix, we present a direct proof of a (recent) result saying that for $r \geq 3$ and $m \geq 2$ there is no construction A of rth order holonomic connections $A(\Gamma) : Y \to J^rY$ on $Y \to M$ from general connections $\Gamma : Y \to J^1Y$ on $Y \to M$.

0. Introduction. Higher order connections were first introduced on groupoids by C. Ehresmann [7]. Then I. Kolář extended this concept to fibred manifolds [10]. It is well-known that higher order connections are a powerful tool in the theory of higher order absolute differentiation [10]. Some other applications of such connections can be found e.g. in [1], [3], [13], [14]. Roughly speaking, higher order connections are sections of bundles of higher order jets. The latter were introduced by C. Ehresmann [6], and they are a powerful tool in differential geometry and in many areas of mathematical physics. Indeed, they globalize the theory of differential systems and play an important role in the calculus of variations and in the theory of partial differential systems (see [24], [25]). The theory of jets and (principal) connections constitutes the geometrical background for field theories and

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theoretical physics (see [16], [23], [15]). The theory of jets and connections is closely connected to the theory of natural operations [12]. Some results devoted to the prolongation of connections can be found e.g. in [3], [4], [5], [8], [9], [11], [13], [18], [19], [22].

Let $p : Y \to M$ be a fibred manifold with m-dimensional base and n-dimensional fibres. We recall that a holonomic *rth order connection on* $p : Y \to M$ is a section $\Theta : Y \to J^rY$ of the rth jet prolongation $J^rY \to Y$ of $p : Y \to M$. For $r = 1$ we obtain the concept of general connections $\Gamma : Y \to J^1Y$ on $p : Y \to M$. For any general connection $\Gamma : Y \to J^1Y$ one can (equivalently) consider the corresponding lifting map $\Gamma : Y \times_M TM \to TY$, $\Gamma(y, v) = T_x\sigma(v)$, $y \in Y_x$, $v \in T_xM$, $x \in M$, $\Gamma(y) = j^1_x\sigma$. If $p : Y \to M$ is a vector bundle and $\Theta : Y \to J^rY$ is a vector bundle map, then Θ is called a *linear rth order connection on $p : Y \to M$*. A linear connection $\lambda : TM \to J^rTM$ on the tangent bundle $p : TM \to M$ of M is called a *linear rth order connection on M*. For $r = 1$, we have the concept of classical linear connection $\lambda : TM \to J^1TM$ on M; equivalently, we can consider the corresponding covariant derivative $\nabla = \nabla(\lambda) : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$. A classical linear connection ∇ on M is called *torsion free* if its torsion tensor $T(X, Y) = \nabla_X Y - \nabla_Y Y - [X, Y]$ is equal to zero.

In Section 1, given a general connection $\Gamma : Y \to J^1Y$ on $p : Y \to M$ (as above) and a torsion free classical linear connection ∇ on M we construct (for any r) an rth order holonomic connection $B^r(\Gamma, \nabla) : Y \to J^rY$ on $p : Y \to M$. In other words, we obtain an $\mathcal{FM}_{m,n}$-natural operator B^r in the sense of [12] (see also below), where $\mathcal{FM}_{m,n}$ is the category of all fibred manifolds with m-dimensional bases and n-dimensional fibres and their local fibred embeddings. We remark (see Remark 3) that $B^r(\Gamma, \nabla)$ can be used in the theory of fields of higher order geometric objects. The main result of Section 1 can be stated as follows. *Any canonical construction B of rth order holonomic connections $B(\Gamma, \nabla) : Y \to J^rY$ on $Y \to M$ from general connections $\Gamma : Y \to J^1Y$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M is equal to B^r.*

In Section 2, applying B^r (where r is arbitrary), given a qth order holonomic connection $\Theta : Y \to J^qY$ on $p : Y \to M$ and a torsion free classical linear connection ∇ on M we construct an rth order holonomic connection $\mathcal{F}^r_q(\Theta, \nabla) : FY \to J^r(FY)$ on $FY \to M$, where $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is an arbitrary bundle functor. Thus for $r = q = 1$ and $\mathrm{ord}(F) = 1$ we recover the classical construction (see Definition 45.4 in [12]) of a general connection $\mathcal{F}(\Gamma, \nabla) : FY \to J^1(FY)$ on $FY \to M$ from a general connection $\Gamma : Y \to J^1Y$ on $Y \to M$ by means of a torsion free classical linear connection ∇ on M. We also study (Proposition 4) the existence of rth order holonomic connections $B(\Theta, \nabla) : FY \to J^r(FY \to Y)$

on $FY \to Y$ coming from qth order holonomic connections $\Theta : Y \to J^qY$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M.

In Section 3, applying B^r we construct a so-called (Γ, ∇)-lift of a wider class of geometric objects. In particular, we construct a (Γ, ∇)-lift of linear rth order differential operators $C^\infty(M) \to C^\infty(M)$ to rth order linear differential operators $C^\infty(Y) \to C^\infty(Y)$.

In Appendix, we present a direct proof of a result from [3] saying that for $r \geq 3$ and $m \geq 2$ there is no canonical construction A of rth order holonomic connections $A(\Gamma) : Y \to J^rY$ on $Y \to M$ from general connections $\Gamma : Y \to J^1Y$ on $Y \to M$.

In what follows we use the terminology and notation from the book [12]. In particular, we denote by $\mathcal{M}f_m$ the category of m-dimensional manifolds and their local diffeomorphisms and by $\mathcal{FM}_{m,n}$ the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and local fibred diffeomorphisms. All canonical constructions are identified with their corresponding natural operators in the sense of [12].

A general concept of natural operators can be found in [12]. In particular, an $\mathcal{FM}_{m,n}$-natural operator B transforming general connections $\Gamma : Y \to J^1Y$ on $Y \to M$ and torsion free classical linear connections ∇ on M into rth order holonomic connections $B(\Gamma, \nabla) : Y \to J^rY$ on $Y \to M$ is an $\mathcal{FM}_{m,n}$-invariant family of regular operators (functions)

$$B_{p:Y\to M} : \mathrm{Con}_{\mathrm{gen}}(p : Y \to M) \times \mathrm{Con}^o_{\mathrm{class}}(M) \to \mathrm{Con}^r_{\mathrm{hol}}(p : Y \to M)$$

for all $\mathcal{FM}_{m,n}$-objects $p : Y \to M$, where $\mathrm{Con}_{\mathrm{gen}}(p : Y \to M)$ is the set of all general connections $\Gamma : Y \to J^1Y$ on $p : Y \to M$, $\mathrm{Con}^o_{\mathrm{class}}(M)$ is the set of all torsion free classical linear connections on M, and $\mathrm{Con}^r_{\mathrm{hol}}(p : Y \to M)$ is the set of all rth order holonomic connections $\Theta : Y \to J^rY$ on $p : Y \to M$. The $\mathcal{FM}_{m,n}$-invariance means that if $(\Gamma, \nabla) \in \mathrm{Con}_{\mathrm{gen}}(p : Y \to M) \times \mathrm{Con}^o_{\mathrm{class}}(M)$ and $(\Gamma_1, \nabla_1) \in \mathrm{Con}_{\mathrm{gen}}(p_1 : Y_1 \to M_1) \times \mathrm{Con}^o_{\mathrm{class}}(M_1)$ are f-related for an $\mathcal{FM}_{m,n}$-map $f : Y \to Y_1$ then so are $B_{p:Y\to M}(\Gamma, \nabla)$ and $B_{p_1:Y_1\to M_1}(\Gamma_1, \nabla_1)$. The regularity means that $B_{p:Y\to M}$ transforms smoothly parametrized families of pairs of connections into smoothly parametrized families of connections.

In the rest of the paper, we assume that any classical linear connection considered is torsion free. This assumption is convenient in classification problems (because of vanishing Christoffel symbols at the centre of normal coordinates), but not essential in existence problems. Indeed, from a classical linear connection ∇ one can produce a corresponding torsion free classical linear connection ∇^{sym}, the torsion free part of ∇. Hence if we have a construction $A(\cdot, \nabla)$ by means of torsion free classical linear connections ∇, then we have a construction $\bar{A}(\cdot, \nabla) := A(\cdot, \nabla^{\mathrm{sym}})$ by means of (not necessarily torsion free) classical linear connections ∇. The converse is obvious.

All manifolds and maps are assumed to be smooth (of class C^∞).

1. Holonomic prolongation of general connections. In [3], we studied the problem whether given a general connection $\Gamma : Y \to J^1Y$ on a fibred manifold $p : Y \to M$ from $\mathcal{FM}_{m,n}$ one can construct ($\mathcal{FM}_{m,n}$-canonically) an rth order holonomic connection $A(\Gamma) : Y \to J^rY$ on $Y \to M$, and obtained the solutions given below.

For $r = 1$ or $r = 2$ or $m = 1$ we have the following constructions.

Example 1. For $r = 1$, we have $A(\Gamma) = \Gamma : Y \to J^1Y$.

Example 2. For $r = 2$, we have the second order holonomic connection

$$\Gamma^{(2)} := C^{(2)} \circ (\Gamma * \Gamma) : Y \to J^2Y$$

on $Y \to M$ from a general connection $\Gamma : Y \to J^1Y$ on $Y \to M$, where $\Gamma * \Gamma = J^1\Gamma \circ \Gamma : Y \to \bar{J}^2Y$ is the Ehresmann semiholonomic prolongation of $\Gamma : Y \to J^1Y$ and $C^{(2)} : \bar{J}^2Y \to J^2Y$ is the well-known symmetrization of semiholonomic second order jets.

Example 3. We have the Ehresmann rth order semiholonomic prolongation

$$\Gamma * \cdots * \Gamma : Y \to \bar{J}^rY$$

of a general connection $\Gamma : Y \to J^1Y$ on $Y \to M$. If $\dim(M) = 1$, then $\bar{J}^rY = J^rY$, and then $\Gamma * \cdots * \Gamma : Y \to J^rY$ is an rth order holonomic connection on $Y \to M$.

In other cases, there are no constructions in question. Indeed, we have the following result.

Proposition 1 ([3]). *Let m, n, r be positive integers with $m \geq 2$ and $r \geq 3$. There is no $\mathcal{FM}_{m,n}$-natural operator A transforming connections $\Gamma : Y \to J^1Y$ into rth order holonomic connections $A(\Gamma) : Y \to J^rY$.*

In [3], we proved Proposition 1 by using a complicated result from [20]. In Appendix, we will present another direct proof of Proposition 1.

Roughly speaking, Proposition 1 says that for the existence of a construction of rth order connections $Y \to J^rY$ from general connections $Y \to J^1Y$ an additional object on $p : Y \to M$ is unavoidable, provided $r \geq 3$ and $m \geq 2$.

On the other hand, in Section 3 of [3] we proved the following lemma.

Lemma 1. *Let $\Gamma : Y \to J^1Y$ be a general connection on a fibred manifold $p : Y \to M$. Let $\tilde{\nabla}$ be a torsion free projectable classical linear connection on $p : Y \to M$ (i.e. $\tilde{\nabla}$ is a (torsion free) classical linear connection on Y*

and there exists a (unique) torsion free classical linear connection $\underline{\tilde{\nabla}}$ on M such that $\tilde{\nabla}$ and $\underline{\tilde{\nabla}}$ are p-related.) Let $y \in Y_x$, $x \in M$.

(i) *There is a normal coordinate system (U, Φ) of $\tilde{\nabla}$ with centre y covering a normal coordinate system $(\underline{U}, \underline{\Phi})$ of $\underline{\tilde{\nabla}}$ with centre x such that $J^1\Phi(\Gamma(y)) = j^1_0(0)$.*

(ii) *If (U, Ψ) is another normal coordinate system $\tilde{\nabla}$ with centre y covering a normal coordinate system $(\underline{U}, \underline{\Psi})$ of $\underline{\tilde{\nabla}}$ with centre x such that $J^1\Psi(\Gamma(y)) = j^1_0(0)$, then there exist $A \in GL(\mathbb{R}^m)$ and $B \in GL(\mathbb{R}^n)$ such that $\Psi = (A \times B) \circ \Phi$ on some neighbourhood of y.*

Then, using Lemma 1, we presented the following construction.

EXAMPLE 4 ([3]). Let $\Gamma : Y \to J^1Y$ be a general connection on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$, and $\tilde{\nabla}$ be a torsion free projectable classical linear connection on $p : Y \to M$. We define an rth order holonomic connection $B^r_1(\Gamma, \tilde{\nabla}) : Y \to J^rY$ on $p : Y \to M$ by

$$B^r_1(\Gamma, \tilde{\nabla})(y) := J^r(\Phi^{-1})(j^r_0(0))$$

where Φ is as in Lemma 1(i). By Lemma 1(ii), the definition of $B^r_1(\Gamma, \tilde{\nabla})(y)$ is independent of the choice of Φ. We see that $B^r_1(\Gamma, \tilde{\nabla})$ is an rth order holonomic extension of Γ, i.e. $\pi^r_1 \circ B^r_1(\Gamma, \tilde{\nabla}) = \Gamma$, where $\pi^r_1 : J^rY \to J^1Y$ is the jet projection.

Because of the canonical character of the construction of $B^r_1(\Gamma, \tilde{\nabla})$, the correspondence $B^r_1 : (\Gamma, \tilde{\nabla}) \to B^r_1(\Gamma, \tilde{\nabla})$ is an $\mathcal{FM}_{m,n}$-natural operator. It is of order 0 in Γ. In other words, we have a well-defined section $D^r_1(\tilde{\nabla}) : J^1Y \to J^rY$ of the jet projection $J^rY \to J^1Y$, given by

$$D^r_1(\tilde{\nabla})(\rho) := B^r_1(\Gamma, \tilde{\nabla})(y),$$

$\rho \in (J^1Y)_y$, $y \in Y$, where $\Gamma : Y \to J^1Y$ is a general connection on $Y \to M$ such that $\Gamma(y) = \rho$. Clearly, $B^r_1(\Gamma, \tilde{\nabla}) = D^r_1(\tilde{\nabla}) \circ \Gamma$.

REMARK 1. In [3], we generalized the above natural operator $B^r_1(\Gamma, \tilde{\nabla})$ directly to higher order general holonomic connections $\Theta : Y \to J^qY$ in place of Γ. Namely, we presented a canonical construction (natural operator) B^r_q for $r > q \geq 1$ of an rth order holonomic connection $B^r_q(\Theta, \tilde{\nabla}) : Y \to J^rY$ on $Y \to M$ from a qth order holonomic connection $\Theta : Y \to J^qY$ on $Y \to M$ by using a torsion free projectable classical linear connection $\tilde{\nabla}$ on $Y \to M$. Then one can define a section $D^r_q(\tilde{\nabla}) : J^qY \to J^rY$ of the jet projection $J^rY \to J^qY$ (a direct generalization of $D^r_1(\tilde{\nabla}) : J^1Y \to J^rY$). Then $B^r_q(\Theta, \tilde{\nabla}) = D^r_q(\tilde{\nabla}) \circ \Theta$.

In [17], we proved the following result.

PROPOSITION 2. *There is no $\mathcal{FM}_{m,n}$-canonical construction of torsion free classical linear connections $B(\Gamma, \underline{\nabla})$ on Y from torsion free classical* *linear connections $\underline{\nabla}$ on M by means of general connections $\Gamma : Y \to J^1Y$ on $Y \to M$.*

So, we have the following natural problem.

PROBLEM 1. *Let $r \geq 1$. Given a general connection $\Gamma : Y \to J^1Y$ on $p : Y \to M$ and a torsion free classical linear connection ∇ on M, construct an rth order holonomic connection $B^r(\Gamma, \nabla) : Y \to J^rY$ on $p : Y \to M$ such that $B^r(\Gamma, \nabla)$ is an extension of Γ, i.e. $\pi^r_1 \circ B^r(\Gamma, \nabla) = \Gamma$.*

Unfortunately, because of Proposition 3 (see below) we have no canonical construction $\tilde{D}^r_1$ of a section $\tilde{D}^r_1(\nabla) : J^1Y \to J^rY$ of the jet projection $J^rY \to J^1Y$ from a torsion free classical linear connection ∇ on M. Therefore in solving Problem 1 we must use a more complicated method than the one used in Example 4.

PROPOSITION 3. *Let m, n be positive integers. For $r > q \geq 1$, there is no $\mathcal{FM}_{m,n}$-natural operator A transforming general connections $\Gamma : Y \to J^1Y$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ and torsion free classical linear connections ∇ on M into sections $A(\Gamma, \nabla) : J^qY \to J^rY$ of the jet projection $J^rY \to J^qY$.*

Proof. Suppose that such an operator A exists. Let $\Gamma_o : \mathbb{R}^{m,n} \to J^1(\mathbb{R}^{m,n})$ be the trivial general connection on $\mathbb{R}^{m,n}$. Let ∇_o be the usual flat torsion free classical linear connection on $\mathbb{R}^m$. Since $A(\Gamma_o, \nabla_o) : J^q\mathbb{R}^{m,n} \to J^r\mathbb{R}^{m,n}$ is a section, $A(\Gamma_o, \nabla_o)(j^q_0(x^1, 0, \ldots, 0)) = j^r_0(x^1 + \sigma^1(x), 0 + \sigma^2(x), \ldots, 0 + \sigma^n(x))$ for some $\sigma^j : \mathbb{R}^m \to \mathbb{R}$ with $j^q_0(\sigma^j) = 0$. The $\mathcal{FM}_{m,n}$-map $\psi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ given by $\psi(x, y) = (x, y^1 + (y^1)^{q+1}, y^2, \ldots, y^n)$, $x \in \mathbb{R}^m$, $y = (y^1, \ldots, y^n) \in \mathbb{R}^n$, preserves Γ_o, ∇_o and $j^q_0(x^1, 0, \ldots, 0)$, but it does not preserve $j^r_0(x^1 + \sigma^1, \sigma^2, \ldots, \sigma^n)$. Contradiction. ∎

To present a local coordinate solution of Problem 1 we need some "special" fibred coordinates presented in Lemma 2 (below). We start with the following notation.

Let $\Phi_r : J^{r-1}_0(T^*\mathbb{R}^m \otimes \mathbb{R}^n) \to J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$ be the composition

$$\begin{aligned} J^{r-1}_0(T^*\mathbb{R}^m \otimes \mathbb{R}^n) &= \bigoplus_{k=0}^{r-1} S^kT^*_0\mathbb{R}^m \otimes T^*_0\mathbb{R}^m \otimes \mathbb{R}^n \\ &\to \bigoplus_{k=0}^{r-1} S^{k+1}T^*_0\mathbb{R}^m \otimes \mathbb{R}^n = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0, \end{aligned}$$

where the arrow is defined by symmetrization and the equalities are the usual ($GL(m)$-invariant) identifications. In other words, Φ_r is the linear map such that

$$\Phi_r(j^{r-1}_0((x^{i_1} \cdots x^{i_k} dx^j)e_s)) = \frac{1}{k+1}\, j^r_0(x^{i_1} \cdots x^{i_k} x^j e_s)$$

for any $i_1, \dots, i_k, j = 1, \dots, m$, $k = 0, \dots, r-1$ and $s = 1, \dots, n$, where e_s is the usual canonical basis in $\mathbb{R}^n$ and $x^1, \dots, x^m$ are the usual coordinates on $\mathbb{R}^m$. Then

$$\Phi_r(j_0^{r-1}(d\sigma)) = j_0^r(\sigma)$$

for any $\sigma : \mathbb{R}^m \to \mathbb{R}^n$ with $\sigma(0) = 0$. Clearly, Φ_r is $GL(m)$-invariant and linear.

LEMMA 2. *Let r, m, n be positive integers. Let $\Gamma : Y \to J^1Y$ be a general connection on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$, ∇ be a torsion free classical linear connection on M, $y_o \in Y$ be a point, $x_o = p(y_o) \in M$.*

(1) *There exists a fibred coordinate system ψ on Y with $\psi(y_o) = (0,0) \in \mathbb{R}^m \times \mathbb{R}^n$ covering a ∇-normal coordinate system $\underline{\psi}$ on M with centre x_o such that*

$$\Phi_r(j_0^{r-1}((\psi_*\Gamma)^0)) = j_0^r(0),$$

where $\psi_*\Gamma : \mathbb{R}^{m,n} \to J^1(\mathbb{R}^{m,n})$ is the image of Γ under ψ and $(\psi_*\Gamma)^0 : \mathbb{R}^m \to T^*\mathbb{R}^m \otimes \mathbb{R}^n$ is the $\mathbb{R}^n$-valued 1-form given by $(\psi_*\Gamma)^0(x) = (\psi_*\Gamma)(x,0) \in J_x^1(\mathbb{R}^m, \mathbb{R}^n)_0 = T_x^*\mathbb{R}^m \otimes \mathbb{R}^n$, $x \in \mathbb{R}^m$, and Φ_r is defined above.

(2) *If ψ^1 is another such coordinate system then $\phi = \psi^1 \circ \psi^{-1}$ is of the form $\phi(x,y) = (A(x), \tilde{\phi}(x,y))$ for some $A \in GL(m)$ and $\tilde{\phi} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ with $j_0^r((\tilde{\phi})^0) = 0$, where $(\tilde{\phi})^0(x) = \tilde{\phi}(x,0)$. In other words, if ψ^1 is another such coordinate system then there exist $A \in GL(m)$ and $\tilde{\phi} : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$j_0^r((\tilde{\phi})^0) = 0 \quad \textit{and} \quad \psi^1 = \phi \circ \psi,$$

where $(\tilde{\phi})^0(x) = \tilde{\phi}(x,0)$ and $\phi(x,y) = (Ax, \tilde{\phi}(x,y))$.

Proof. Because of the existence of ∇-normal coordinates, we may assume $Y = \mathbb{R}^{m,n}$, $y_o = (0,0)$ and $\mathrm{id}_{\mathbb{R}^m}$ is a ∇-normal coordinate system with centre 0. Let $\Gamma = \sum_{i=1}^m dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i=1}^m \sum_{k=1}^n \Gamma_i^k dx^i \otimes \frac{\partial}{\partial y^k}$ be the lifting (affinor) presentation of the general connection, where $x^1, \dots, x^m, y^1, \dots, y^n$ are the usual coordinates on $\mathbb{R}^{m,n}$. We may assume that $\Gamma_i^k : \mathbb{R}^{m,n} \to \mathbb{R}$ are polynomials of degree $\leq r-1$.

(1) We will apply induction with respect to r.

(i) *The case $r = 1$.* Let $\Gamma(0,0) = j_0^1(\sigma)$, where $\sigma : \mathbb{R}^m \to \mathbb{R}^n$ is the linear map. Define $\psi : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$, $\psi(x,y) = (x, y - \sigma(x))$. Then $(\psi_*\Gamma)(0,0) = j_0^1(0)$. Hence $j_0^0((\psi_*\Gamma)^0) = (\psi_*\Gamma)^0(0) = 0$. Consequently, $\Phi_1(j_0^0(\psi_*\Gamma)^0) = j_0^1(0)$, as well.

(ii) *The inductive step.* By the inductive assumption for $r-1 \geq 1$ we may additionally assume that $\Phi_{r-1}(j_0^{r-2}\Gamma^0) = 0$. Let $\sigma = \sigma^s e_s : \mathbb{R}^m \to \mathbb{R}^n$ be a map such that $j_0^r(\sigma) = \Phi_r(j_0^{r-1}(\Gamma^0))$, where Γ^0 is the 1-form given by $\Gamma^0(x) = \Gamma(x,0) \in J_x^1(\mathbb{R}^m, \mathbb{R}^n)_0 = T^*\mathbb{R}^m \otimes \mathbb{R}^n$, $x \in \mathbb{R}^m$. Then (by the additional inductive assumption), $j_0^{r-1}(\sigma)=0$. Define $\psi:\mathbb{R}^{m,n}\to\mathbb{R}^{m,n}$ by $\psi(x,y)=(x,y-\sigma(x))$. We prove that ψ satisfies part (1) of the lemma for r. We see that ψ preserves x^i and $\frac{\partial}{\partial y^s}$, sends $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial x^i}-\sum_{s=1}^n\frac{\partial\sigma^s}{\partial x^i}\frac{\partial}{\partial y^s}$ and y^s to $y^s+\sigma^s$. Therefore ψ sends $\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}$ to $\sum_{i=1}^m dx^i\otimes\frac{\partial}{\partial x^i}-\sum_{s=1}^n d\sigma^s\otimes\frac{\partial}{\partial y^s}$, and Γ to $\psi_*\Gamma=\Gamma-\sum_{s=1}^n d\sigma^s\otimes\frac{\partial}{\partial y^s}+\cdots$, the dots standing for a linear combination of terms of degree $\geq r$. Then $j_0^{r-1}((\psi_*\Gamma)^0)=j_0^{r-1}(\Gamma^0)-j_0^{r-1}(d\sigma)$. Therefore $\Phi_r(j_0^{r-1}((\psi_*\Gamma)^0)=\Phi_r(j_0^{r-1}(\Gamma^0))-\Phi_r(j_0^{r-1}(d\sigma))=j_0^r(\sigma)-j_0^r(\sigma)$ $=0$, as well.

(2) We may assume $\psi=\mathrm{id}_{\mathbb{R}^{m,n}}$. Then $\Phi_r(j_0^{r-1}(\Gamma^0))=0$, $\phi=\psi^1$ and $\underline{\phi}$ is a ∇-normal coordinate system with centre 0. Then $\underline{\phi}=A$ for some $A\in GL(m)$ (as $\mathrm{id}_{\mathbb{R}^m}$ is a ∇-coordinate system with centre 0). Since A preserves $j_0^r(0)$ and A commutes with Φ_r, we may assume $A=\mathrm{id}$. Denote $\Gamma^0=(\Gamma^0)^s e_s$. Let $\sigma:=(\tilde{\phi})^0$. Let ϕ_1 be given by $\phi_1(x,y)=\phi(x,y)-(0,\sigma(x))$. Then $\phi_1(x,0)=(x,0)$. Consequently, $j_0^{r-1}(((\phi_1)_*\Gamma)^0)=j_0^{r-1}(\Gamma^0+\cdots)$, where the dots denote a linear combination of terms of the form $x^\beta(\Gamma^0)^s_k e_{s_1}$, with $(\Gamma^0)^s_k$ denoting the homogeneous part of $(\Gamma^0)^s$ of degree k. Clearly, by the definition of Φ_r we have $\Phi_r(j_0^{r-1}(x^\beta((\Gamma^0)^s_k e_{s_1})))=c_k\Phi_r(j_0^{r-1}((\Gamma^0)^s_k e_{s_1}))j_0^r(x^\beta)=j_0^r(0)$, where c_k is some real number. Then $\Phi_r(j_0^{r-1}((\phi_1)_*\Gamma)^0)=j_0^r(0)$. So, replacing ϕ by $\phi\circ(\phi_1)^{-1}$, we may assume that $\phi(x,y)=(x,y+\sigma(x))$. It remains to prove that $j_0^r(\sigma)=0$. We will proceed by induction with respect to r.

(i) *The case* $r=1$. Clearly, $\Gamma(0,0)=j_0^1(0)$. Consequently, $(\phi_*\Gamma)^0(0)=d_0\sigma$. Then $0=\Phi_1(j_0^0((\phi_*\Gamma)^0))=\Phi_1(j_0^0(d\sigma))=j_0^1(\sigma)$.

(ii) *The inductive step.* By the inductive assumption for $r-1\geq 1$ we may additionally assume that $j_0^{r-1}(\sigma)=0$. Then by the same reason as in the inductive step for (1) (with ϕ instead of ψ and $-\sigma$ instead of σ) we have $0=\Phi_r(j_0^{r-1}((\phi_*\Gamma)^0))=\Phi_r(j_0^{r-1}(\Gamma^0))+j_0^r(\sigma)=j_0^r(\sigma)$, as well. ∎

Now, we are in a position to present the following local coordinate solution of Problem 1.

Example 5. Let $\Gamma:Y\to J^1Y$ be a general connection on an $\mathcal{FM}_{m,n}$-object $p:Y\to M$. Let ∇ be a torsion free classical linear connection on M. We define an rth order holonomic connection $B^r(\Gamma,\nabla):Y\to J^rY$ on $Y\to M$ as follows. Consider an arbitrary point $y\in Y_x$, $x\in M$. Let ψ be as in Lemma 2(1) (for $\Gamma,\nabla,y_o=y$ as above). We put

$$B^r(\Gamma,\nabla)(y):=J^r(\psi^{-1})(j_0^r(0)),$$

where $j_0^r(0)\in J_0^r(\mathbb{R}^m,\mathbb{R}^n)_0=J^r_{(0,0)}(\mathbb{R}^{m,n})$ is the r-jet at $0\in\mathbb{R}^m$ of the zero-section of $\mathbb{R}^{m,n}$. Using Lemma 2(2) we see that the definition of $B^r(\Gamma,\nabla)(y)$ is correct (independent of the choice of ψ). Since we can choose such coordinates ψ (for y) smoothly in y (as follows from the construction of ψ in the proof of Lemma 2), $B^r(\Gamma,\nabla) : Y \to J^rY$ is smooth, and so it is an rth order holonomic connection on $Y \to M$. From the above construction, we see that $\pi^r_1 \circ B^r(\Gamma,\nabla) = \Gamma$, i.e. $B^r(\Gamma,\nabla)$ is an extension of Γ.

We also have the following strictly geometric solution of Problem 1.

EXAMPLE 6. Let $\Gamma : Y \to J^1Y$ be a general connection on $p : Y \to M$ and ∇ be a torsion free classical linear connection on M. Define an rth order holonomic connection $\tilde{B}^r(\Gamma,\nabla) : Y \to J^rY$ on $Y \to M$ as follows. Let $y_o \in Y$ and $x_o = p(y_o)$. Let $\exp_{x_o} : T_{x_o}M \supset U_{0_{x_o}} \to W_{x_o} \subset M$ be the exponent of ∇ at x_o, where $U_{0_{x_o}}$ is sufficiently small and convex. Define a smooth map $\phi : (-2,2) \times W_{x_o} \to M$ by

$$\phi(t,x) = \exp_{x_o}(t(exp_{x_o})^{-1}(x)),$$

$t \in (2,2)$, $x \in W_{x_o}$. Given $x \in W_{x_o}$, let $\phi_x : (-2,2) \to M$ be the curve given by $\phi_x(t) = \phi(x,t)$, $t \in (-2,2)$. Let $\tilde{\phi}^{y_o}_x : (-2,2) \to Y$ be the Γ-horizontal lifting of ϕ_x passing through y_o, i.e. the curve such that $p\circ\tilde{\phi}^{y_o}_x = \phi_x$, $\tilde{\phi}^{y_o}_x(0) = y_o$ and $\frac{d}{dt}(\tilde{\phi}^{y_o}_x)(t)$ is Γ-horizontal for any t (see the proof of Theorem 9.8 in [12] for the existence and uniqueness). Then we have a smooth local section $\sigma^{y_o} : W_{x_o} \to Y$ of $Y \to M$ defined by $\sigma^{y_o}(x) = \tilde{\psi}^{y_o}_x(1)$, $x \in W_{x_o}$ (see Theorem 9.8 in [12] for smoothness). We put

$$\tilde{B}^r(\Gamma,\nabla)(y_o) = j^r_{x_o}(\sigma^{y_o}).$$

Then $\tilde{B}^r(\Gamma,\nabla) : Y \to J^rY$ is smooth (see Theorem 9.8 in [12]). Since $\sigma^{y_o}(x_o) = y_o$, $\tilde{B}^r(\Gamma,\nabla) : Y \to J^rY$ is an rth order holonomic connection on $Y \to M$.

REMARK 2. (1) By Theorem 1 below, the constructions B^r and $\tilde{B}^r$ from Examples 5 and 6 are equal.

(2) The construction $\tilde{B}^r(\Gamma,\nabla)$ from Example 6 is more "economic" than $A^r(\Gamma,\tilde{\nabla}) = B^r_1(\Gamma,\tilde{\nabla})$ from Example 4. Indeed, to obtain $\tilde{B}^r(\Gamma,\nabla)$ we need a torsion free classical linear connection ∇ on M instead of a torsion free projectable classical linear one $\tilde{\nabla}$ on $p : Y \to M$ as in the case of $A^r(\Gamma,\tilde{\nabla})$ from Example 4.

(3) We have the following coordinate expression of $\tilde{B}^2(\Gamma,\nabla)$ from Example 5. Denote by (x^i, y^p) the canonical coordinates on Y and let

$$\left(x^i, y^p, y^p_i = \frac{\partial y^p}{\partial x^i}, y^p_{ij} = \frac{\partial y^p_i}{\partial x^j}\right)$$

be the induced coordinates on $\bar{J}^2Y$. If $y^p_i = \Gamma^p_i(x,y)$ is the coordinate expression of Γ, then its Ehresmann prolongation $\Gamma * \Gamma : Y \to \bar{J}^2Y$ has equations

$$y^p_i = \Gamma^p_i, \qquad y^p_{ij} = \frac{\partial \Gamma^p_i}{\partial x^j} + \frac{\partial \Gamma^p_i}{\partial y^q}\Gamma^q_j.$$

Then $\Gamma^{(2)}$ (from Example 2) has the expression

$$y^p_i = \Gamma^p_i, \quad y^p_{ij} = \frac{1}{2}\left(\frac{\partial \Gamma^p_i}{\partial x^j} + \frac{\partial \Gamma^p_i}{\partial y^q}\Gamma^q_j + \frac{\partial \Gamma^p_j}{\partial x^i} + \frac{\partial \Gamma^p_j}{\partial y^q}\Gamma^q_i\right).$$

By Theorem 1 below, $\tilde{B}^2(\Gamma, \nabla) = \Gamma^{(2)}$.

(4) It is difficult to obtain the coordinate expression of $\tilde{B}^r(\Gamma, \nabla)$ from Example 6 for $r \geq 3$. By the construction from Example 5, we know such expressions at the centres of "special coordinates" only.

Remark 3. Reformulating suitable parts of Remark 1 in [3], the construction B^r from Example 5 can be used for the geometric description of fields of higher geometric objects as follows. Let $P \to M$ be a principal G-bundle with m-dimensional basis. Its rth principal prolongation $W^r P$ is defined as the space of all r-jets $j^r_{(0,e)}\varphi$ of local trivializations $\varphi : \mathbb{R}^m \times G \to P$, where $e \in G$ is the unit. By [12], $W^r P \to M$ is a principal bundle with the structure group $W^r_m G = J^r_{(0,e)}(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_{(0,-)}$, and the fibred manifold $W^r P \to M$ coincides with the fibred product $W^r P = P^r M \times_M J^r P$, where $P^r M = \mathrm{inv}\, J^r_0(\mathbb{R}^m, M)$ is the rth order frame bundle of M. Let $\Gamma : P \to J^1 P$ be a principal (i.e. G-invariant) connection on $P \to M$ and Λ be a classical linear connection on M. By [12], we have a reduction of principal bundles $\mu_\Gamma : P^1 M \times_M P \to P^1 M \times_M J^1 P = W^1 P$, $\mu_\Gamma(l, p) = (l, \Gamma(p))$, with the obvious group monomorphism $GL(m) \times G \to W^1_m G$. Using right translations, we can extend the product connection $\Lambda \times \Gamma$ on $P^1 M \times_M P$ to a principal connection $p(\Gamma, \Lambda)$ on $W^1 P \to M$. It is well known that the bundle functor W^r_m plays a fundamental role in the theory of gauge natural operators and in mathematical physics. Moreover, the reduction μ_Γ has applications in the coordinate description of fields of geometric objects (see [12]).

Now let $B^r(\Gamma, \Lambda) : P \to J^r P$ be the connection from Example 5. Because of the canonical character of the construction B^r with respect to fibred embeddings, $B^r(\Gamma, \Lambda)$ is G-right invariant (as right translations of G on P are fibre embeddings). Then taking into account the exponential extension of λ^r_Λ (see Section 2 below) instead of Λ, we can generalize μ_Γ to the reduction $\mathrm{id}_{P^r M} \times B^r(\Gamma, \Lambda) : P^r M \times_M P \to W^r P$, the connection $\Lambda \times \Gamma$ to the connection $\lambda^r_\Lambda \times \Gamma$ on $P^r M \times_M P$ and the principal connection $p(\Gamma, \Lambda)$ on $W^1 P \to M$ to the principal connection $p^r(\Gamma, \Lambda)$ on $W^r P \to M$ (extending $\lambda^r_\Lambda \times \Gamma$). This provides us with the geometric background for the description of fields of higher order geometric objects.

In the rest of this section we prove the following theorem.

Theorem 1. *Let m, n, r be natural numbers. Any $\mathcal{FM}_{m,n}$-natural operator B transforming general connections $\Gamma : Y \to J^1 Y$ on $Y \to M$ and torsion free classical linear connections ∇ on M into rth order holonomic* *connections $B(\Gamma,\nabla) : Y \to J^rY$ on $Y \to M$ is equal to $B^r(\Gamma,\nabla)$ from Example* 5. *In particular, the constructions B^r and $\tilde{B}^r$ from Examples* 5 *and* 6 *coincide.*

To prove Theorem 1 we need the following lemma.

LEMMA 3. *Any $\mathcal{FM}_{m,n}$-natural operator Δ transforming general connections $\Gamma : Y \to J^1Y$ on $Y \to M$ and torsion free classical linear connections ∇ on M into tensor fields $\Delta(\Gamma,\nabla) : Y \to S^rT^*M \otimes VY$ is equal to zero.*

Proof of Theorem 1. We proceed by induction with respect to r.

(i) $r = 1$. Let $B(\Gamma,\nabla) : Y \to J^1Y$ be an operator as in the statement. We have the well-known affine bundle structure on $\pi^1_0 : J^1Y \to Y$ with the corresponding vector bundle $T^*M \otimes VY$. Consider the difference

$$\Delta(\Gamma,\nabla) := B(\Gamma,\nabla) - B^1(\Gamma,\nabla) : Y \to T^*M \otimes VY.$$

It is equal to zero, by Lemma 3 for $r = 1$. Therefore $B(\Gamma,\nabla) = B^1(\Gamma,\nabla)$.

(ii) *The inductive step.* Let $r \geq 2$. Let $B(\Gamma,\nabla) : Y \to J^rY$ be as in the statement. Then $\pi^r_{r-1}\circ B(\Gamma,\nabla) = B^{r-1}(\Gamma,\nabla) : Y \to J^{r-1}Y$ by the inductive assumption (as $\pi^r_{r-1} \circ B(\Gamma,\nabla)$ is an $\mathcal{FM}_{m,n}$-natural operator of the type considered for $r-1$). Moreover, by the same argument, $\pi^r_{r-1} \circ B^r(\Gamma,\nabla) = B^{r-1}(\Gamma,\nabla) : Y \to J^{r-1}Y$. We have the well known affine bundle structure on $\pi^r_{r-1} : J^rY \to J^{r-1}Y$ with the corresponding vector bundle $S^rT^*M\otimes VY$ over $J^{r-1}Y$ (where the pull back with respect to $\pi^{r-1}_0 : J^{r-1}Y \to Y$ is not indicated). Then we have the difference

$$\Delta(\Gamma,\nabla) := B(\Gamma,\nabla) - B^r(\Gamma,\nabla) : Y \to S^rT^*M \otimes VY.$$

It is zero by Lemma 3. Hence $B(\Gamma,\nabla) = B^r(\Gamma,\nabla)$. ∎

Proof of Lemma 3. It is sufficient to show that the contraction

$$\langle \Delta(\Gamma,\nabla)(y), w \otimes v\rangle = 0$$

for any $y \in Y_x$, $x \in M$, $w \in S^rT_xM$, $v \in (V_yY)^*$. Because of the $\mathcal{FM}_{m,n}$-invariance of Δ and Lemma 2 and the linearity in $w \otimes v$, we may assume that $(Y \to M) = \mathbb{R}^{m,n}$, $y = (0,0)$, $x = 0$, $w = \odot^r \frac{\partial}{\partial x^1}(0)$ and $v = d_0y^1$, and that the identity map $\mathrm{id}_{\mathbb{R}^{m,n}}$ is a "special" fibred coordinate system as in Lemma 2 for $\Gamma,\nabla,\tilde{r}$ and $y_o = (0,0)$, $x_o = 0$, where $\tilde{r}$ is arbitrarily large. Then $\Phi_{\tilde{r}}((\Gamma)^0) = 0$, and then (in particular) $\Gamma(0,0) = j^1_0(0)$, and $\nabla(0) = \nabla_o(0)$ (as the Christoffel symbols of torsion free classical linear connections vanish at the centre of normal coordinates), where ∇_o is the usual flat torsion free classical linear connection on $\mathbb{R}^m$. Because of Corollary 19.8 in [12] of the non-linear Peetre-like theorem, we may replace Γ and ∇ by new ones with the same q-jets at $(0,0)$ and 0, where q is sufficiently large (depending on Γ,∇). Then we may assume that the Christoffel symbols of ∇ and Γ in the identity chart are polynomials of degree q, where q is a natural number. Then we can assume that Γ is of the form

$$\Gamma = \Gamma_o + \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{1\le|(\alpha,\beta)|\le q} a^k_{\alpha\beta i}x^\alpha y^\beta dx^i \otimes \frac{\partial}{\partial y^k} \tag{1}$$

and ∇ is of the form

$$\nabla^{i_1}_{i_2 i_3} = \sum_{1\le|\gamma|\le q} c^{i_1}_{i_2 i_3\gamma}x^\gamma, \tag{2}$$

where the coefficients $a^k_{\alpha\beta i}$ and $c^{i_1}_{i_2i_3\gamma}$ are real numbers and Γ_o denotes the trivial general connection on $\mathbb{R}^{m,n}$. As $\Phi_{\tilde r}((\Gamma)^0) = 0$, we may additionally assume

$$a^k_{(b,0,\dots,0)(0)1} = 0 \tag{3}$$

for all $b = 0, 1, \dots, q$ and $k = 1, \dots, n$. Because at the centre of ∇-normal coordinates $\nabla^i_{1,1;1,\dots,1}(0) = 0$ for all $i = 1, \dots, m$ and arbitrarily long $1, \dots, 1$, we may additionally assume

$$c^{i_1}_{1,1,(b,0,\dots,0)} = 0 \tag{4}$$

for any $i_1 = 1, \dots, m$ and any $b = 1, \dots, q$. Given q, the space W of all systems (Γ, Δ) of the form (1) and (2) satisfying (3) and (4) is obviously a finite-dimensional vector space, which is invariant with respect to the $\mathcal{FM}_{m,n}$-maps $(t^1x^1, \dots, t^mx^m, \tau^1y^1, \cdots, \tau^ny^n)$ for $t^i > 0$ and $\tau^j > 0$. The coefficients $a^k_{\alpha\beta i}$ and $c^{i_1}_{i_2i_3\gamma}$ as in (3) and (4) will be called *inessential*, and the others from (1) and (2) will be called *essential*. Let K be the set of all indices of essential coefficients. Define $\Delta : \mathbb{R}^K \to \mathbb{R}$ by

$$\Delta(c^{i_1}_{i_2i_3\gamma}, a^k_{\alpha\beta i}) := \left\langle \Delta(\Gamma,\nabla)(0,0), \odot^r \frac{\partial}{\partial x^1}(0) \odot d_0y^1 \right\rangle \in \mathbb{R}, \tag{5}$$

where (Γ, ∇) is the unique system from W given by (1) and (2) for $(c^{i_1}_{i_2i_3\gamma}, a^k_{\alpha\beta i}) \in \mathbb{R}^K$. Fixing an ordering in K we may assume that $K \in \mathbb{N}$ is the number of elements in K. The function $\Delta : \mathbb{R}^K \to \mathbb{R}$ is defined on the whole $\mathbb{R}^K$. By the regularity of the $\mathcal{FM}_{m,n}$-natural operator Δ, the function $\Delta : \mathbb{R}^K \to \mathbb{R}$ is smooth. Applying the invariance of Δ with respect to the homotheties $\frac{1}{\tau}\mathrm{id}_{\mathbb{R}^{m,n}}$ for $\tau > 0$, we obtain the homogeneity condition

$$\Delta(\tau^{|\gamma|+1}c^{i_1}_{i_2i_3\gamma}, \tau^{|(\alpha,\beta)|}a^k_{\alpha\beta i}) = \tau^{r-1}\Delta(c^{i_1}_{i_2i_3\gamma}, a^k_{\alpha\beta i}).$$

If $r = 1$ then putting $\tau \to 0$ we see that Δ is constant, and then using the invariance with respect to the base homotheties we get $\Delta = 0$. If $r \ge 2$ then by the homogeneous function theorem [12], this type of homogeneity implies that $\Delta(c^{i_1}_{i_2i_3\gamma}, a^k_{\alpha\beta i})$ is a linear combination (with real coefficients) of monomials of the form

$$(6) \qquad \prod_{s=1}^{S} c^{i_1^s}_{i_2^s i_3^s \gamma^s} \prod_{p=1}^{P} a^{k^p}_{\alpha^p \beta^p i^p}$$

where $c^{i_1^s}_{i_2^s i_3 \gamma^s}$ and $a^{k^p}_{\alpha^p \beta^p i^p}$ are essential (where $\prod_{s=1}^{S} c^{i_1^s}_{i_2^s i_3^s \gamma^s} := 1$ if $S = 0$, and similarly if $P = 0$). Suppose that the coefficient of some monomial (6) in the linear combination expressing Δ is not zero. By the invariance of Δ with respect to fibre homotheties we deduce that $P \geq 1$ and $\beta^p = (0)$ for some $p = 1, \ldots, P$. Then $\alpha^p + e_{i^p} = (b_1, \ldots, b_m)$ with $b_j \geq 1$ for some $j = 2, \ldots, m$ (because in the other case $a^{k^p}_{\alpha^p \beta^p i^p}$ is inessential, see (3)). Then using the invariance of Δ with respect to $(x^1, \tau x^2, \ldots, \tau x^m, y^1, \ldots, y^n)$ we deduce that $S \geq 1$ and for some $s = 1, \ldots, S$ there is b_s such that $e_{i_2^s} + e_{i_3^s} + \gamma^s = (b_s, 0, \ldots, 0)$. But such a $c^{i_s}_{i_2^s i_3^s \gamma^s}$ is inessential (see (4)). Therefore, $\Delta(c^{i_1}_{i_2 i_3 \gamma}, a^k_{\alpha\beta i}) = 0$. ■

2. Prolongation of higher order holonomic connections by bundle functors. In Definition 45.4 of [12], the authors defined a general connection $\mathcal{F}(\Gamma, \lambda) : FY \to J^1(FY)$ on $FY \to M$ from a general connection $\Gamma : Y \to J^1 Y$ on $p : Y \to M$ by means of kth order linear connection $\lambda : TM \to J^k TM$ on M, where $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is a bundle functor of order k. More precisely, the corresponding lifting map $\mathcal{F}(\Gamma, \lambda) : FY \times_M TM \to TFY$ of $\mathcal{F}(\Gamma, \lambda)$ is defined by

$$\mathcal{F}(\Gamma, \lambda)(u, v) = \mathcal{F}(X^\Gamma)(u),$$

$u \in (FY)_x$, $v \in T_x M$, $x \in M$, where X is a vector field on M such that $\lambda(x) = j^r_x(X)$, X^Γ is the Γ-lift of X to Y (i.e. a projectable vector field on $Y \to M$ defined by $X^\Gamma(y) = \Gamma(X(x), y)$, $y \in Y_x$, $x \in M$) and $\mathcal{F}(Z)$ denotes the flow prolongation of a projectable vector field Z on $Y \to M$. (Since the flow prolongation $\mathcal{F}$ is linear and of order k, $\mathcal{F}(\Gamma, \nabla)$ is a well defined lifting map of a general connection on $FY \to M$; see [12] for details.) But given a torsion free classical linear connection ∇ on M, using the exponent of ∇, one can produce a kth order linear connection $\lambda^k_\nabla : TM \to J^k TM$ on M (see e.g. [18]). More precisely,

$$\lambda^k_\nabla(v) = j^k_x((\exp^\nabla_x)_* \tilde{v}),$$

$v \in T_x M$, $x \in M$, where $\tilde{v} \in \mathcal{X}(T_x M)$ is the constant vector field on $T_x M$ given by $\tilde{v}(w) = \frac{d}{dt}_0 (w + tv)$, $w \in T_x M$, and $\exp^\nabla_x : T_x M \supset U \to W \subset M$ is the exponent of ∇ at x (from some open neighbourhood U of $0 \in T_x M$ onto some open neighbourhood W of x). (Probably $\lambda^k_\nabla = \tilde{B}^k(\lambda, \lambda)$, where $\tilde{B}^k(\Gamma, \nabla)$ is the operator as in Example 6 for k instead of r, $\Gamma = \lambda : Y = TM \to J^1 Y = J^1 TM$ and $\nabla = \lambda$.) Then given a general connection $\Gamma : Y \to J^1 Y$ on $p : Y \to M$ and a torsion free classical linear connection ∇ on

M we have the general connection

$$\mathcal{F}(\Gamma,\nabla) := \mathcal{F}(\Gamma,\lambda^k_\nabla) : FY \to J^1(FY)$$

on $FY \to M$. So, we have the following natural problem.

Problem 2. *Let r,q,m,n be positive integers. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Given a qth order holonomic connection $\Theta : Y \to J^qY$ on $p : Y \to M$ and a torsion free classical linear connection ∇ on M, construct an rth order holonomic connection $\mathcal{F}^r_q(\Theta,\nabla) : FY \to J^r(FY)$ on $FY \to M$.*

In [4], we solved Problem 2 in the special case $F = J^s : \mathcal{FM}_{m,n} \to \mathcal{FM}$ and $q = r$. A solution of Problem 2 (in the general case) is given in the following example.

Example 7. Let $\Theta : Y \to J^qY$ be a qth order holonomic connection on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$ and let ∇ be a torsion free classical linear connection on M. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Define an rth order holonomic connection $\mathcal{F}^r_q(\Theta,\nabla) : FY \to J^r(FY)$ on $FY \to M$ as follows. Let k be the order of F. Let $\Gamma : Y \to J^1Y$ be the underlying connection of Θ. Let $\mathcal{F}(\Gamma,\nabla) : FY \to J^1(FY)$ be the general connection on $FY \to M$ induced by Γ and ∇ (recalled above). We put

$$\mathcal{F}^r_q(\Theta,\nabla) := B^r(\mathcal{F}(\Gamma,\nabla),\nabla),$$

where the operator B^r is as in Example 5 for $FY \to M$ instead of $Y \to M$.

Using the construction $\mathcal{F}^r_q(\Theta,\nabla)$ from Example 7, we immediately obtain a solution of the next problem.

Problem 3. *Let r,q,m,n be positive integers. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Given a qth order holonomic connection $\Theta : Y \to J^qY$ on $p : Y \to M$ and a torsion free projectable classical linear connection $\tilde\nabla$ on $p : Y \to M$, construct an rth order holonomic connection $\tilde{\mathcal{F}}^r_q(\Theta,\tilde\nabla) : FY \to J^r(FY)$ on $FY \to M$.*

Indeed, we have the following example.

Example 8. Let $\Theta : Y \to J^qY$ be a general qth order holonomic connection on an $\mathcal{FM}_{m,n}$-object $p : Y \to M$ and let $\tilde\nabla$ be a torsion free projectable classical linear connection on $p : Y \to M$. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Define an rth order holonomic connection

$$\tilde{\mathcal{F}}^r_q(\Theta,\tilde\nabla) := \mathcal{F}^r_q(\Theta,\underline{\tilde\nabla}) : FY \to J^r(FY)$$

on $FY \to M$, where $\underline{\tilde\nabla}$ is the torsion free classical linear connection on M underlying $\tilde\nabla$ and $\mathcal{F}^r_q$ is as in Example 7.

Remark 4. Problem 3 solved above was open and rather difficult. Indeed, in [19] we presented a bundle functor $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ such that there is no construction of a torsion free projectable classical linear connection $B^F(\tilde{\nabla})$ on FY from a torsion free projectable classical linear connection $\tilde{\nabla}$ on $p : Y \to M$, and hence we could not solve Problem 3 by putting $\tilde{\mathcal{F}}^r_q(\Theta, \tilde{\nabla}) = B^r_1(\mathcal{F}(\Gamma, \underline{\tilde{\nabla}}), B^F(\tilde{\nabla})) : FY \to J^r(FY)$ (for such F), where $\Gamma : Y \to J^1 Y$ is the general connection underlying Θ and $B^r_1(\Gamma, \tilde{\nabla})$ is the operator from Example 4.

In the rest of this section, using the operator B^r from Example 5 and the main result from [21], we also solve the following classification problem.

PROBLEM 4. *Let r, q, m, n be positive integers. Describe all bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ such that there is a canonical construction A of rth order holonomic connections $A(\Theta, \nabla) : FY \to J^r(FY \to Y)$ on $FY \to Y$ from qth order holonomic connections $\Theta : Y \to J^q Y$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M.*

We start with the following example.

EXAMPLE 9. Let $G : \mathcal{M}f_m \to \mathcal{FM}$ be a natural bundle of (finite) order k. We have a bundle functor $p^*G : \mathcal{FM}_{m,n} \to \mathcal{FM}$, where $(p^*G)(Y) := p^*(G(M) \to M)$ for $\mathcal{FM}_{m,n}$-objects $p : Y \to M$ is the pull-back of $G(M) \to M$ with respect to $p : Y \to M$, and where $(p^*G)(f) : (p^*G)(Y) \to (p^*G)(Y_1)$ is obviously defined from $G(\underline{f}) : G(M) \to G(M_1)$ for $\mathcal{FM}_{m,n}$-morphisms $f : Y \to Y_1$ covering $\underline{f} : M \to M_1$. Let $p : Y \to M$ be an $\mathcal{FM}_{m,n}$-object and ∇ be a torsion free classical linear connection on M. We construct an rth order holonomic connection $A(\nabla)$ on $(p^*G)(Y) \to Y$ as follows. The connection ∇ induces a general connection $\Gamma^{\nabla}_G : GM \times_M TM \to TGM$ on $G(M) \to M$ by $\Gamma^{\nabla}_G(v, w) = \mathcal{G}(X)(v)$, $v \in F_x M$, $w \in T_x M$, $x \in M$, where $\lambda^k_{\nabla} : TM \to J^k TM$ is the "exponential" extension of ∇ (see the beginning of this section), $j^k_x(X) = \lambda^k_{\nabla}(w)$ and $\mathcal{G}(X)$ is the flow lifting of a vector field X on M to GM. Applying the operator B^r from Example 5, we have the rth order holonomic connection $B^r_G(\nabla) := B^r(\Gamma^{\nabla}_G, \nabla)$ on $G(M) \to M$. Consider an arbitrary element $w = (v, y) \in (p^*G)(Y)$, $y \in Y_x$, $v \in G_x(M)$, $x \in M$. Let $\sigma : M \to G(M)$ be a section of $G(M) \to M$ such that $j^r_x(\sigma) = B^r_G(\nabla)(v)$. We put $A(\nabla)(w) := j^r_y(\tilde{\sigma})$, where $\tilde{\sigma} : Y \to (p^*G)(Y)$ is defined by $\tilde{\sigma}(z) = (\sigma(p(z)), z)$, $z \in Y$. Clearly, $j^r_y(\tilde{\sigma})$ is determined by $j^r_x(\sigma)$. Hence the definition of $A(\nabla)(w)$ is correct. Clearly, $A(\nabla)$ is an rth order holonomic connection on $(p^*G)(Y) \to Y$.

PROPOSITION 4. *Let r, q, m, n be positive integers. Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. There exists a canonical construction A of rth order holonomic connections $A(\Theta, \nabla)$ on $F(Y) \to Y$ from qth order holonomic connections Θ on $Y \to M$ by means of torsion free classical linear connections ∇ on M if and only if F is isomorphic to p^*G for some natural bundle $G : \mathcal{M}f_m \to \mathcal{FM}$.*

Proof. In [21], we proved this proposition in the case $q = r = 1$. Now, let q and r be arbitrary. If F is isomorphic to p^*G then such an A exists in view of Example 9. If F is not isomorphic to any p^*G then by the proposition for $q = r = 1$ there is no canonical construction B of general connections $B(\Gamma, \nabla)$ on $F(Y) \to Y$ from general connections Γ on $Y \to M$ by means of classical linear connections ∇ on M. Suppose that such an A exists. Then we can define a general connection $B(\Gamma, \nabla)$ on $FY \to Y$ to be the underlying connection of the rth order connection $A(B^q(\Gamma, \nabla), \nabla)$ on $FY \to Y$, where $B^q(\Gamma, \nabla)$ is as in Example 5 for q instead of r. Contradiction. ∎

Roughly speaking, Proposition 4 says that (in contrast to Example 7) the class of bundle functors $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ for which we can construct canonically an rth order holonomic connection $A(\Theta, \nabla) : FY \to J^r(FY \to Y)$ on $FY \to Y$ from a qth order holonomic connection $\Theta : Y \to J^qY$ on $Y \to M$ by using a torsion free classical linear connection ∇ on M is very narrow. For example, from Proposition 4 we immediately have the following corollaries.

Corollary 1. *Let $s \geq 1$. There is no $\mathcal{FM}_{m,n}$-canonical construction of rth order holonomic connections $A(\Theta, \nabla) : J^sY \to J^r(J^sY \to Y)$ on $J^sY \to Y$ from qth order holonomic connections $\Theta : Y \to J^qY$ on $Y \to M$ by means of torsion free classical linear connections ∇ on M.*

Corollary 2. *There is no $\mathcal{FM}_{m,n}$-canonical construction of rth order holonomic connections $A(\Theta, \nabla) : TY \to J^r(TY \to Y)$ on $TY \to Y$ from qth order holonomic connections $\Theta : Y \to J^qY$ on $Y \to M$ by using torsion free classical linear connections ∇ on M.*

3. An application of prolongations of connections to lifting of geometric objects. Let $\Gamma : Y \to J^1Y$ be a general connection on $p : Y \to M$.

It is a well-known classical fact that using Γ one can lift vector fields on M to vector fields on Y. This classical construction can be presented in the following way. Let X be a vector field on M. We define a vector field X^Γ on Y by $X^\Gamma(y) = T_x(\sigma)(X(x))$, $y \in Y_x$, $x \in M$, where $\sigma : M \to Y$ is a section such that $j^1_x(\sigma) = \Gamma(y)$. The definition of $X^\Gamma(y)$ is correct because $T_x(\sigma) : T_xM \to T_{\sigma(x)}Y$ depends on the first jet of σ at x only.

The above construction of X^Γ can be directly generalized to all first order bundle functors $F : \mathcal{M}f \to \mathcal{FM}$ in place of the tangent functor $T : \mathcal{M}f \to \mathcal{FM}$, where $\mathcal{M}f$ is the category of all smooth manifolds and all smooth maps. Namely, we have the following example.

Example 10. Let $\rho : M \to F(M)$ be a section of $F(M) \to M$. We define a section $\rho^\Gamma : Y \to F(Y)$ of $F(Y) \to Y$ by $\rho^\Gamma(y) = F_x(\sigma)(\rho(x))$, $y \in Y_x$, $x \in M$, where $\sigma : M \to Y$ is a section such that $j^1_x(\sigma) = \Gamma(y)$. If F is of order 1, the definition of $\rho^{\Gamma}(y)$ is correct because $F_x(\sigma) : F_xM \to F_{\sigma(x)}(Y)$ depends on the first jet of σ at x only.

Clearly, if $F : \mathcal{M}f \to \mathcal{FM}$ is of the minimal order $r > 1$, then the construction from Example 10 is not correct. However, if we apply an rth order holonomic connection $\Theta : Y \to J^rY$ instead of $\Gamma : Y \to J^1Y$ then the construction as in Example 10 will be correct. Namely we have the following example.

EXAMPLE 11. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor of finite order r. Let $\Theta : Y \to J^rY$ be an rth order holonomic connection on a fibred manifold $Y \to M$. Let $\rho : M \to F(M)$ be a section of $F(M) \to M$. We define a section $\rho^{\Theta} : Y \to F(Y)$ of $F(Y) \to Y$ by $\rho^{\Theta}(y) = F_x(\sigma)(\rho(x))$, $y \in Y_x$, $x \in M$, where $\sigma : M \to Y$ is a section such that $j^r_x(\sigma) = \Theta(y)$.

So, given a general connection $\Gamma : Y \to J^1Y$ and an rth order bundle functor $F : \mathcal{M}f \to \mathcal{FM}$ one can lift sections of $F(M) \to M$ to sections of $F(Y) \to Y$ if we can produce an rth order holonomic connection $\Theta : Y \to J^rY$ from Γ. This is possible if $r = 2$ (in this case we have $\Theta = \Gamma^{(2)} : Y \to J^2Y$, see Example 2). Thus we have the following example.

EXAMPLE 12. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor of order 2. Let $\Gamma : Y \to J^1Y$ be a general connection on $Y \to M$. Let $\rho : M \to F(M)$ be a section of $F(M)$. We have a section $\rho^{\Gamma} := \rho^{\Gamma^{(2)}} : Y \to F(Y)$, where $\rho^{\Gamma^{(2)}}$ is defined in Example 11 for $\Theta = \Gamma^{(2)}$ (from Example 2).

REMARK 5. An important example of a second order bundle functor is the vector second order tangent bundle functor $T^{(2)} : \mathcal{M}f \to \mathcal{FM}$. We recall that $T^{(2)}(M) = (J^2(M,\mathbb{R})_0)^*$ (see [12]). Thus sections $\sigma : M \to T^{(2)}(M)$ of $T^{(2)}(M) \to M$ are in fact linear second order differential operators $\sigma : C^{\infty}(M) \to C^{\infty}(M)$ (annihilating constants), $\sigma(f)(x) := \sigma(x)(j^2_x(f - f(x)))$, $x \in M$, $f \in C^{\infty}(M)$. (To omit the assumption on annihilating constants it is sufficient to take the extended vector second order tangent bundle functor $E^{(2)} = T^{(2)} \times \mathbb{R}$ instead of $T^{(2)}$.) Thus Example 12 for $F = T^{(2)}$ shows that given a general connection $\Gamma : Y \to J^1Y$ one can lift linear second order differential operators on $C^{\infty}(M)$ to linear second order differential operators on $C^{\infty}(Y)$.

REMARK 6. In [2], we produced (in a rather complicated way) a so-called (2)-connection $\tilde{\Gamma}^{(2)} : Y \times_M T^{(2)}(M) \to T^{(2)}Y$ on $Y \to M$ from a given general connection $\Gamma : Y \times_M TM \to TY$ on $Y \to M$. Of course, this (2)-connection can be (equivalently) interpreted as the corresponding lift of sections ρ of $T^{(2)}(M)$ to sections $\rho^{\tilde{\Gamma}(2)}$ of $T^{(2)}(Y)$, by $\rho^{\tilde{\Gamma}^{(2)}}(y) = \tilde{\Gamma}^{(2)}(y, \rho(x))$, $y \in Y_x$, $x \in M$. In [2], we also proved that the natural operator $\Gamma \to \tilde{\Gamma}^{(2)}$ is the unique natural operator sending general connections on $Y \to M$ to

(2)-connections on $Y \to M$. Hence, the lift ρ^{Γ} of Example 12 for $F = T^{(2)}$ coincides with $\rho^{\tilde{\Gamma}^{(2)}}$. Now, we see that the above fact holds because $\Gamma \to \Gamma^{(2)}$ is the only natural operator sending general connections on $Y \to M$ to second order holonomic connections on $Y \to M$. In [2], we could not generalize the construction $\tilde{\Gamma}^{(2)}$ to all r. Now, we see that the reason was that for $r \geq 3$ there is no rth order holonomic connection on $Y \to M$ coming from a general connection on $Y \to M$.

Substituting $B^r(\Gamma, \nabla)$ of Example 5 for Θ in Example 11 we have the following example.

Example 13. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor of finite order r. Let $\Gamma : Y \to J^1Y$ be a general connection on a fibred manifold $Y \to M$ and ∇ be a torsion free linear classical connection on M. Let $\rho : M \to F(M)$ be a section of $F(M) \to M$. We have a section $\rho^{(\Gamma,\nabla)} := \rho^{B^r(\Gamma,\nabla)} : Y \to F(Y)$ of $F(Y) \to Y$, where $\rho^{B^r(\Gamma,\nabla)}$ is defined in Example 11 for $\Theta = B^r(\Gamma, \nabla)$ and $B^r(\Gamma, \nabla) : Y \to J^rY$ is defined in Example 5. Thus given (Γ, ∇) we have the (Γ, ∇)-lift $\rho \to \rho^{(\Gamma,\nabla)}$. In particular, for $F = T^{(r)} : \mathcal{M}f \to \mathcal{FM}$ we obtain the (Γ, ∇)-lifting of linear rth order differential operators $C^\infty(M) \to C^\infty(M)$ (annihilating constants) to rth order linear differential operators $C^\infty(Y) \to C^\infty(Y)$ (annihilating constants). For $F = (J^rT^*)^* : \mathcal{M}f \to \mathcal{FM}$ we obtain the (Γ, ∇)-lifting of linear rth order differential operators $\Omega^1(M) \to C^\infty(M)$ to rth order ones $\Omega^1(Y) \to C^\infty(Y)$.

Remark 7. Let $\Gamma : Y \to J^1Y$ be a general connection on a fibred manifold $p : Y \to M$ with non-vanishing curvature. We see that for $r \geq 2$, there is no operator lifting rth order linear differential operators $\rho : C^\infty(M) \to C^\infty(M)$ to rth order linear differential operators $\rho^{\Gamma} : C^\infty(Y) \to C^\infty(Y)$ such that $(X_1 \circ \cdots \circ X_k)^{\Gamma} = X_1^{\Gamma} \circ \cdots \circ X_k^{\Gamma}$ for all $X_1, \ldots, X_k \in \mathcal{X}(M)$ and $k = 1, \ldots, r$, where X^{Γ} denotes the Γ-horizontal lift of X (see the beginning of this section). Indeed, if Γ has non-vanishing curvature, then there are two commuting vector fields $X_1, X_2 \in \mathcal{X}(M)$ such that $X_1^{\Gamma}, X_2^{\Gamma} \in \mathcal{X}(Y)$ are not commuting, and then we would have $(X_1 \circ X_2)^{\Gamma} = X_1^{\Gamma} \circ X_2^{\Gamma} \neq X_2^{\Gamma} \circ X_1^{\Gamma} = (X_2 \circ X_1)^{\Gamma} = (X_1 \circ X_2)^{\Gamma}$.

We apply the construction of Example 13 in the case $Y = G(M) \to M$, where $G : \mathcal{M}f_m \to \mathcal{FM}$ is a natural bundle over m-manifolds and their embeddings. We have the following example.

Example 14. Let $F : \mathcal{M}f \to \mathcal{FM}$ be a bundle functor of finite order r. Let $G : \mathcal{M}f_m \to \mathcal{FM}$ be a natural bundle. Denote $k = \mathrm{ord}(G)$. Let ∇ be a torsion free classical linear connection on an m-manifold M. This connection ∇ induces the "exponential" extension $\lambda^k_\nabla : TM \to J^kTM$ (recalled in Section 2). It is a well-known fact that this kth order linear connection λ^k_∇ on M induces a general connection $\Gamma^\nabla_G : G(M) \to J^1(G(M))$ on

$G(M) \to M$ with the lifting form $\Gamma_G^\nabla : G(M) \times_M TM \to TG(M)$ given by $\Gamma_G^\nabla(w,v) = \mathcal{G}(X)(w)$, $w \in G_x(M)$, $v \in T_xM$, $x \in M$, where X is the vector field on M such that $j^k_x(X) = \lambda^k_\nabla(v)$ and $\mathcal{G}(Z)$ denotes the flow prolongation of the vector field Z on M to $G(M)$. Now, let $\rho : M \to F(M)$ be a section of $F(M) \to M$. Then we have a section $\rho^\nabla := \rho^{(\Gamma_G^\nabla,\nabla)} : G(M) \to F(G(M))$ of $F(G(M)) \to G(M)$ (see Example 13). In particular, for $F = T^{(r)}$, we obtain the ∇-lifting of rth order linear differential operators $C^\infty(M) \to C^\infty(M)$ (annihilating constants) to rth order linear differential operators $C^\infty(G(M)) \to C^\infty(G(M))$ (annihilating constants).

4. Appendix. In [3], we proved Proposition 1 by using a complicated result from [20]. Below, we present another direct proof.

Proof of Proposition 1. By a jet projection argument we may assume $r = 3$. Suppose that A is an operator as in the statement. Then (we repeat the relevant part of the proof from [3]) we have an $\mathcal{M}f_m$-natural operator $B : T^* \rightsquigarrow T^{*3}$ transforming 1-forms on m-manifolds M into sections of $T^{*3}M = J^3(M,\mathbb{R})_0$ defined by

$$B_M(\omega)(x) = \mathrm{pr}_1 \circ A\Big(\Gamma_M + \omega \otimes \frac{\partial}{\partial y^1}\Big)(x,0) \in T^{*3}_xM = J^3_x(M,\mathbb{R})_0,$$

where $\omega \in \Omega^1(M)$, $x \in M$, $\Gamma_M : TM \times_M (M \times \mathbb{R}^n) \to T(M \times \mathbb{R}^n)$ is the trivial connection on the trivial bundle $M \times \mathbb{R}^n \to M$ and $\mathrm{pr}_1 : J^3_x(M,\mathbb{R}^n)_0 = \times^n J^3_x(M,\mathbb{R})_0 \to J^3_x(M,\mathbb{R})_0$ is the projection onto the first factor. By Proposition 23.5 in [12], B is of finite order. This operator is linear because of the invariance of A with respect to fiber homotheties and the homogeneous function theorem. We consider the 1-form $\omega_o = x^2dx^1 - x^1dx^2$. Using the invariance of A with respect to the base homotheties $(t_1x^1,\dots,t_mx^m)$ we easily deduce that $B(\omega_o)(0) = j^3_0(ax^1x^2)$ for some $a \in \mathbb{R}$. Then using the invariance of A with respect to the permutation of x^1 and x^2 we see that $a = 0$ (because this permutation sends ω_o to $-\omega_o$ and preserves x^1x^2). Then using the invariance of A with respect to $\varphi = (x^1, x^2 + (x^1)^2, x^3,\dots,x^m,y^1,\dots,y^n)$ we deduce that $B(\omega_o - (x^1)^2dx^1)(0) = j^3_0(0)$ (because φ^{-1} sends ω_o into $\omega_o - (x^1)^2dx^1$ and preserves $j^3_0(0)$). Then $B((x^1)^2dx^1)(0) = 0$. But $\mathrm{pr}_1 \circ A(\Gamma_M)(0,0) = B(0)(0) = j^3_0(0)$. Then using the invariance of A with respect to $\psi = (x^1,\dots,x^m,y^1 + (x^1)^3, y^2,\dots,y^n)$ we deduce that $0 = 3B((x^1)^2dx^1)(0) = \mathrm{pr}_1 \circ A\big(\Gamma_M + 3(x^1)^2dx^1 \otimes \frac{\partial}{\partial y^1}\big)(0,0) = j^3_0((x^1)^3)$ (because ψ sends Γ_M to $\Gamma_M + 3(x^1)^2dx^1 \otimes \frac{\partial}{\partial y^1}$, and $j^3_0(0)$ to $j^1_0((x^1)^3)$). But $j^3_0((x^1)^3) \neq 0$. Contradiction. ∎

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