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# The Laplace transform on a Boehmian space

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**Abstract.** In the literature a Boehmian space containing all right-sided Laplace transformable distributions is defined and studied. Besides obtaining basic properties of this Laplace transform, an inversion formula is also obtained. In this paper we shall improve upon two theorems one of which relates to the continuity of this Laplace transform and the other is concerned with the inversion formula.

**1. Introduction.** Generalizing the concept of distributions, several generalized function spaces, Boehmian spaces and more general generalized quotient spaces are developed and studied in the literature (see [1, 2, 3, 7]). Further several integral transforms are also introduced and studied in this context. In particular, the theory of Laplace transform has been extensively studied in the recent literature (see [4, 5, 6]).

In [5], Nemzer introduces a Boehmian space $\beta$ and investigates the properties of Laplace transforms of certain elements of $\beta$. In this paper we shall first modify this Boehmian space without loss of generality as follows:

Let $G$ be the set of all complex-valued smooth functions $f$ defined on the real axis, $\mathbb{R}$, with the property that

(i) $\operatorname{supp} f \subset [0,\infty)$,
(ii) $f(t) = O(e^{\alpha t})$ as $t \to \infty$ for some real number $\alpha$.

Instead of equipping $G$ with the usual topology of uniform convergence on compact subsets of $\mathbb{R}$, we shall use the following notion of convergence in $G$. A sequence $\{f_n\}$ in $G$ is said to converge to $f \in G$ if

(i) $|f_n(t)| \le Me^{\alpha t}$ for all $n$,
(ii) $f_n \to f$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$.

Let $S$ $(\subset G)$ be the set of all smooth functions on $\mathbb{R}$ with compact support. We shall also use the usual convolution as a map from $G \times S$ to $G$. Further, a sequence $\{\delta_n\}$ in $S$ will be called a *delta sequence* if

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(i) $\delta_n(t) \geq 0$ for all $t$ and $n = 1, 2, \dots$,
(ii) $\int_0^\infty \delta_n(t)\,dt = 1$ for all $n$,
(iii) $\operatorname{supp} \delta_n \to 0$ as $n \to \infty$.

With these notations we can construct (as in [2]) a Boehmian space which will be called $\beta_L$ and is identical with the Boehmian space constructed in [5] for Laplace transformable elements of $\beta$ (as defined in [5]). In this Boehmian space we can easily prove that $x * \delta_n \to x$ as $n \to \infty$ for any $x \in \beta_L$ and any $\delta$-sequence $\{\delta_n\}$.

Recall that the Laplace transform $L(x)$ of $x \in \beta_L$ ($x = [\frac{f_n}{\delta_n}]$) is the unique analytic function defined by $L(x)L(\delta_n) = L(f_n)$ for all $n$. Obviously for each $x \in \beta_L$ there exists a half-plane in which $L(x)$ is an analytic function.

In [5] there is no mention of continuity except in Theorem 2.5 where it is proved that if $\{x_n\}$ is a sequence of Boehmians in $\beta_L$ such that $x_n \to x$ in the delta sense and $\operatorname{supp} x_n \subset [0, T]$ for all $n$ (for some $T \geq 0$), then $L(x_n) \to L(x)$ uniformly on compact subsets of the plane.

Further, an inversion formula is also obtained in which $x \in \beta_L$ is obtained as a delta limit of a sequence of functions which uses $L(x * \delta_n)$ instead of $L(x)$ only. As such, this inversion formula is rather obvious because $x * \delta_n \to x$ in the delta sense always.

In this paper we shall define the concept of continuity of a map between $\beta_L$ and a suitable space $\mathcal{H}$ (which we are going to construct) and show that the Laplace transform defined above is continuous from $\beta_L$ into $\mathcal{H}$. Further, we shall also show that above inversion formula can be considerably improved using only $L(x)$ in its representation.

In order to motivate the concept of convergence in the Boehmian space $\beta_L$, we shall first consider the following example.

Let $\{f_n(t)\}$ be a sequence in $G$ defined as follows:

$$f_n(t) = \begin{cases} 0 & \text{for } t \leq n, \\ e^{nt} - e^{n^2} & \text{for } t \geq n,\ n = 1, 2, \dots. \end{cases}$$

It is obvious that each $f_n$ is Laplace transformable and

$$f_n(t) \to f(t) \equiv 0$$

uniformly on compact subsets of $\mathbb{R}$, and that $f$ is also Laplace transformable. But there is no common half-plane in which $L(f_n)$ can be defined for all sufficiently large $n$. Thus we are motivated to define the following concept of convergence in $\beta_L$.

Definition 1.1. We say that a sequence $\{x_n\}$ of Boehmians in $\beta_L$ converges to $x$ in $\beta_L$ (written $x_n \xrightarrow{\delta} x$) if there exist $\alpha \in \mathbb{R}$, $M > 0$ and a delta sequence $\{\delta_n\}$ such that $x_n * \delta_k, x * \delta_k \in G$ and

(i) $|(x_n * \delta_k)(t)| \le Me^{\alpha t}$ for all $n$ and $k$,
(ii) $|(x * \delta_k)(t)| \le Me^{\alpha t}$ for all $k$,
(iii) $x_n * \delta_k \to x * \delta_k$ as $n \to \infty$ in $G$ for each $k$.

NOTE 1.2. We observe that the above assumptions in particular imply that there exists a common half-plane in which $L(x_n * \delta_k)$ and $L(x * \delta_k)$ are defined and analytic.

DEFINITION 1.3. Let $\mathcal{H}$ denote the space of all analytic functions defined in half-planes (which can vary with the elements of $\mathcal{H}$). We shall say that a sequence $\{F_n\}$ in $\mathcal{H}$ *converges to* $F$ *in* $\mathcal{H}$ (written $F_n \xrightarrow{\mathcal{H}} F$) if all these functions are defined in a common half-plane $H$ and $F_n \to F$ uniformly on compact subsets of $H$.

DEFINITION 1.4. A map $h : \beta_L \to \mathcal{H}$ is said to be *continuous* if whenever $x_n \xrightarrow{\delta} x$ in $\beta_L$, we have $h(x_n) \xrightarrow{\mathcal{H}} h(x)$.

## 2. Main results

THEOREM 2.1. *$\Phi : \beta_L \to \mathcal{H}$ defined by $\Phi(x) = L(x)$ is continuous.*

*Proof.* From the definitions, we have, for each $k = 1, 2, \ldots$,

$$f_{nk} = x_n * \delta_k \to f_k = x * \delta_k \quad \text{in } G \text{ as } n \to \infty.$$

Further, for $n \ge 1$ and with $\alpha$ as in Definition 1.1, we have

$$L(x_n)(z)L(\delta_k)(z) = L(f_{nk})(z) \quad (\operatorname{Re} z > \alpha),$$
$$L(x)(z)L(\delta_k)(z) = L(f_k)(z) \quad (\operatorname{Re} z > \alpha).$$

Since $\{\delta_k\}$ is a delta sequence, $L(\delta_k)(z) \to 1$ uniformly on compact subsets of the complex plane. Fix $z_0 \in H = \{\operatorname{Re} z > \alpha\}$. We can choose a positive integer $k$, $\epsilon_1 > 0$ and a neighborhood $N_{z_0}$ of $z_0$ such that $|L(\delta_k)(z)| \ge \epsilon_1 > 0$ for all $z \in \overline{N}_{z_0}$. Hence $L(x_n)(z) = L(f_{nk})(z)/L(\delta_k)(z)$, $L(x)(z) = L(f_k)(z)/L(\delta_k)(z)$ are defined and analytic in $N_{z_0}$. On the other hand, using the estimates available in the definitions we see that for each $\epsilon > 0$, there exist $T_1$ and $T_2$ such that

$$\int_{T_1}^{\infty} |f_{nk}(t)|e^{-\gamma t}\,dt < \epsilon\epsilon_1/3, \qquad \int_{T_2}^{\infty} |f_k(t)|e^{-\gamma t}\,dt < \epsilon\epsilon_1/3.$$

Using the uniform convergence of $f_{nk}$ to $f_k$ on compact subsets of $\mathbb{R}$, we get $\int_0^T |f_{nk}(t) - f_k(t)|e^{-\gamma t}\,dt \le \epsilon\epsilon_1/3$ where $T = \max\{T_1, T_2\}$, $n$ large. We now have, for $\operatorname{Re} z \ge \gamma > \alpha$ and for large $n$,

$$\begin{aligned}
&|L(f_{nk})(z) - L(f_k)(z)| \\
&\quad \le \int_0^\infty |f_{nk}(t) - f_k(t)| e^{-\gamma t}\,dt \\
&\quad \le \int_0^T |f_{nk}(t) - f_k(t)| e^{-\gamma t}\,dt + \int_T^\infty |f_{nk}(t) - f_k(t)| e^{-\gamma t}\,dt \\
&\quad \le \int_0^T |f_{nk}(t) - f_k(t)| e^{-\gamma t}\,dt + \int_T^\infty |f_{nk}(t)| e^{-\gamma t}\,dt + \int_T^\infty |f_k(t)| e^{-\gamma t}\,dt \\
&\quad < \epsilon\epsilon_1.
\end{aligned}$$

Therefore $L(f_{nk})(z) \to L(f_k)(z)$ uniformly on compact subsets of $H$.

We now observe that (for $\epsilon > 0, z \in \overline{N}_{z_0}$ and $n$ large)

$$|L(x_n)(z) - L(x)(z)| = \left| \frac{L(f_{nk})(z) - L(f_k)(z)}{L(\delta_k)(z)} \right| \le \frac{1}{\epsilon_1}\,\epsilon\epsilon_1 = \epsilon.$$

This shows that $L(x_n)(z) \to L(x)(z)$ uniformly on $\overline{N}_{z_0}$. Since $z_0$ is arbitrary, we see that each $z_0 \in H$ has a neighborhood on whose closure $L(x_n)(z) \to L(x)(z)$ uniformly as $n \to \infty$. Using standard compactness arguments we get the uniform convergence of $\{L(x_n)(z)\}$ on compact subsets of $H$. This completes the proof.

LEMMA 2.2. *Let $x \in \beta_L$, $L(x) = F$, and suppose $F$ is analytic in a half-plane* $\operatorname{Re} z > \alpha$. *Let $\gamma > \alpha$. Define, for $n \in \mathbb{N}$,*

$$f_n(t) = \begin{cases} \int_{-n}^{n} F(\gamma + iu) e^{(\gamma + iu)t}\,du & \text{for } t > 0, \\ 0 & \text{for } t \le 0. \end{cases}$$

*Then $f_n$ is a Laplace transformable distribution for each $n \in \mathbb{N}$.*

*Proof.* It is clear from the definition that $f_n$ is a locally integrable function on $\mathbb{R}$ and hence a distribution on $\mathbb{R}$. We shall show that $e^{-\gamma t} f_n(t) \in \mathscr{S}'$ (here $\mathscr{S}$ denotes the usual space of rapidly decreasing functions, and $\mathscr{S}'$ the space of tempered distributions) for each $n \in \mathbb{N}$. Denoting the action of $u \in \mathscr{S}'$ on $\phi \in \mathscr{S}$ by $(u, \phi)$, we have, for $\phi \in \mathscr{S}$,

$$\begin{aligned}
(e^{-\gamma t} f_n(t), \phi) &= \int_{\mathbb{R}} e^{-\gamma t} f_n(t)\phi(t)\,dt = \int_{\mathbb{R}} \int_{-n}^{n} e^{-\gamma t} F(\gamma + iu) e^{(\gamma + iu)t}\,du\,\phi(t)\,dt \\
&= \int_{\mathbb{R}} \int_{-n}^{n} F(\gamma + iu) e^{iut}\,du\,\phi(t)\,dt \\
&= \int_{-n}^{n} \int_{\mathbb{R}} \phi(t) e^{iut}\,dt\,F(\gamma + iu)\,du \text{ (by Fubini's Theorem)} \\
&= \int_{-n}^{n} \hat{\phi}(-u) F(\gamma + iu)\,du < \infty.
\end{aligned}$$

Hence for each fixed $n$, $e^{-\gamma t} f_n(t)$ acts on $\mathscr{S}$. Since the linearity is obvious, we shall only prove the continuity of $e^{-\gamma t} f_n(t)$ on $\mathscr{S}$. Suppose $\psi_m \to \psi$ in $\mathscr{S}$. Then using Fubini's theorem and the continuity of the Fourier transform on $\mathscr{S}$ we have

$$\begin{aligned}
\Big| \int_{\mathbb{R}} e^{-\gamma t} f_n(t)\psi_m(t)\,dt - \int_{\mathbb{R}} e^{-\gamma t} f_n(t)\psi(t)\,dt\Big| \\
&= \Big| \int_{\mathbb{R}} e^{-\gamma t}\Big(\int_{-n}^{n} F(\gamma+iu)e^{(\gamma+iu)t}\,du\Big)\psi_m(t)\,dt \\
&\quad - \int_{\mathbb{R}} e^{-\gamma t}\Big(\int_{-n}^{n} F(\gamma+iu)e^{(\gamma+iu)t}\,du\Big)\psi(t)\,dt\Big| \\
&= \Big| \int_{-n}^{n} F(\gamma+iu)\hat{\psi}_m(-u)\,du - \int_{-n}^{n} F(\gamma+iu)\hat{\psi}(-u)\,du\Big| \\
&\le \int_{-n}^{n} |F(\gamma+iu)||\hat{\psi}_m(-u) - \hat{\psi}(-u)|\,du.
\end{aligned}$$

This last integral tends to 0 as $m \to \infty$. This completes the proof.

THEOREM 2.3. *Under the assumptions of Lemma 2.2, each $f_n$ can be considered as an element of $\beta_L$, and $f_n \xrightarrow{\delta} x$.*

*Proof.* By Lemma 2.2, $f_n(t)$ is a Laplace transformable distribution and hence can be considered as an element of $\beta_L$ (by the usual identification $f_n \leftrightarrow \big[\frac{f_n * \delta_k}{\delta_k}\big]$ where $\{\delta_k\}$ is any delta sequence). Let $x = \big[\frac{g_n}{\delta_n}\big]$ so that $L(x)L(\delta_k) = L(g_k)$ for all $k \in \mathbb{N}$. Fix $k = 1, 2, \ldots$. Using Fubini's theorem, we have

$$\begin{aligned}
(f_n * \delta_k)(x) &= \int_0^\infty f_n(y)\delta_k(x-y)\,dy \\
&= \int_0^\infty \int_{-n}^{n} F(\gamma+iu)e^{(\gamma+iu)y}\,du\,\delta_k(x-y)\,dy \\
&= \int_0^\infty \int_{-n}^{n} F(\gamma+iu)e^{-(\gamma+iu)(x-y)}\,du\,\delta_k(x-y)e^{(\gamma+iu)x}\,dy \\
&= \int_{-n}^{n} \Big\{\int_0^\infty \delta_k(x-y)e^{-(\gamma+iu)(x-y)}\,dy\Big\} F(\gamma+iu)e^{(\gamma+iu)x}\,du \\
&= \int_{-n}^{n} L(\delta_k)(\gamma+iu)F(\gamma+iu)e^{(\gamma+iu)x}\,du \\
&= \int_{-n}^{n} L(g_k)(\gamma+iu)e^{(\gamma+iu)x}\,du.
\end{aligned}$$

Let $K$ be a fixed compact subset of $[0,\infty)$ and $x \in K$. Using the above result and the Fourier inversion theorem for elements of $\mathscr{S}$, we have

$$\begin{aligned}
&|(f_n * \delta_k)(x) - g_k(x)| \\
&\quad = \Big| \int_{-n}^{n} L(g_k)(\gamma + iu) e^{(\gamma+iu)x}\, du - g_k(x) \Big| \\
&\quad = \Big| \int_{-n}^{n} (e^{-\gamma t} g_k(t))^\wedge(u) e^{\gamma x} e^{iux}\, du - g_k(x) \Big| \\
&\quad = \Big| e^{\gamma x} \int_{\mathbb{R}} (e^{-\gamma t} g_k(t))^\wedge(u) e^{iux}\, du - e^{\gamma x} \int_{|u|\geq n} (e^{-\gamma t} g_k(t))^\wedge(u) e^{iux}\, du - g_k(x) \Big| \\
&\quad = \Big| - e^{\gamma x} \int_{|u|\geq n} (e^{-\gamma t} g_k(t))^\wedge(u) e^{iux}\, du \Big| \leq M \int_{|u|\geq n} |(e^{-\gamma t} g_k(t))^\wedge(u)|\, du
\end{aligned}$$

for a suitable constant $M$. Since $(e^{-\gamma t} g_k(t))^\wedge \in \mathscr{S} \subset L^1(\mathbb{R})$, we see that the right hand side of the above inequality tends to 0 as $n \to \infty$. Thus $f_n * \delta_k \to x * \delta_k$ as $n \to \infty$ (for each $k = 1, 2, \ldots$) uniformly on compact subsets of $[0,\infty)$. This completes the proof of our theorem.

NOTE 2.4. In the above inversion formula, we have defined the functions $f_n(t)$ depending on a parameter $\gamma > \alpha$. In general for different values of $\gamma$, we will get different sequences. However, all these sequences will converge (in the $\delta$-sense) to $x$. Further, in this representation of $f_n(t)$ we use only the values of $L(x)(z)$, and not $L(x * \delta_n)$ (as in [5]) whose values may not be available if only $L(x)$ is given. This gives the desired improvement of the inversion formula.

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