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On locally bounded solutions of Schilling's problem

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Abstract. We prove that for some parameters $q \in (0,1)$ every solution $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)]$$

which vanishes outside the interval $[-q/(1-q), q/(1-q)]$ and is bounded in a neighbourhood of a point of that interval vanishes everywhere.

Introduction. Considering a physical problem R. Schilling [18] came to the functional equation

$$f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)], \tag{1}$$

where $q \in (0,1)$ is a fixed number, and to its solutions $f : \mathbb{R} \to \mathbb{R}$ satisfying the boundary condition

$$f(x) = 0 \quad \text{for } |x| > Q \tag{2}$$

where

$$Q = \frac{q}{1-q}.$$

The physical background of this problem can also be found in [9] by G. Derfel and R. Schilling and in [11] by R. Girgensohn.

In what follows any solution $f : \mathbb{R} \to \mathbb{R}$ of (1) satisfying (2) will be called a *solution of Schilling's problem*.

The first nontrivial continuous solution of Schilling's problem was given by R. Schilling himself for $q = 1/2$. This solution is defined by

$$f_1(x) = \max\{1 - |x|, 0\} \quad \text{for } x \in \mathbb{R}.$$

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K. Baron, A. Simon and P. Volkmann [3] showed that if n is a positive integer and $q = 1/\sqrt[n]{2}$, then the convolution

$$f_1(x) \star f_1(qx) \star \ldots \star f_1(q^{n-1}x)$$

is a nontrivial continuous solution of Schilling's problem. They also proved that if $q \in (0, 1/2)$ and f is a nontrivial Lebesgue integrable solution of Schilling's problem, then

$$\int_0^{\varepsilon} |f(x)|^{(\log q)/\log(2q)}\, dx = +\infty$$

for every $\varepsilon > 0$. In particular, for every $q \in (0, 1/2)$ every bounded Lebesgue measurable solution of Schilling's problem vanishes almost everywhere. (Note that in [3] by K. Baron, A. Simon and P. Volkmann and in [19] by A. Simon and P. Volkmann distributional solutions of Schilling's problem are considered.) The case $q \in (1/2, 1)$ is quite different. Namely, from the paper [9] by G. Derfel and R. Schilling it follows that for almost all $q \in (1/2, 1)$ there are nontrivial continuous solutions. However, if the inverse of q is a Salem number [6], then such solutions do not exist (cf. also [15] where nonexistence of nontrivial continuous solutions of Schilling's problem was proved for the golden ratio $q = (\sqrt{5} - 1)/2$).

K. Baron and P. Volkmann [4] (see also [8] by I. Daubechies and J. C. Lagarias) proved that for every $q \in (0, 1)$ the vector space of Lebesgue integrable solutions of Schilling's problem is at most one-dimensional. (The same concerns Riemann integrable solutions; see [10] by W. Förg-Rob.) It is known that the vector space of Lebesgue integrable solutions of Schilling's problem is zero-dimensional for $q \in (0, 1/(2\sqrt{2}))$ (see [16] by Y. Peres and B. Solomyak) and also for those $q \neq 1/2$ for which the inverse of q is a Pisot number (see [7] by J. M. Borwein and R. Girgensohn). However, it is one-dimensional for almost all $q \in (1/(2\sqrt{2}), 1)$ (see [16] by Y. Peres and B. Solomyak). Up to now the only explicitly given q's for which the vector space of integrable solutions is one-dimensional are $1/\sqrt[n]{2}$ given by K. Baron, A. Simon and P. Volkmann [3]. If the vector space of Lebesgue integrable solutions of Schilling's problem is one-dimensional, then every nonzero function from this space is either positive or negative (almost everywhere) on its support (see [12]) and according to [5] by L. Bartłomiejczyk, Schilling's problem has also strange solutions; e.g. such that their graph meets every Borel subset of $[-Q, Q] \times \mathbb{R}$ with uncountable vertical projection.

Bounded solutions interesting from the physical point of view were first examined by K. Baron [5]. His result says that for $q \in (0, \sqrt{2} - 1]$ the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of the origin. Generalizations of this result can be found in [14] and [13] where it is proved among other things that for $q \leq 1/3$ the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of a point of the set

$$\Big\{\varepsilon\sum_{i=1}^{n} q^i : n\in\mathbb{N}\cup\{0,+\infty\},\ \varepsilon\in\{-1,1\}\Big\} \tag{3}$$

and no point outside (3) has this property. Note that for $q=1/3$ the set (3) coincides with the interval $[-Q,Q]$.

More details on Schilling's problem can be found in [11] by R. Girgensohn, in [2, Section 5] by K. Baron and W. Jarczyk and in [17].

In the present paper we are interested in finding parameters $q\in(1/3,1/2)$ for which the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of a point of $[-Q,Q]$. We make the following definition.

Definition. Let $x\in[-Q,Q]$.

We say $x\in B_q$ if the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of x.

We say $x\in C_q$ if the zero function is the only solution of Schilling's problem which is continuous at x.

We say $x\in Z_q$ if the zero function is the only solution of Schilling's problem which vanishes in a neighbourhood of x.

It is easily seen that

$$B_q\subset C_q\subset Z_q\subset[-Q,Q]$$

for every $q\in(0,1)$.

Main results. For the convenience of the reader we repeat four relevant facts from [13] without proofs.

Remark 1. *Assume f is a solution of Schilling's problem. If $q\neq 1/4$, then $f(-Q)=f(Q)=0$. If $q<1/2$, then $f(0)=0$.*

Remark 2. *If f is a solution of Schilling's problem, then so is the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=f(-x)$.*

Lemma 1. *Assume $q\in(0,1/2)$. If a solution of Schilling's problem vanishes either on $(-q,0)$ or on $(0,q)$, then it vanishes everywhere.*

Lemma 2. *Assume $q\in(0,1/2)$. If f is a solution of Schilling's problem, then*

$$f\Big(q^{N+M}x+\varepsilon\sum_{m=1}^{M}q^m\Big)=\Big(\frac{1}{2}\Big)^M\Big(\frac{1}{2q}\Big)^{N+M}f(x)$$

for every $x\in(Q-1,1-Q)$ (for every $x\in[Q-1,1-Q]$ if $q\neq 1/4$), every $\varepsilon\in\{-1,1\}$, and any nonnegative integers M and N.

We first deal with the number $q = (3-\sqrt{5})/2$.

LEMMA 3. *Let $q = (3-\sqrt{5})/2$. If f is a solution of Schilling's problem, then*

$$f(q^N x - q^N) = \frac{1}{2}\left(\frac{1}{2q}\right)^N f(x) \tag{4}$$

for every $x \in [0, 1-Q]$ and every positive integer N.

Proof. Observe that $q \in (1/3, 1/2)$,

$$q^2 - 3q + 1 = 0 \quad \text{and} \quad Q = 1 - q. \tag{5}$$

Fix $x_0 \in [0, 1-Q]$ and put $x = x_0 - 1$. Since $x - 1 < x \le -Q$, by (1), (2) and Remark 1 we have

$$f(qx_0 - q) = f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] = \frac{1}{4q} f(x_0).$$

Fix now a positive integer N and assume that (4) holds for every $x \in [0, 1-Q]$. Fixing $x_0 \in [0, 1-Q]$ and putting $x = q^N x_0 - q^N$ we see that $x - 1 < -1 < -Q$ and $x + 1 \ge -q + 1 = Q$. Consequently,

$$\begin{aligned} f(q^{N+1}x_0 - q^{N+1}) &= \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \\ &= \frac{1}{2q} f(x) = \frac{1}{2q} f(q^N x_0 - q^N) = \frac{1}{2}\left(\frac{1}{2q}\right)^{N+1} f(x_0). \end{aligned}$$

LEMMA 4. *Let $q = (3-\sqrt{5})/2$. If f is a solution of Schilling's problem, then for any nonnegative integers k and l satisfying*

$$|k - lq| < Q \tag{6}$$

there exist a positive real $\alpha_{k,l}$ and a positive integer $n_{k,l}$ such that

$$f(q^n x + k - lq) = \alpha_{k,l}\left(\frac{1}{2q}\right)^n f(x), \tag{7}$$

$$f(q^n x - q^n - k + lq) = \frac{1}{2}\alpha_{k,l}\left(\frac{1}{2q}\right)^n f(x), \tag{8}$$

for every integer $n \ge n_{k,l}$ and every $x \in [0, 1-Q]$.

Proof. With the help of (5) we check at once that if nonnegative integers k and $l \le 4$ satisfy (6), then

$$(k, l) \in \{(0,0), (0,1), (1,2), (1,3), (1,4), (2,4)\}.$$

Put

$$\alpha_{0,0} = 1, \quad \alpha_{0,1} = \alpha_{1,3} = 1/2, \quad \alpha_{1,2} = \alpha_{1,4} = 1/4, \quad \alpha_{2,4} = 1/8,$$
$$n_{0,0} = n_{0,1} = n_{1,2} = n_{1,3} = n_{1,4} = n_{2,4} = 4.$$

If $x \in [0, 1-Q]$, then using Lemma 2, (5), (1), (2), Remark 1 and Lemma 3 we find that for every integer $n \geq 2$ the following equalities hold:

$$\begin{aligned}
f(q^n x) &= \left(\frac{1}{2q}\right)^n f(x) = \alpha_{0,0}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x - q) &= \frac{1}{2}\left(\frac{1}{2q}\right)^n f(x) = \alpha_{0,1}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x + 1 - 2q) &= f(q^n x - q^2 + q)\\
&= \frac{1}{4q}[f(q^{n-1}x - q) + f(q^{n-1}x - q + 2)\\
&\quad + 2f(q^{n-1}x - q + 1)]\\
&= \frac{1}{4q} f(q^{n-1}x - q)\\
&= \frac{1}{4q}\frac{1}{2}\left(\frac{1}{2q}\right)^{n-1} f(x)\\
&= \alpha_{1,2}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x + 1 - 3q) &= f(q^2(q^{n-2}x) - q^2) = \frac{1}{2}\left(\frac{1}{2q}\right)^2 f(q^{n-2}x)\\
&= \frac{1}{2}\left(\frac{1}{2q}\right)^2\left(\frac{1}{2q}\right)^{n-2} f(x)\\
&= \alpha_{1,3}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x + 1 - 4q) &= f(q^n x - q^2 - q) = \left(\frac{1}{2}\right)^2\left(\frac{1}{2q}\right)^n f(x)\\
&= \alpha_{1,4}\left(\frac{1}{2q}\right)^n f(x),
\end{aligned}$$

and if $n \geq 3$, then using also the third of the above equalities we get

$$\begin{aligned}
f(q^n x + 2 - 4q) &= f(q^n x - 2q^2 + 2q)\\
&= \frac{1}{4q}[f(q^{n-1}x - 2q + 1)\\
&\quad + f(q^{n-1}x - 2q + 3) + 2f(q^{n-1}x - 2q + 2)]\\
&= \frac{1}{4q} f(q^{n-1}x + 1 - 2q)\\
&= \frac{1}{4q}\alpha_{1,2}\left(\frac{1}{2q}\right)^{n-1} f(x)\\
&= \alpha_{2,4}\left(\frac{1}{2q}\right)^n f(x);
\end{aligned}$$

similarly we obtain equalities which correspond to (8):

$$
\begin{aligned}
f(q^n x - q^n) &= \frac{1}{2}\left(\frac{1}{2q}\right)^n f(x) = \frac{1}{2}\alpha_{0,0}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x - q^n + q) &= \frac{1}{4q}[f(q^{n-1}x - q^{n-1})\\
&\quad + f(q^{n-1}x - q^{n-1} + 2) + 2f(q^{n-1}x - q^{n-1} + 1)]\\
&= \frac{1}{4q} f(q^{n-1}x - q^{n-1}) = \frac{1}{4q}\frac{1}{2}\left(\frac{1}{2q}\right)^{n-1} f(x)\\
&= \frac{1}{2}\alpha_{0,1}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x - q^n - 1 + 2q) &= f(q^n x - q^n + q^2 - q)\\
&= \frac{1}{4q}[f(q^{n-1}x - q^{n-1} + q - 2)\\
&\quad + f(q^{n-1}x - q^{n-1} + q)\\
&\quad + 2f(q^{n-1}x - q^{n-1} + q - 1)]\\
&= \frac{1}{4q} f(q^{n-1}x - q^{n-1} + q)\\
&= \frac{1}{4q}\frac{1}{2}\alpha_{0,1}\left(\frac{1}{2q}\right)^{n-1} f(x)\\
&= \frac{1}{2}\alpha_{1,2}\left(\frac{1}{2q}\right)^n f(x),\\
f(q^n x - q^n - 1 + 3q) &= f(q^n x - q^n + q^2)\\
&= \frac{1}{4q}[f(q^{n-1}x - q^{n-1} + q - 1)\\
&\quad + f(q^{n-1}x - q^{n-1} + q + 1)\\
&\quad + 2f(q^{n-1}x - q^{n-1} + q)]\\
&= \frac{1}{2q} f(q^{n-1}x - q^{n-1} + q)\\
&= \frac{1}{2q}\frac{1}{2}\alpha_{0,1}\left(\frac{1}{2q}\right)^{n-1} f(x)\\
&= \frac{1}{2}\alpha_{1,3}\left(\frac{1}{2q}\right)^n f(x),
\end{aligned}
$$

$$\begin{aligned}
f(q^n x - q^n - 1 + 4q) &= f(q^n x - q^n + q^2 + q) \\
&= \frac{1}{4q}[f(q^{n-1}x - q^{n-1} + q) \\
&\quad + f(q^{n-1}x - q^{n-1} + q + 2) \\
&\quad + 2f(q^{n-1}x - q^{n-1} + q + 1)] \\
&= \frac{1}{4q} f(q^{n-1}x - q^{n-1} + q) \\
&= \frac{1}{4q}\frac{1}{2}\alpha_{0,1}\left(\frac{1}{2q}\right)^{n-1} f(x) \\
&= \frac{1}{2}\alpha_{1,4}\left(\frac{1}{2q}\right)^{n} f(x), \\
f(q^n x - q^n - 2 + 4q) &= f(q^n x - q^n + 2q^2 - 2q) \\
&= \frac{1}{4q}[f(q^{n-1}x - q^{n-1} + 2q - 3) \\
&\quad + f(q^{n-1}x - q^{n-1} + 2q - 1) \\
&\quad + 2f(q^{n-1}x - q^{n-1} + 2q - 2)] \\
&= \frac{1}{4q} f(q^{n-1}x - q^{n-1} + 2q - 1) \\
&= \frac{1}{4q}\frac{1}{2}\alpha_{1,2}\left(\frac{1}{2q}\right)^{n-1} f(x) \\
&= \frac{1}{2}\alpha_{2,4}\left(\frac{1}{2q}\right)^{n} f(x).
\end{aligned}$$

Fix now a nonnegative integer $L \geq 5$ and assume that for any nonnegative integers k and $l < L$ satisfying (6) there exist a positive real $\alpha_{k,l}$ and a positive integer $n_{k,l}$ such that (7) and (8) hold for every integer $n \geq n_{k,l}$ and every $x \in [0, 1 - Q]$. Let k be a nonnegative integer such that

$$|k - Lq| < Q.$$

Then

$$1 < k < L. \tag{9}$$

Putting

$$y = \frac{k - Lq}{q}$$

and using (5) we see that $|y| < Q/q = Q+1$ and

$$y = \frac{(3q-q^2)k - Lq}{q} = 3k - L - kq.$$

Applying (5) again we see that y belongs to one of the intervals

$$(10)\quad (-Q-1,-Q),\ (-Q,Q-1),\ (Q-1,1-Q),\ (1-Q,Q),\ (Q,Q+1).$$

It follows that there exists a positive integer N such that for every integer $n \geq N$ and $x \in [0,1-Q]$ the number y belongs to one of the intervals (10) together with the numbers

$$y + q^{n-1}x, \qquad y - q^{n-1}x + q^{n-1}.$$

Moreover, making also use of (9) we have: if $y > -Q-1$, then

$$3k - L + 1 = y + kq + 1 > -Q + kq = -1 + (k+1)q \geq -1 + 3q > 0;$$

if $y > -Q$, then

$$3k - L = y + kq > -Q + kq > 0;$$

and if $y > 1-Q$, then

$$3k - L - 1 = y + kq - 1 > -Q + kq > 0.$$

This allows us to define $\alpha_{k,L}$ and $n_{k,L}$ by

$$\alpha_{k,L} = \begin{cases} \frac{1}{2}\alpha_{3k-L+1,k} & \text{if } -Q-1 < y < -Q, \\ \frac{1}{2}[\alpha_{3k-L+1,k} + 2\alpha_{3k-L,k}] & \text{if } -Q < y < Q-1, \\ \alpha_{3k-L,k} & \text{if } Q-1 < y < 1-Q, \\ \frac{1}{2}[\alpha_{3k-L-1,k} + 2\alpha_{3k-L,k}] & \text{if } 1-Q < y < Q, \\ \frac{1}{2}\alpha_{3k-L-1,k} & \text{if } Q < y < Q+1, \end{cases}$$

$$n_{k,L} = \begin{cases} \max\{n_{3k-L+1,k}+1, N\} & \text{if } -Q-1 < y < -Q, \\ \max\{n_{3k-L+1,k}+1, n_{3k-L,k}+1, N\} & \text{if } -Q < y < Q-1, \\ \max\{n_{3k-L,k}+1, N\} & \text{if } Q-1 < y < 1-Q, \\ \max\{n_{3k-L-1,k}+1, n_{3k-L,k}+1, N\} & \text{if } 1-Q < y < Q, \\ \max\{n_{3k-L-1,k}+1, N\} & \text{if } Q < y < Q+1. \end{cases}$$

If $n \geq n_{k,L}$ is an integer and $x \in [0,1-Q]$, then putting

$$w = y + q^{n-1}x, \qquad z = y - q^{n-1}x + q^{n-1},$$

we have

$$qw = q^n x + qy = q^n x + k - Lq, \qquad qz = -(q^n x - q^n - k + Lq)$$

and, in consequence,

$$f(q^n x + k - Lq) = f(qw) = \frac{1}{4q}[f(w-1) + f(w+1) + 2f(w)]$$
$$= \begin{cases} \frac{1}{4q} f(w+1) & \text{if } -Q-1 < w < -Q, \\ \frac{1}{4q}[f(w+1) + 2f(w)] & \text{if } -Q < w < Q-1, \\ \frac{1}{2q} f(w) & \text{if } Q-1 < w < 1-Q, \\ \frac{1}{4q}[f(w-1) + 2f(w)] & \text{if } 1-Q < w < Q, \\ \frac{1}{4q} f(w-1) & \text{if } Q < w < Q+1, \end{cases}$$
$$= \begin{cases} \frac{1}{4q} f(w+1) & \text{if } -Q-1 < y < -Q, \\ \frac{1}{4q}[f(w+1) + 2f(w)] & \text{if } -Q < y < Q-1, \\ \frac{1}{2q} f(w) & \text{if } Q-1 < y < 1-Q, \\ \frac{1}{4q}[f(w-1) + 2f(w)] & \text{if } 1-Q < y < Q, \\ \frac{1}{4q} f(w-1) & \text{if } Q < y < Q+1, \end{cases}$$
$$= \alpha_{k,L} \left(\frac{1}{2q}\right)^n f(x),$$

and

$$f(q^n x - q^n - k + Lq) = f(-qz) = \frac{1}{4q}[f(-z-1) + f(-z+1) + 2f(-z)]$$
$$= \begin{cases} \frac{1}{4q} f(-z-1) & \text{if } -Q-1 < y < -Q, \\ \frac{1}{4q}[f(-z-1) + 2f(-z)] & \text{if } -Q < y < Q-1, \\ \frac{1}{2q} f(-z) & \text{if } Q-1 < y < 1-Q, \\ \frac{1}{4q}[f(-z+1) + 2f(-z)] & \text{if } 1-Q < y < Q, \\ \frac{1}{4q} f(-z+1) & \text{if } Q < y < Q+1, \end{cases}$$
$$= \frac{1}{2} \alpha_{k,L} \left(\frac{1}{2q}\right)^n f(x).$$

THEOREM 1. *If* $q = (3 - \sqrt{5})/2$, *then*

$$B_q = C_q = Z_q = [-Q, Q]. \tag{11}$$

Proof. Assume f is a solution of Schilling's problem which is bounded in a neighbourhood of $x_0 \in [-Q, Q]$. Since $\mathbb{Z} + q\mathbb{Z}$ is a dense subset of the real line, we may (and do) assume that x_0 is of the form $k - lq$, where k and l are integers satisfying (6). This jointly with (5) implies that $k \cdot l \geq 0$.

If $x \in [0, 1-Q]$, then either the left-hand side of (7) or the left-hand side of (8) is bounded with respect to n. From Lemma 4 we then infer that f vanishes on $[0, 1-Q]$. Now by the second part of (5) and Lemma 1 it is obvious that f vanishes everywhere.

Putting $x = 0$ in (7) and using Remarks 1 and 2 we get one of the main results of [15].

Corollary 1. *If $q = (3-\sqrt{5})/2$, then every solution of Schilling's problem vanishes on $\mathbb{Z} + q\mathbb{Z}$.*

In the second part of this paper we will show that in Theorem 1 the number $(3-\sqrt{5})/2$ may be replaced by any $q \in (1/3, 1/2)$ satisfying

$$2\sum_{k=1}^{K} q^k + \lambda q^{K+1} = 1 \tag{12}$$

with a positive integer K and a $\lambda \in \{1, 2\}$.

Lemma 5. *If there exist $K \in \mathbb{N}$ and $\lambda \in \{1, 2\}$ satisfying* (12), *then*

$$1 - Q \neq \sum_{n=0}^{N} \varepsilon_n q^n$$

for all $N \in \mathbb{N} \cup \{0\}$ and $\varepsilon_0, \ldots, \varepsilon_N \in \{-1, 0, 1\}$.

Proof. Suppose that there exist $N \in \mathbb{N} \cup \{0\}$ and $\varepsilon_0, \ldots, \varepsilon_N \in \{-1, 0, 1\}$ such that

$$1 - Q = \sum_{n=0}^{N} \varepsilon_n q^n \tag{13}$$

and put

$$x_0 = \sum_{n=0}^{N} \varepsilon_n q^n. \tag{14}$$

We conclude from (12) that

$$1 - Q = 2\sum_{n=1}^{K} q^n + \lambda q^{K+1} - \sum_{n=1}^{\infty} q^n,$$

hence

$$x_0 = \sum_{n=1}^{K} q^n + (\lambda - 1)q^{K+1} - \sum_{n=K+2}^{\infty} q^n. \tag{15}$$

Moreover, $q < 1/2$ implies

$$\sum_{n=K+2}^{\infty} q^n < q^{K+1} < q^K,$$

which jointly with (15) gives

$$\sum_{n=1}^{K-1} q^n < \sum_{n=1}^{K} q^n - \sum_{n=K+2}^{\infty} q^n \leq x_0. \tag{16}$$

We are now in a position to show

(17) $$\varepsilon_0 = 0, \quad N \geq K, \quad \varepsilon_n = 1 \quad \text{for } n \in \{1, \dots, K-1\}.$$

In fact, if $\varepsilon_0 \neq 0$, then using (13) and (14) we have $x_0 - \varepsilon_0 = 1 - Q - \varepsilon_0 \in \{-Q, 2 - Q\}$ which contradicts $x_0 - \varepsilon_0 \in (-Q, Q)$ resulting from (14). Thus $\varepsilon_0 = 0$. Hence (14) yields

$$x_0 = \sum_{n=1}^{N} \varepsilon_n q^n \leq \sum_{n=1}^{N} q^n.$$

By (16) we therefore get $N \geq K$. Suppose now that $\varepsilon_i \neq 1$ for some $i \in \{1, \dots, K-1\}$. Then

$$x_0 = \sum_{n=1}^{N} \varepsilon_n q^n \leq \sum_{n=1}^{i-1} q^n + \sum_{n=i+1}^{N} q^n \leq \sum_{n=1}^{K-2} q^n + \sum_{n=K}^{N} q^n < \sum_{n=1}^{K-1} q^n,$$

contrary to (16).

From (14) and (17) we have

$$x_0 = \sum_{n=1}^{K-1} q^n + \sum_{n=K}^{N} \varepsilon_n q^n,$$

which jointly with (15) gives

(18) $$\sum_{n=0}^{N-K} \varepsilon_{K+n} q^n = 1 + (\lambda - 1)q - \sum_{n=2}^{\infty} q^n.$$

Consider first the case $\lambda = 2$. Then (18) reads

(19) $$\sum_{n=0}^{N-K} \varepsilon_{K+n} q^n = 1 + q - \sum_{n=2}^{\infty} q^n.$$

If $\varepsilon_K \neq 1$, then

$$\sum_{n=0}^{N-K} \varepsilon_{K+n} q^n \leq \sum_{n=1}^{N-K} \varepsilon_{K+n} q^n < Q < 1 < 1 + q - \sum_{n=2}^{\infty} q^n.$$

This contradicts (19), so $\varepsilon_K = 1$. Thus (19) leads to

$$\sum_{n=0}^{N-K-1} \varepsilon_{K+1+n} q^n = 1 - \sum_{n=1}^{\infty} q^n = 1 - Q,$$

i.e.,

(20) $$1 - Q = \sum_{n=0}^{N-K} \varepsilon'_n q^n,$$

where $\varepsilon'_n = \varepsilon_{K+1+n}$ for $n \in \{0,\dots,N-K-1\}$ and $\varepsilon'_{N-K} = 0$. Of course

$$\varepsilon'_n \in \{-1,0,1\} \quad \text{for } n \in \{0,\dots,N-K\}. \tag{21}$$

Assume now that $\lambda = 1$. Then (18) reduces to

$$\sum_{n=0}^{N-K} \varepsilon_{K+n} q^n = 1 - \sum_{n=2}^{\infty} q^n. \tag{22}$$

In particular, $N \neq K$. This and (17) give $N \geq K+1$.

If $\varepsilon_{K+1} \neq -1$, then $\varepsilon_{K+1} - 1 \in \{-1,0\}$ and (22) can be written in the form

$$\varepsilon_K + (\varepsilon_{K+1} - 1)q + \sum_{n=2}^{N-K} \varepsilon_{K+n} q^n = 1 - Q,$$

i.e., in the form (20), where now $\varepsilon'_0 = \varepsilon_K$, $\varepsilon'_1 = \varepsilon_{K+1} - 1$ and $\varepsilon'_n = \varepsilon_{K+n}$ for $n \in \{2,\dots,N-K\}$. Moreover, (21) holds as well.

Finally assume that $\varepsilon_{K+1} = -1$. From (22) we have

$$\varepsilon_K - q + \sum_{n=2}^{N-K} \varepsilon_{K+n} q^n = 1 - \sum_{n=2}^{\infty} q^n > 1 - q.$$

Hence

$$1 < \varepsilon_K + \sum_{n=2}^{N-K} \varepsilon_{K+n} q^n < \varepsilon_K + 1.$$

Therefore $\varepsilon_K = 1$ and equality (22) can also be written in the form

$$\sum_{n=1}^{N-K-1} \varepsilon_{K+1+n} q^n = 1 - Q.$$

In each of the cases considered we have represented $1-Q$ in the form (20) with (21). Consequently, we have shown that if $1-Q$ is of the form (13), then $N \geq K$ and $1-Q$ is of the form (20) as well. Consequently, $N \geq mK$ for every positive integer m, a contradiction.

Lemma 6. *Assume that* (12) *holds for some* $K \in \mathbb{N}$ *and* $\lambda \in \{1,2\}$. *If* N *is a positive integer*, $\varepsilon_0 \in \{-1,1\}$, $\varepsilon_1,\dots,\varepsilon_N \in \{-1,0,1\}$, *and the number* x_0 *defined by* (14) *belongs to* $[-Q,Q]$, *then* $N \geq K$ *and* $\varepsilon_n = -\varepsilon_0$ *for all* $n \in \{1,\dots,K\}$. *Moreover*, *if* $\lambda = 2$, *then* $N \geq K+1$ *and* $\varepsilon_{K+1} \neq \varepsilon_0$.

Proof. Combining (14) with (12) we obtain

$$|x_0| = \Big|\varepsilon_0 + \sum_{n=1}^{N} \varepsilon_0 q^n\Big| \geq 1 - \sum_{n=1}^{N} q^n = 2\sum_{n=1}^{K} q^n + \lambda q^{K+1} - \sum_{n=1}^{N} q^n. \tag{23}$$

Suppose that $N < K$. Then

$$|x_0| \geq \sum_{n=1}^{K} q^n + \sum_{n=N+1}^{K} q^n + \lambda q^{K+1} > \sum_{n=1}^{K} q^n + q^K > Q,$$

a contradiction.

Suppose now that $\varepsilon_i \neq -\varepsilon_0$ for some $i \in \{1, \ldots, K\}$. Then

$$\begin{aligned}
|x_0| &= \left|\varepsilon_0 + \sum_{n=1}^{i-1} \varepsilon_n q^n + \varepsilon_i q^i + \sum_{n=i+1}^{N} \varepsilon_n q^n\right| \\
&\geq 1 - \sum_{n=1}^{i-1} q^n - \sum_{n=i+1}^{N} q^n \\
&= 2\sum_{n=1}^{K} q^n + \lambda q^{K+1} - \sum_{n=1,\, n\neq i}^{N} q^n \\
&\geq 2\sum_{n=1}^{K} q^n + \lambda q^{K+1} - \sum_{n=1,\, n\neq K}^{N} q^n \\
&= \sum_{n=1}^{K} q^n + \lambda q^{K+1} + q^K - \sum_{n=K+1}^{N} q^n \\
&> \sum_{n=1}^{K} q^n + q^{K+1} + q^K - \sum_{n=K+1}^{\infty} q^n \\
&= Q - 2\sum_{n=K+1}^{\infty} q^n + q^{K+1} + q^K.
\end{aligned}$$

Moreover, since (12) implies $2q + q^2 \leq 1$, we have

$$2\sum_{n=K+1}^{\infty} q^n = 2\frac{q^{K+1}}{1-q} \leq q^K + q^{K+1}.$$

Hence $|x_0| > Q$, which contradicts our assumption.

Assume now $\lambda = 2$. If $N = K$, then from (23) we get

$$|x_0| \geq \sum_{n=1}^{K} q^n + 2q^{K+1} = \sum_{n=1}^{K+1} q^n + q^{K+1} > Q,$$

a contradiction which shows that $N \geq K + 1$. It remains to show that

$\varepsilon_{K+1} \neq \varepsilon_0$. Indeed, if $\varepsilon_{K+1} = \varepsilon_0$, then

$$\begin{aligned}
|x_0| &= \Big|\varepsilon_0 + \sum_{n=1}^{K} \varepsilon_n q^n + \varepsilon_0 q^{K+1} + \sum_{n=K+2}^{N} \varepsilon_n q^n\Big| \\
&\geq 1 - \sum_{n=1}^{K} q^n + q^{K+1} - \sum_{n=K+2}^{N} q^n \\
&= 2\sum_{n=1}^{K} q^n + 2q^{K+1} - \sum_{n=1}^{K} q^n + q^{K+1} - \sum_{n=K+2}^{N} q^n \\
&> \sum_{n=1}^{K+1} q^n + 2q^{K+1} - \sum_{n=K+2}^{\infty} q^n \\
&> \sum_{n=1}^{K+1} q^n + q^{K+1} > Q,
\end{aligned}$$

contrary to the assumption.

Lemma 7. *Assume $q \in (0, 1/2)$. If f is a solution of Schilling's problem, then*

$$f\Big(q^{N+1}y + 2\varepsilon q^{N+1} + \varepsilon \sum_{n=1}^{N} q^n\Big) = \left(\frac{1}{4q}\right)^{N+1} f(y+\varepsilon) \tag{24}$$

for all $N \in \mathbb{N} \cup \{0\}$, $\varepsilon \in \{-1, 1\}$, and $|y| \leq 1$.

Proof. Notice that if $\varepsilon = -1$, then $y + 2\varepsilon \leq -1$, and if $\varepsilon = 1$, then $y + 2\varepsilon \geq 1$. This gives

$$\begin{aligned}
f(qy + 2\varepsilon q) &= \frac{1}{4q}[f(y + 2\varepsilon - 1) + f(y + 2\varepsilon + 1) + 2f(y + 2\varepsilon)] \\
&= \frac{1}{4q} f(y+\varepsilon)
\end{aligned}$$

and (24) remains true for $N = 0$.

Assuming (24) to hold for a nonnegative integer N, put

$$z = q^{N+1}y + 2\varepsilon q^{N+1} + \varepsilon \sum_{n=1}^{N} q^n$$

and observe that if $\varepsilon = -1$, then

$$z = q^{N+1}y - 2q^{N+1} - \sum_{n=1}^{N} q^n \leq -\sum_{n=1}^{N+1} q^n < 0 < 1 - Q,$$

while if $\varepsilon = 1$, then

$$z = q^{N+1}y + 2q^{N+1} + \sum_{n=1}^{N} q^n \geq \sum_{n=1}^{N+1} q^n > 0 > Q - 1.$$

It follows that

$$f\Big(q^{N+2}y + 2\varepsilon q^{N+2} + \varepsilon \sum_{n=1}^{N+1} q^n\Big) = f(qz + q\varepsilon)$$
$$= \frac{1}{4q}[f(z+\varepsilon-1) + f(z+\varepsilon+1) + f(z+\varepsilon)] = \frac{1}{4q}f(z)$$
$$= \frac{1}{4q}f\Big(q^{N+1}y + 2\varepsilon q^{N+1} + \varepsilon\sum_{n=1}^{N} q^n\Big) = \Big(\frac{1}{4q}\Big)^{N+2} f(y+\varepsilon).$$

LEMMA 8. *Assume that* (12) *holds for some* $K \in \mathbb{N}$ *and* $\lambda \in \{1,2\}$, n *is a nonnegative integer*, $\varepsilon_0, \ldots, \varepsilon_N \in \{-1,0,1\}$ *and the number* x_0 *defined by* (14) *belongs to* $[-Q,Q]$. *If* f *is a solution of Schilling's problem*, *then there exist* $\alpha_{\varepsilon_0,\ldots,\varepsilon_N} > 0$ *and a positive integer* $n_{\varepsilon_0,\ldots,\varepsilon_N}$ *such that*

$$f(q^{N+n}x + x_0) = \alpha_{\varepsilon_0,\ldots,\varepsilon_N}\Big(\frac{1}{2q}\Big)^{N+n} f(x) \tag{25}$$

for every integer $n \geq n_{\varepsilon_0,\ldots,\varepsilon_N}$ *and* $x \in [Q-1, 1-Q]$.

Proof. If $N = 0$, then $|\varepsilon_0| = |x_0| \leq Q < 1$. Consequently, $\varepsilon_0 = 0$ and $x_0 = 0$. Hence for $N = 0$ it is enough to put $\alpha_{\varepsilon_0} = 1$, $n_{\varepsilon_0} = 1$ and use Lemma 2.

Fix now a nonnegative integer M and assume that for every nonnegative integer $N \leq M$ and every $\varepsilon_0, \ldots, \varepsilon_N \in \{-1,0,1\}$ such that x_0 belongs to $[-Q,Q]$ there exist $\alpha_{\varepsilon_0,\ldots,\varepsilon_N} > 0$ and a positive integer $n_{\varepsilon_0,\ldots,\varepsilon_N}$ such that (25) holds for all $n \geq n_{\varepsilon_0,\ldots,\varepsilon_N}$ and $x \in [Q-1, 1-Q]$.

Fix $\varepsilon_0, \ldots, \varepsilon_{M+1} \in \{-1,0,1\}$ and assume that

$$y_0 = \sum_{m=0}^{M+1} \varepsilon_m q^m \in [-Q,Q].$$

Consider the following three cases:

(i) $\varepsilon_0 = 0$,
(ii) $|\varepsilon_0| = 1$ and $\lambda = 1$,
(iii) $|\varepsilon_0| = 1$ and $\lambda = 2$.

In case (i) from Lemma 5 we see that $y = q^{-1}y_0$ belongs to one of the intervals (10). Then there exists a positive integer L such that for every $n \geq L$ and every $x \in [Q-1, 1-Q]$ the number y belongs to one of the intervals (10) together with $q^{M+n}x + y$. Observe also that if $y < Q - 1$, then $\varepsilon_1 \neq 1$, and if $y > 1 - Q$, then $\varepsilon_1 \neq -1$. In particular, we can define

$$\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} \frac{1}{2}\alpha_{\varepsilon_1+1,\varepsilon_2,\dots,\varepsilon_{M+1}} & \text{if } -Q-1<y<-Q, \\ \frac{1}{2}[\alpha_{\varepsilon_1+1,\varepsilon_2,\dots,\varepsilon_{M+1}} + 2\alpha_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{M+1}}] & \text{if } -Q<y<Q-1, \\ \alpha_{\varepsilon_1,\dots,\varepsilon_{M+1}} & \text{if } Q-1<y<1-Q, \\ \frac{1}{2}[\alpha_{\varepsilon_1-1,\varepsilon_2,\dots,\varepsilon_{M+1}} + 2\alpha_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{M+1}}] & \text{if } 1-Q<y<Q, \\ \frac{1}{2}\alpha_{\varepsilon_1-1,\varepsilon_2,\dots,\varepsilon_{M+1}} & \text{if } Q<y<Q+1, \end{cases}$$

$$n_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} \max\{n_{\varepsilon_1+1,\varepsilon_2,\dots,\varepsilon_{M+1}}, L\} & \text{if } -Q-1<y<-Q, \\ \max\{n_{\varepsilon_1+1,\varepsilon_2,\dots,\varepsilon_{M+1}}, n_{\varepsilon_1,\dots,\varepsilon_{M+1}}, L\} & \text{if } -Q<y<Q-1, \\ \max\{n_{\varepsilon_1,\dots,\varepsilon_{M+1}}, L\} & \text{if } Q-1<y<1-Q, \\ \max\{n_{\varepsilon_1-1,\varepsilon_2,\dots,\varepsilon_{M+1}}, n_{\varepsilon_1,\dots,\varepsilon_{M+1}}, L\} & \text{if } 1-Q<y<Q, \\ \max\{n_{\varepsilon_1-1,\varepsilon_2,\dots,\varepsilon_{M+1}}, L\} & \text{if } Q<y<Q+1. \end{cases}$$

If $n \ge n_{\varepsilon_0,\dots,\varepsilon_{M+1}}$ and $x \in [Q-1, 1-Q]$, then putting $w = q^{M+n}x + y$ we have

$$\begin{aligned} f(q^{M+1+n}x + y_0) = f(qw) &= \frac{1}{4q}[f(w-1) + f(w+1) + 2f(w)] \\ &= \begin{cases} \frac{1}{4q}f(w+1) & \text{if } -Q-1<y<-Q, \\ \frac{1}{4q}[f(w+1)+2f(w)] & \text{if } -Q<y<Q-1, \\ \frac{1}{2q}f(w) & \text{if } Q-1<y<1-Q, \\ \frac{1}{4q}[f(w-1)+2f(w)] & \text{if } 1-Q<y<Q, \\ \frac{1}{4q}f(w-1) & \text{if } Q<y<Q+1, \end{cases} \\ &= \alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}}\left(\frac{1}{2q}\right)^{M+1+n} f(x). \end{aligned}$$

Consider now case (ii). According to Lemma 6, $M+1 \ge K$, and $\varepsilon_m = -\varepsilon_0$ for $m \in \{1,\dots,K\}$. Applying now (12) we see that

$$y_0 = \varepsilon_0 - \sum_{m=1}^{K} \varepsilon_0 q^m + \sum_{m=K+1}^{M+1} \varepsilon_m q^m = \sum_{m=1}^{K+1} \varepsilon_0 q^m + \sum_{m=K+1}^{M+1} \varepsilon_m q^m. \tag{26}$$

Put

$$\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} \left(\frac{1}{2}\right)^{K+1} & \text{if } M+1=K, \\ \alpha_{0,\underbrace{\varepsilon_0,\dots,\varepsilon_0}_{K},\varepsilon_{K+1}+\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } M+1>K, \\ & \quad \varepsilon_{K+1}\in\{0,-\varepsilon_0\}, \\ \left(\frac{1}{2}\right)^{K+1}\alpha_{\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } M+1>K,\ \varepsilon_{K+1}=\varepsilon_0, \end{cases}$$

$$n_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} 1 & \text{if } M+1=K, \\ n_{0,\underbrace{\varepsilon_0,\dots,\varepsilon_0}_{K},\varepsilon_{K+1}+\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } M+1>K, \\ & \quad \varepsilon_{K+1}\in\{0,-\varepsilon_0\}, \\ n_{\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } M+1>K,\ \varepsilon_{K+1}=\varepsilon_0, \end{cases}$$

and fix $n \ge n_{\varepsilon_0,\dots,\varepsilon_{M+1}}$ and $x \in [Q-1, 1-Q]$.

If $M+1=K$, then by (26) and Lemma 2 we get

$$f(q^{M+1+n}x+y_0)=f\Big(q^{K+n}x+\varepsilon_0\sum_{m=1}^{K+1}q^m\Big)=\Big(\frac{1}{2}\Big)^{K+1}\Big(\frac{1}{2q}\Big)^{K+n}f(x)$$
$$=\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}}\Big(\frac{1}{2q}\Big)^{M+1+n}f(x).$$

If $M+1>K$ and $\varepsilon_{K+1}\in\{0,-\varepsilon_0\}$, then by (26) and the proof of case (i) we have

$$f(q^{M+1+n}x+y_0)=\alpha_{0,\varepsilon_0,\dots,\varepsilon_0,\varepsilon_{K+1}+\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}}\Big(\frac{1}{2q}\Big)^{M+1+n}f(x)$$
$$=\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}}\Big(\frac{1}{2q}\Big)^{M+1+n}f(x).$$

If $M+1>K$ and $\varepsilon_{K+1}=\varepsilon_0$, then (26) takes the form

$$y_0=\sum_{m=1}^{K}\varepsilon_0q^m+2\varepsilon_0q^{K+1}+\sum_{m=K+2}^{M+1}\varepsilon_mq^m, \tag{27}$$

and, since $y_0\in[-Q,Q]$ and $Q=q^{K+1}(Q+1)+\sum_{m=1}^{K}q^m$, we have

$$2\varepsilon_0+\sum_{m=1}^{M-K}\varepsilon_{K+1+m}q^m\in[-Q-1,Q+1].$$

Hence, because $|\varepsilon_0|=1$ and the remaining ε's are from $\{-1,0,1\}$, we get

$$\varepsilon_0+\sum_{m=1}^{M-K}\varepsilon_{K+1+m}q^m\in[-Q,Q].$$

Applying (27), Lemma 7 and the induction hypothesis for $x\in[Q-1,1-Q]$ we obtain

$$\begin{aligned}f(q^{M+1+n}x+y_0)&=f\Big(q^{K+1}\Big(q^{M-K+n}x+\sum_{m=1}^{M-K}\varepsilon_{K+1+m}q^m\Big)\\&\qquad+2\varepsilon_0q^{K+1}+\varepsilon_0\sum_{m=1}^{K}q^m\Big)\\&=\Big(\frac{1}{4q}\Big)^{K+1}f\Big(q^{M-K+n}x+\sum_{m=1}^{M-K}\varepsilon_{K+1+m}q^m+\varepsilon_0\Big)\\&=\Big(\frac{1}{4q}\Big)^{K+1}\alpha_{\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}}\Big(\frac{1}{2q}\Big)^{M-K+n}f(x)\\&=\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}}\Big(\frac{1}{2q}\Big)^{M+1+n}f(x).\end{aligned}\tag{28}$$

Finally assume (iii) holds. Now Lemma 6 says that $M+1 \geq K+1$, $\varepsilon_m = -\varepsilon_0$ for all $m \in \{1,\dots,K\}$ and $\varepsilon_{K+1} \in \{0, -\varepsilon_0\}$. Applying (12) again, we obtain

$$y_0 = \varepsilon_0 - \sum_{m=1}^{K} \varepsilon_0 q^m + \varepsilon_{K+1} q^{K+1} + \sum_{m=K+2}^{M+1} \varepsilon_m q^m \tag{29}$$
$$= \sum_{m=1}^{K} \varepsilon_0 q^m + (2\varepsilon_0 + \varepsilon_{K+1}) q^{K+1} + \sum_{m=K+2}^{M+1} \varepsilon_m q^m.$$

Put

$$\alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} \left(\frac{1}{2}\right)^{K+1} \alpha_{\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } \varepsilon_{K+1} = 0, \\ \alpha_{0,\underbrace{\varepsilon_0,\dots,\varepsilon_0}_{K+1},\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } \varepsilon_{K+1} = -\varepsilon_0, \end{cases}$$

$$n_{\varepsilon_0,\dots,\varepsilon_{M+1}} = \begin{cases} n_{\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } \varepsilon_{K+1} = 0, \\ n_{0,\underbrace{\varepsilon_0,\dots,\varepsilon_0}_{K+1},\varepsilon_{K+2},\dots,\varepsilon_{M+1}} & \text{if } \varepsilon_{K+1} = -\varepsilon_0, \end{cases}$$

and fix $n \geq n_{\varepsilon_0,\dots,\varepsilon_{M+1}}$ and $x \in [Q-1, 1-Q]$.

If $\varepsilon_{K+1} = 0$, then (29) implies (27) and hence also (28).

If $\varepsilon_{K+1} = -\varepsilon_0$, then using (29) and part (i) we get

$$f(q^{M+1+n} x + y_0) = \alpha_{0,\varepsilon_0,\dots,\varepsilon_0,\varepsilon_{K+2},\dots,\varepsilon_{M+1}} \left(\frac{1}{2q}\right)^{M+1+n} f(x)$$
$$= \alpha_{\varepsilon_0,\dots,\varepsilon_{M+1}} \left(\frac{1}{2q}\right)^{M+1+n} f(x).$$

THEOREM 2. *If there exist $K \in \mathbb{N}$ and $\lambda \in \{1,2\}$ satisfying* (12), *then* (11) *holds.*

Proof. Assume f is a solution of Schilling's problem which is bounded in a neighbourhood of $x_0 \in [-Q, Q]$. Since

$$\left\{ \sum_{n=0}^{N} \varepsilon_n q^n : \varepsilon_0, \dots, \varepsilon_N \in \{-1,0,1\},\ N \in \mathbb{N} \right\} \tag{30}$$

is dense in $[-Q, Q]$, applying Lemma 8 and arguing as in Theorem 1 we see that f vanishes on $[Q-1, 1-Q]$. We will show that f vanishes on $[0, q)$, which jointly with Lemma 1 will complete the proof.

From (12) we get $2q + q^2 \leq 1$, so $q^{-1}(1-Q) \geq Q$. If $x \in (1-Q, q)$, then $Q \leq q^{-1}(1-Q) < q^{-1}x$ and $Q - 1 < q^{-1}x - 1 < 0 < 1 - Q$, whence

$$f(x) = \frac{1}{4q}[f(q^{-1}x - 1) + f(q^{-1}x + 1) + f(q^{-1}x)] = 0.$$

We conclude with the following simple consequence of Lemma 8 and Remark 1.

COROLLARY 2. *If there exist* $K \in \mathbb{N}$ *and* $\lambda \in \{1, 2\}$ *satisfying* (12), *then every solution of Schilling's problem vanishes on the set* (30).

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