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Asymptotic stability of a system of randomly connected transformations on Polish spaces

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Abstract. We give sufficient conditions for the existence of a matrix of probabilities $[p_{ik}]_{i,k=1}^{N}$ such that a system of randomly chosen transformations $\Pi_k,\ k = 1,\ldots,N$, with probabilities p_{ik} is asymptotically stable.

0. Introduction. We consider a system of randomly connected transformations on a Polish space Y (see K. Horbacz [2]).

Let Y be a Polish space, i.e. a separable, complete metric space, which is the phase space of some dynamical system. In the deterministic case the dynamics can be described in terms of a function $\Pi : \mathbb{R}_+ \times Y \to Y$ so that a point starting from $x_0 \in Y$ at time t_0 is in position $\Pi(t - t_0, x_0)$ at time $t > t_0$.

In this paper we consider a random dynamics. We assume that a point can move according to one of the transformations $\Pi_k : \mathbb{R}_+ \times Y \to Y$ from a set $\{\Pi_1,\ldots,\Pi_N\}$. The choice of the initial transformations is random and changes randomly at random moments t_k. This system is called a *system of randomly connected transformations*. The probabilities determining the frequency with which the maps Π_k can be chosen are described by means of a stochastic matrix $[p_{ik}(x)]_{i,k=1}^{N}$. We give sufficient conditions for the existence of a stochastic matrix as above such that the system Π_k, $k = 1,\ldots,N$, with probabilities p_{ik} is asymptotically stable.

It should be underlined that our stability criterion is valid in a general class of metric spaces (Polish spaces) which are not necessarily locally compact. Thus these results are applicable to infinite-dimensional systems.

In the case when the transformation Π_k does not depend on the variable t and $p_{ij} = p_i$ for $j = 1,\ldots,N$, we obtain an Iterated Function System with probabilities. In [4] A. Lasota and J. Myjak gave sufficient conditions for the existence of probabilities $\{p_i : i \in I\},\ p_i : Y \to (0,1]$, such that an Iterated

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Function System on measures is asymptotically stable. Our method is based on their ideas.

The organization of the paper is as follows. Section 1 contains some notation and definitions from the theory of Markov operators. In Section 2 we specify the problem to be considered. Sufficient conditions for the existence of $[p_{ik}]_{i,k=1}^N$ such that the corresponding system of randomly connected trasformations is asymptotically stable are given in Section 3.

1. Preliminaries. Let (Y,ϱ) be a Polish space. Throughout this paper, $K(x,r)$ stands for the closed ball in Y with center at x and radius r.

We denote by $\mathcal{C}_\varepsilon(Y)$, $\varepsilon>0$, the family of all sets $C\subset Y$ for which there exists a finite set $\{x_1,\ldots,x_n\}\subset Y$ such that

$$C=\bigcup_{i=1}^n K(x_i,\varepsilon).$$

We denote by $\mathcal{B}(Y)$ the σ-algebra of all Borel subsets of Y and by $\mathcal{M}(Y)$ the family of all finite Borel measures (nonnegative, σ-additive) on Y. By $\mathcal{M}_1(Y)$ we denote the subset of $\mathcal{M}(Y)$ such that $\mu(Y)=1$ for $\mu\in\mathcal{M}_1(Y)$. The elements of $\mathcal{M}_1(Y)$ will be called *distributions.* Further

$$\mathcal{M}_{\mathrm{sig}}(Y)=\{\mu_1-\mu_2:\mu_1,\mu_2\in\mathcal{M}(Y)\}$$

is the space of all finite *signed measures.*

Let $\Theta\subset\mathcal{M}_1(Y)$. We call Θ *tight* if for every $\varepsilon>0$ there exists a compact set $K\subset Y$ such that $\mu(K)\ge 1-\varepsilon$ for all $\mu\in\Theta$.

As usual, $B(Y)$ denotes the space of all bounded Borel measurable functions $f:Y\to\mathbb{R}$, and $C(Y)$ the subspace of all bounded continuous functions. Both spaces are considered with the supremum norm $\|\cdot\|_0$.

For $f\in B(Y)$ and $\mu\in\mathcal{M}_{\mathrm{sig}}(Y)$ we write

$$\langle f,\mu\rangle=\int_Y f(x)\,\mu(dx).$$

We say that a sequence $\{\mu_n\}_{n\ge1}$, $\mu_n\in\mathcal{M}_1(Y)$, *converges weakly* to a measure $\mu\in\mathcal{M}_1(Y)$ if

$$\lim_{n\to\infty}\langle f,\mu_n\rangle=\langle f,\mu\rangle \quad \text{for } f\in C(Y).$$

In the space $\mathcal{M}_{\mathrm{sig}}(Y)$ we introduce the *Fortet–Mourier norm* (see [1, 5]) by setting

$$\|\mu\|_{\mathcal{F}}=\sup\{\langle f,\mu\rangle: f\in\mathcal{F}\},$$

where

$$\mathcal{F}=\{f\in C(Y):\|f\|_0\le 1 \text{ and } |f(x)-f(y)|\le\varrho(x,y) \text{ for } x,y\in Y\}.$$

The space $\mathcal{M}_1(Y)$ with the distance $\|\mu_1 - \mu_2\|_{\mathcal{F}}$ is a complete metric space and the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0 \quad \text{for } \mu_n, \mu \in \mathcal{M}_1(Y)$$

is equivalent to the weak convergence of $\{\mu_n\}_{n\ge 1}$ to μ .

For $A \subset Y$ we denote by $\operatorname{diam}_\varrho A$ the diameter of the set A, i.e. $\operatorname{diam}_\varrho A = \sup\{\varrho(x,y) : x, y \in A\}$.

A linear mapping $P : \mathcal{M}_{\mathrm{sig}}(Y) \to \mathcal{M}_{\mathrm{sig}}(Y)$ is called a *Markov operator* if $P(\mathcal{M}_1(Y)) \subset \mathcal{M}_1(Y)$. A measure $\mu_* \in \mathcal{M}(Y)$ is called *invariant* or *stationary* for P if $P\mu_* = \mu_*$. A stationary probability measure is called a *stationary distribution.*

We define

$$\omega(\mu) = \{\nu \in \mathcal{M}_1(Y) : \exists_{\{n_k\}_{k\ge 1}},\ n_k \to \infty,\ P^{n_k}\mu \to \nu\},$$

$$\varGamma = \bigcup_{\mu\in\mathcal{M}_1(Y)} \omega(\mu).$$

A Markov operator $P : \mathcal{M}_{\mathrm{sig}}(Y) \to \mathcal{M}_{\mathrm{sig}}(Y)$ is called a *Feller operator* if there is an operator $U : B(Y) \to B(Y)$ (*dual* to P) such that

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in B(Y),\ \mu \in \mathcal{M}_{\mathrm{sig}}(Y) \tag{1.1}$$

and

$$Uf \in C(Y) \quad \text{for } f \in C(Y). \tag{1.2}$$

Setting $\mu = \delta_x$ in (1.1) we obtain

$$Uf(x) = \langle f, P\delta_x\rangle \quad \text{for } f \in B(Y),\ x \in Y, \tag{1.3}$$

where $\delta_x \in \mathcal{M}_1(Y)$ is the point (Dirac) measure supported at x.

A Markov operator $P : \mathcal{M}_{\mathrm{sig}}(Y) \to \mathcal{M}_{\mathrm{sig}}(Y)$ is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\|_{\mathcal{F}} \le \|\mu_1 - \mu_2\|_{\mathcal{F}} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1(Y); \tag{1.4}$$

semi-concentrating if for every $\varepsilon > 0$ there exist $C \in \mathcal{C}_\varepsilon$ and $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(C) > \alpha \quad \text{for } \mu \in \mathcal{M}_1(Y); \tag{1.5}$$

and *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1(Y)$ such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{\mathcal{F}} = 0 \quad \text{for } \mu \in \mathcal{M}_1(Y). \tag{1.6}$$

2. Formulation of the problem. Let (Y, ϱ) be a Polish space. Suppose we are given a sequence of continuous transformations $\varPi_k : \mathbb{R}_+ \times Y \to Y$, $k = 1, \ldots, N$, and a sequence of random variables $\{t_n\}_{n\ge 1}$ such that the increments

$$\varDelta t_1 = t_1 - t_0,\ \ldots,\ \varDelta t_n = t_n - t_{n-1},\ \ldots \quad (t_0 = 0) \tag{2.1}$$

are independent and have the same density distribution function $g(t) = ae^{-at}$.

Assume moreover that Borel measurable functions $p_i, p_{ik} : Y \to \mathbb{R}_+$ satisfy

$$p_i(x) \geq 0, \quad \sum_{i=1}^{N} p_i(x) = 1 \quad \text{for } x \in Y$$

and

$$p_{ij}(x) \geq 0, \quad \sum_{j=1}^{N} p_{ij}(x) = 1 \quad \text{for } x \in Y \text{ and } i, j = 1, \ldots, N.$$

The action of randomly chosen transformations can be roughly described as follows. We choose an initial point $x_0 \in Y$. Next we randomly select an integer k_1 from $\{1, \ldots, N\}$ with some probability $p_{k_1}(x_0)$. We define

$$x_1 = \Pi_{k_1}(t_1, x_0).$$

Next, we select k_2 with probability $p_{k_1 k_2}(x_1)$ and define

$$x_2 = \Pi_{k_2}(t_2 - t_1, x_1)$$

and so on. Thus

$$x_n = \Pi_s(t_n - t_{n-1}, x_{n-1})$$

with probability $p_{ks}(x_{n-1})$ if $x_{n-1} = \Pi_k(t_{n-1} - t_{n-2}, x_{n-2})$.

The system of randomly chosen Π_k with probabilities p_{ik} is denoted by $[\Pi, p]$.

Denote by $\mu_n,\ n = 0, 1, \ldots,$ the distribution of x_n, i.e.

$$\mu_n(A) = \operatorname{prob}(x_n \in A) \quad \text{for } A \in \mathcal{B}(Y),\ n = 0, 1, \ldots \tag{2.2}$$

We will give sufficient conditions for the existence of a matrix of probabilities $[p_{ik}]_{i,k=1}^{N}$, $p_{ik} : Y \to (0, 1]$, such that the sequence $\{\mu_n\}_{n\geq 1}$ is weakly convergent to a unique measure μ_*.

We change the space Y in order to be able to describe the evolution of measures under some Markov operator.

Let $\overline{Y} = Y \times \{1, \ldots, N\}$ with the metric $\overline{\varrho}$ given by

$$\overline{\varrho}((x, i), (y, j)) = \varrho(x, y) + \varrho_0(i, j) \quad \text{for } x, y \in Y,\ i, j \in \{1, \ldots, N\},$$

where ϱ_0 is some metric in $\{1, \ldots, N\}$.

We define a new sequence of transformations

$$\overline{\Pi}_k : \mathbb{R}_+ \times \overline{Y} \to \overline{Y} \quad \text{for } k = 1, \ldots, N$$

by

$$\overline{\Pi}_k(t, (x, s)) = (\Pi_k(t, x), k).$$

Now, for an initial point x_0 we randomly select an integer k with probability $p_k(x_0)$ and we define $x_1 = \Pi_k(t_1, x_0)$. Next we randomly select $s \in \{1, \ldots, N\}$ with probability $p_{ks}(x_1)$, and we define

$$(x_2, s) = \overline{\Pi}_s(t_2 - t_1, (x_1, k))$$

and so on. Hence

$$(x_n, s) = \overline{\Pi}_s(\Delta t_n, (x_{n-1}, k)), \quad n = 2, 3, \ldots,$$

with probability $p_{ks}(x_{n-1})$.

The evolution of the distributions $\overline{\mu}_n$ on the space $\overline{Y}$, where

$$\overline{\mu}_n(A \times \{s\}) = \mathrm{prob}(x_n \in A \text{ and } x_n = \Pi_s(\Delta t_n, x_{n-1})), \quad n = 1, 2, \ldots,$$

can be described by a Feller operator P, i.e. $\overline{\mu}_{n+1} = P\overline{\mu}_n$. It is called the *transition operator* for this system. To find the explicit form of P, we look for the dual operator U. A straightforward calculation shows that

$$\begin{aligned} (2.3) \quad Uf(x,k) &= \sum_{s=1}^{N} \int_0^\infty f(\overline{\Pi}_s(t, (x,k)))ae^{-at}p_{ks}(x)\,dt \\ &= \sum_{s=1}^{N} \int_0^\infty f(\Pi_s(t,x), s)ae^{-at}p_{ks}(x)\,dt \quad \text{for } f \in B(\overline{Y}). \end{aligned}$$

Thus (see [3]), we may find P by the formula

$$P\mu(A) = \langle \mathbf{1}_A, P\mu \rangle = \langle U\mathbf{1}_A, \mu \rangle.$$

This gives

$$(2.4) \qquad P\mu(A) = \sum_{s=1}^{N} \int_{\overline{Y}} \int_0^\infty \mathbf{1}_A(\overline{\Pi}_s(t, (x,k)))ae^{-at}\,dt\,p_{ks}(x)\,d\mu(x,k)$$

for $\mu \in \mathcal{M}(\overline{Y})$ and $A \in \mathcal{B}(\overline{Y})$.

The weak convergence of the sequence $\{\mu_n\}_{n\ge1}$ will follow from the asymptotic stability of the operator P.

To prove the latter we need the following three lemmas. The first was proved by T. Szarek [6].

LEMMA 2.1. *Let $P : \mathcal{M}_{\mathrm{sig}}(\overline{Y}) \to \mathcal{M}_{\mathrm{sig}}(\overline{Y})$ be a nonexpansive semi-concentrating Markov operator. Then*

(i) *P has an invariant distribution*,
(ii) *$\omega(\mu) \neq \emptyset$ for every $\mu \in \mathcal{M}_1(\overline{Y})$*,
(iii) *$\Gamma = \bigcup_{\mu \in \mathcal{M}_1(Y)} \omega(\mu)$ is tight.*

In [5] A. Lasota and J. A. Yorke proved the following result.

LEMMA 2.2. *Let P be a nonexpansive Markov operator. Assume that for every $\varepsilon > 0$ there is a $\Delta > 0$ with the following property*: *for every*

$\mu_1, \mu_2 \in \mathcal{M}_1(\overline{Y})$ there exists a Borel measurable set A with $\mathrm{diam}_{\overline{\varrho}}(A) \le \varepsilon$ *and an integer n_0 such that*

$$P^{n_0}\mu_i(A) \ge \Delta \quad \textit{for } i = 1, 2. \tag{2.5}$$

Then P satisfies

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_{\mathcal{F}} = 0 \quad \textit{for } \mu_1, \mu_2 \in \mathcal{M}_1(\overline{Y}). \tag{2.6}$$

We also need the following elementary lemma whose proof is left to the reader.

LEMMA 2.3. *Let $q : \mathbb{R}_+ \to (\delta, \infty)$, $\delta \ge 0$, be a nonincreasing function and let $\varepsilon > 0$. Then there exists a nonincreasing function $\overline{q} : \mathbb{R}_+ \to (\delta, \delta+\varepsilon)$ such that*

$$\delta < \overline{q}(t) < q(t) \quad \textit{and} \quad |\overline{q}(t) - \overline{q}(s)| \le \varepsilon|t-s| \quad \textit{for } t, s \ge 0.$$

3. Main result.

We now formulate the main result of this paper.

THEOREM 3.1. *Assume that the sequence of transformations $\Pi_k : \mathbb{R}_+ \times Y \to Y$ satisfies*

$$\varrho(\Pi_k(t,x), \Pi_k(t,y)) \le L_k e^{-\lambda t}\varrho(x,y) \tag{3.1}$$

for $x, y \in A$, $t \ge 0$ and $k = 1, \ldots, N$, on every bounded set $A \subset Y$. Assume moreover that the positive constants a, λ and L_1 satisfy

$$L_1 - \lambda/a < 1. \tag{3.2}$$

If in addition there is a point $x_* \in Y$ such that

$$\sup\{\varrho(\Pi_k(t,x_*), x_*) : t \ge 0\} < \infty \quad \textit{for } k = 1, \ldots, N, \tag{3.3}$$

then there exists a matrix of probabilities $p_{ik} : Y \to (0,1]$ satisfying

$$\sum_{k=1}^{N} p_{ik}(x) = 1$$

for $x \in Y$ and $i = 1, \ldots, N$ such that the sequence $\{\mu_n\}_{n\ge1}$ defined by (2.2) *is weakly convergent to a distribution μ_*.*

Proof. For $x \in Y$ set

$$|x|_* = \varrho(x, x_*).$$

Let $0 < \delta < (1 + \lambda/a - L_1)/N$. For $i \in \{2, \ldots, N\}$ and $r \ge 0$ define

$$\sigma_i(r) = \sup\{\varrho(\Pi_i(t,x), x_*) : |x|_* \le r,\ t \ge 0\},$$

$$r_i(r) = \sup\left\{\frac{\varrho(\Pi_i(t,x), \Pi_i(t,y))}{e^{-\lambda t}\varrho(x,y)} : |x|_*, |y|_* \le r,\ x \ne y \text{ and } t \ge 0\right\},$$

$$q_i(r) = \min\left\{\frac{\delta}{1+r_i(r)}, \frac{\delta}{1+\sigma_i(r)}\right\}, \qquad \sup\emptyset = 0.$$

Fix $\varepsilon > 0$ such that $\delta + \varepsilon < 1$. Using Lemma 2.3 for ε, choose a sequence of functions $\overline{q}_i$, $i = 2, \ldots, N$. Define

$$p_{i1}(x) = 1 - \sum_{s=2}^{N} \overline{q}_s(|x|_*) \quad \text{for } i = 1, \ldots, N,$$
$$p_{ik}(x) = \overline{q}_k(|x|_*) \quad \text{for } i = 1, \ldots, N \text{ and } k = 2, \ldots, N.$$

Consider now the resulting system $[\Pi, p]$ and let P and U be given by (2.4) and (2.3), respectively.

Claim I. *There exists a metric ϱ_K on $\overline{Y}$ such that P is nonexpansive with respect to ϱ_K.*

Set

$$K = \frac{2(N-1)N\varepsilon}{1 - L_1 a/(a+\lambda)}.$$

Define

$$\varrho_K((x,i),(y,j)) = K(\varrho(x,y) + \varrho_0(i,j))$$

for $x, y \in Y$ and $i, j \in \{1, \ldots, N\}$, where

$$\varrho_0(i,j) = \begin{cases} c & \text{for } i \neq j, \\ 0 & \text{for } i = j, \end{cases}$$

for c such that $cK \geq 2$. Denote by $\|\cdot\|_K$ the Fortet–Mourier norm in $\mathcal{M}_1(\overline{Y})$ given by $\|\mu_1 - \mu_2\|_K = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}_K\}$, where $\mathcal{F}_K$ is the set of functions f such that $\|f\|_0 \leq 1$ and

$$|f(x,i) - f(y,j)| \leq \varrho_K((x,i),(y,j))$$

for $x, y \in Y$, $i, j \in \{1, \ldots, N\}$. To prove the nonexpansiveness it is sufficient to show that $U(\mathcal{F}_K) \subset \mathcal{F}_K$. Fix an $f \in \mathcal{F}_K$. Evidently $\|Uf\|_0 \leq 1$, so we have to prove that

$$|Uf(x,i) - Uf(y,j)| \leq K\overline{\varrho}((x,i),(y,j)) \tag{3.4}$$

for $x, y \in Y$ and $i, j \in \{1, \ldots, N\}$. Since $\varrho_0(i,j) = c$ for $i \neq j$ and $Kc \geq 2$, the condition (3.4) is satisfied for $i \neq j$. For $i = j$ we have

$$|Uf(x,i) - Uf(y,i)| \leq \sum_{k=1}^{N} \int_0^\infty |f(\Pi_k(t,x),k)| a e^{-at} |p_{ik}(x) - p_{ik}(y)|\, dt$$
$$+ \sum_{k=1}^{N} \int_0^\infty |f(\Pi_k(t,x),k) - f(\Pi_k(t,y),k)| p_{ik}(y) a e^{-at}\, dt.$$

Since $f \in \mathcal{F}_K$, we obtain

$$(3.5)\quad |Uf(x,i) - Uf(y,i)| \le \sum_{k=1}^{N} |p_{ik}(x) - p_{ik}(y)| + \sum_{k=1}^{N} \int_0^{\infty} K\varrho(\varPi_k(t,x), \varPi_k(t,y)) p_{ik}(y) a e^{-at}\, dt.$$

Without any loss of generality we may assume that $|x|_* \le |y|_*$. For $k \ge 2$, we have

$$\varrho(\varPi_k(t,x), \varPi_k(t,y)) \le r_k(|y|_*) e^{-\lambda t} \varrho(x,y)$$

and

$$p_{ik}(y) = \bar{q}_k(|y|_*) \le q_k(|y|_*) \le \frac{\delta}{1 + r_k(|y|_*)}.$$

Thus

$$\begin{aligned}\sum_{k=2}^{N} p_{ik}(y) \varrho(\varPi_k(t,x), \varPi_k(t,y)) &\le \sum_{k=2}^{N} \frac{\delta}{1 + r_k(|y|_*)} r_k(|y|_*) e^{-\lambda t} \varrho(x,y) \\ &\le \sum_{k=2}^{N} \delta e^{-\lambda t} \varrho(x,y) = (N-1)\delta e^{-\lambda t} \varrho(x,y).\end{aligned}$$

Moreover

$$p_{i1}(y) \varrho(\varPi_1(t,x), \varPi_1(t,y)) \le p_{i1}(y) L_1 e^{-\lambda t} \varrho(x,y),$$

thus

$$\begin{aligned}\sum_{k=1}^{N} p_{ik}(y) \varrho(\varPi_k(t,x), \varPi_k(t,y)) &\le (p_{i1}(y) L_1 + (N-1)\delta) e^{-\lambda t} \varrho(x,y) \\ &\le (L_1 + (N-1)\delta) e^{-\lambda t} \varrho(x,y) \\ &\le L e^{-\lambda t} \varrho(x,y)\end{aligned}$$

where $L = 1 + \lambda/a - (1 + \lambda/a - L_1)/N$.

From (3.5) it now follows that

$$\begin{aligned}(3.6)\quad |Uf(x,i) - Uf(y,i)| &\le \sum_{k=1}^{N} |p_{ik}(x) - p_{ik}(y)| + K \int_0^{\infty} L a e^{-(a+\lambda)t} \varrho(x,y)\, dt \\ &\le \sum_{k=1}^{N} |p_{ik}(x) - p_{ik}(y)| + K r \varrho(x,y)\end{aligned}$$

where $r = aL/(a+\lambda)$. Moreover from Lemma 2.3 we obtain

$$|p_{ik}(x) - p_{ik}(y)| = |\bar{q}_k(|x|_*) - \bar{q}_k(|y|_*)| \le \varepsilon \varrho(x,y) \quad \text{for } k = 2, \ldots, N.$$

Thus

$$\sum_{k=1}^{N}|p_{ik}(x)-p_{ik}(y)| \le |p_{i1}(x)-p_{i1}(y)| + \sum_{k=2}^{N}|p_{ik}(x)-p_{ik}(y)|$$
$$\le 2\sum_{k=2}^{N}|\bar{q}_k(|x|_*)-\bar{q}_k(|y|_*)| \le 2(N-1)\varepsilon\varrho(x,y).$$

From (3.6) we finally obtain

$$|Uf(x,i)-Uf(y,i)| \le 2(N-1)\varepsilon\varrho(x,y)+Kr\varrho(x,y),$$

which reduces to

$$|Uf(x,i)-Uf(y,i)| \le K\varrho(x,y)$$

by the definition of K, and completes the proof of the nonexpansiveness.

Claim II. *The operator P is semi-concentrating.*

Fix $\gamma>0$. Consider the function

$$V(x,k)=\varrho(x,x_*) \quad \text{for } x\in Y \text{ and } k=1,\ldots,N.$$

By (2.3) and the definition of V, p_{ik} and σ_i we have

$$\begin{aligned}
UV(x,k) &= \sum_{s=1}^{N}\int_0^\infty \varrho(\Pi_s(t,x),x_*)ae^{-at}p_{ks}(x)\,dt\\
&\le \int_0^\infty \varrho(\Pi_1(t,x),x_*)ae^{-at}p_{k1}(x)\,dt+\sum_{s=2}^{N}\sigma_s(|x|_*)\frac{\delta}{1+\sigma_s(|x|_*)}\\
&\le \int_0^\infty \varrho(\Pi_1(t,x),\Pi_1(t,x_*))ae^{-at}p_{k1}(x)\,dt\\
&\quad+\int_0^\infty \varrho(\Pi_1(t,x_*),x_*)ae^{-at}p_{k1}(x)\,dt+(N-1)\delta\\
&\le L_1\frac{a}{\lambda+a}\varrho(x,x_*)+M+(N-1)\delta,
\end{aligned}$$

where

$$M=\max_{1\le k\le N}\sup_{t\ge 0}\varrho(\Pi_k(t,x_*),x_*).$$

Setting $b=M+(N-1)\delta$ and $\beta=L_1a/(\lambda+a)$, we have

$$UV(x,k)\le \beta V(x,k)+b. \tag{3.7}$$

Now define

$$m_n = \langle V, \overline{\mu}_n \rangle, \qquad n = 0, 1, \ldots$$

Consider first the case $m_0 < \infty$. Using the recurrence formula $\overline{\mu}_{n+1} = P\overline{\mu}_n$ and (3.7) we have

$$m_{n+1} = \langle V, P\overline{\mu}_n \rangle = \langle UV, \overline{\mu}_n \rangle \le \langle \beta V + b, \overline{\mu}_n \rangle = \beta m_n + b.$$

By an induction argument this gives

$$m_{n+1} \le \beta^n m_0 + \frac{b}{1-\beta}. \tag{3.8}$$

Define

$$R = \frac{2b}{\gamma(1-\beta)}.$$

Using the Chebyshev inequality we get

$$P^n \overline{\mu}_0(B) = \overline{\mu}_n(B) \ge 1 - \gamma \quad \text{for } n \ge n_0 \text{ and } \overline{\mu}_0 \in \mathcal{M}_1(\overline{Y}) \tag{3.9}$$

where $B = K(x_*, R) \times \{1, \ldots, N\}$. The general case $m_0 \le \infty$ can be reduced to the previous one as follows. For given $\delta > 0$ we choose a bounded Borel set $A \subset \overline{Y}$ such that $\overline{\mu}_0(A) \ge 1 - \delta$. Setting

$$\nu_0(D) = \frac{\overline{\mu}_0(A \cap D)}{\overline{\mu}_0(A)}$$

we define a probability measure ν_0 supported on A for which the initial moment $\overline{m}_0 = \langle V, \nu_0 \rangle$ is finite. Thus

$$P^n \nu_0(B) \ge 1 - \gamma \quad \text{for } n \ge n_0.$$

Since $\overline{\mu}_0(D) \ge \overline{\mu}_0(D \cap A)$, we have

$$P^n \overline{\mu}_0(B) \ge \overline{\mu}_0(A) P^n \nu_0(B) \ge (1-\delta)(1-\gamma).$$

Choosing δ sufficiently small we obtain

$$P^n \overline{\mu}_0(B) \ge 1 - \overline{\gamma} \quad \text{for } n \ge n_0.$$

Now we define the families of functions $\Pi^{t_n \ldots t_1}_{k_n \ldots k_1} : Y \to Y$ and $\overline{\Pi}^{t_n \ldots t_1}_{k_n \ldots k_1} : \overline{Y} \to \overline{Y}$ ($t_i \in \mathbb{R}_+$, $k_i \in \{1, \ldots, N\}$ for $i = 1, \ldots, n$) by the recurrence relations

$$\Pi^{t_1}_{k_1}(y) = \Pi_{k_1}(t_1, y),$$
$$\Pi^{t_n \ldots t_1}_{k_n \ldots k_1}(y) = \Pi_{k_n}(t_n, \Pi^{t_{n-1} \ldots t_1}_{k_{n-1} \ldots k_1}(y)) \quad \text{for } y \in Y$$

and

$$\overline{\Pi}^{t_1}_{k_1}(y, s) = (\Pi^{t_1}_{k_1}(y), k_1),$$
$$\overline{\Pi}^{t_n \ldots t_1}_{k_n \ldots k_1}(y, s) = (\Pi^{t_n \ldots t_1}_{k_n \ldots k_1}(y), k_n) \quad \text{for } (y, s) \in \overline{Y}.$$

Using equation (2.3) n times, we obtain

$$(3.10)\quad U^n f(y,i) = \sum_{k_1,\ldots,k_n} \underbrace{\int_{\mathbb{R}_+} \ldots \int_{\mathbb{R}_+}}_{n} p_{ik_1}(y) p_{k_1k_2}(\Pi_{k_1}^{t_1}(y)) \ldots p_{k_{n-1}k_n}(\Pi_{k_{n-1}\ldots k_1}^{t_{n-1}\ldots t_1}(y)) \times f(\overline{\Pi}_{k_n\ldots k_1}^{t_n\ldots t_1}(y,i)) a^n e^{-a(t_1+\ldots+t_n)}\, dt_1 \ldots dt_n.$$

By (3.9) there exists a bounded Borel set $F \subset \overline{Y}$ such that for every $\mu \in \mathcal{M}_1(\overline{Y})$ there exists an integer $n_1 = n_1(\mu)$ for which

$$(3.11)\qquad P^n\mu(F) \ge 1/2 \quad \text{for } n \ge n_1.$$

Let $\bar{t} > 0$ be such that

$$(3.12)\qquad r_0 = \sup_{1\le k\le N} L_k e^{-\lambda \bar{t}} < 1.$$

Fix $\alpha > 0$. We can find an integer m such that

$$(3.13)\qquad r_0^m \operatorname{diam}_{\overline{\varrho}} F \le \alpha/2.$$

Fix $(y,s) \in F$ and set

$$C = \bigcup_{j_1,\ldots,j_m=1}^{N} K(\overline{\Pi}_{j_m\ldots j_1}^{\bar{t}\ldots\bar{t}}(y,s),\alpha).$$

Obviously $C \in \mathcal{C}_\alpha$. By continuity there exists a constant τ, $0 < \tau < \bar{t}$, such that

$$(3.14)\qquad \overline{\varrho}(\overline{\Pi}_{j_m\ldots j_1}^{\bar{t}\ldots\bar{t}}(y,s), \overline{\Pi}_{j_m\ldots j_1}^{t_m\ldots t_1}(y,s)) < \alpha/2$$

for all sequences $(j_1,\ldots,j_m) \in \{1,\ldots,N\}^m$ and $t_1,\ldots,t_m \in [\bar{t},\bar{t}+\tau]$. Set

$$\sigma = \inf\{p_{ij}(x) : x \in Y,\ i,j \in \{1,\ldots,N\}\}.$$

From Lemma 2.3 it follows that $\sigma > 0$.

We now prove that

$$(3.15)\qquad \liminf_{n\to\infty} P^n\mu(C) \ge \frac{1}{2}\left(\frac{\sigma}{N}\tau a e^{-a(\bar{t}+\tau)}\right)^m$$

for all $\mu \in \mathcal{M}_1(\overline{Y})$. To do this fix $\mu \in \mathcal{M}_1(\overline{Y})$. There exists an integer $n_1 = n_1(\mu)$ for which (3.11) holds. Let $n = \overline{n} + m$ for some $\overline{n} \ge n_1$. Using (3.10) we get

$$
\begin{aligned}
(3.16)\quad & P^n\mu(C) \\
&= \int_{\overline{Y}} \sum_{k_1,\dots,k_m} \underbrace{\int_{\mathbb{R}_+} \dots \int_{\mathbb{R}_+}}_{m} p_{sk_1}(w) p_{k_1k_2}(\Pi^{t_1}_{k_1}(w)) \dots p_{k_{m-1}k_m}(\Pi^{t_{m-1}\dots t_1}_{k_{m-1}\dots k_1}(w)) \\
&\quad \times \mathbf{1}_C(\overline{\Pi}^{t_m\dots t_1}_{k_m\dots k_1}(w,r)) a^m e^{-a(t_1+\dots+t_m)}\, dt_1 \dots dt_m\, dP^{\overline{n}}\mu(w,r).
\end{aligned}
$$

Consider the space

$$Z = F \times \underbrace{[\,\overline{t}, \overline{t}+\tau] \times \dots \times [\,\overline{t}, \overline{t}+\tau]}_{m \text{ times}}$$

with the product measure $P^{\overline{n}}\mu \otimes \underbrace{l_1 \otimes \dots \otimes l_1}_{m \text{ times}}$, where l_1 denotes the Lebesgue measure. Define

$$
\begin{aligned}
Z_{j_1\dots j_m} = \{((w,r), t_1, \dots, t_m) \in Z :& \\
&\overline{\varrho}(\overline{\Pi}^{t_m\dots t_1}_{j_m\dots j_1}(y,s), \overline{\Pi}^{t_m\dots t_1}_{j_m\dots j_1}(w,r)) \le r_0^m \overline{\varrho}((w,r),(y,s))\}
\end{aligned}
$$

for $j_1, \dots, j_m \in \{1, \dots, N\}$. Applying (3.1) ($m$ times) we see that for every $(w,r) \in F$ and $(t_1, \dots, t_m) \in [\,\overline{t}, \overline{t}+\tau]$ there exists a sequence $(i_1, \dots, i_m) \in \{1, \dots, N\}^m$ such that $((w,r), t_1, \dots, t_m) \in Z_{i_1\dots i_m}$.

Hence we deduce that

$$Z = \bigcup_{j_1,\dots,j_m=1}^{N} Z_{j_1\dots j_m}. \tag{3.17}$$

Since

$$(P^{\overline{n}}\mu \otimes l_1 \otimes \dots \otimes l_1)(Z) \ge \tfrac{1}{2}\tau^m,$$

there exists a sequence $(k_1, \dots, k_m) \in \{1, \dots, N\}^m$ such that

$$(P^{\overline{n}}\mu \otimes l_1 \otimes \dots \otimes l_1)(Z_{k_1\dots k_m}) \ge \frac{\tau^m}{2N^m}. \tag{3.18}$$

Combining (3.13) and (3.14) with (3.16) and (3.18) we obtain

$$
\begin{aligned}
(3.19)\qquad P^n\mu(C) &\ge \sigma^m \int \dots \int_{Z_{k_1\dots k_m}} \mathbf{1}_C(\overline{\Pi}^{t_m\dots t_1}_{k_m\dots k_1}(w,r)) \\
&\qquad \times a^m e^{-a(t_1+\dots+t_m)} dt_1 \dots dt_m\, dP^{\overline{n}}(w,r) \\
&\ge \frac{1}{2}\left[\frac{\sigma}{N}\tau a e^{-a(\overline{t}+\tau)}\right]^m,
\end{aligned}
$$

which finishes the proof of Claim II.

From Lemma 2.1(i) it follows that the operator P given by (2.4) has an invariant measure.

Claim III. *The operator P satisfies a lower bound condition, namely, for every $\beta > 0$ there is $\Delta > 0$ such that for any $\mu_1, \mu_2 \in \mathcal{M}_1(\overline{Y})$ there exists a*

Borel measurable set A *with* $\operatorname{diam}_{\bar\varrho} A \le \beta$ *and an integer* n_0 *for which*

$$P^{n_0}\mu_k(A) \ge \Delta \quad \textit{for } k = 1, 2. \tag{3.20}$$

Fix $\beta > 0$. Since P is semi-concentrating, Lemma 2.1 shows that there exists a compact set $H \subset \overline{Y}$ such that

$$\mu(H) \ge 4/5 \quad \text{for } \mu \in \Gamma. \tag{3.21}$$

Now let $p \in \mathbb{N}$ be such that

$$r_0^p \operatorname{diam}_{\bar\varrho} H \le \beta/3, \tag{3.22}$$

where r_0 is defined by (3.12). For $(y, s) \in H$ and $(k_1, \ldots, k_p) \in \{1, \ldots, N\}^p$ we define the open sets

$$O_{(y,s)} = \bigcap_{k_1,\ldots,k_p=1}^{N} \{(w, r) \in \overline{Y} : \overline{\Pi}^{\bar t\ldots\bar t}_{k_p\ldots k_1}((w, r)) \in K(\overline{\Pi}^{\bar t\ldots\bar t}_{k_p\ldots k_1}(y, s), \beta/3)\}. \tag{3.23}$$

Take $(y_1, s_1), \ldots, (y_h, s_h) \in H$ such that

$$H \subset \bigcup_{i=1}^{h} O_{(y_i,s_i)} \tag{3.24}$$

and define

$$G = \bigcup_{i=1}^{h} O_{(y_i,s_i)}. \tag{3.25}$$

By continuity there exists $\overline{\gamma} > 0$ such that for all $(y, s) \in H$ and all sequences $(k_1, \ldots, k_p) \in \{1, \ldots, N\}^p$ we have

$$\bar\varrho(\overline{\Pi}^{t_p\ldots t_1}_{k_p\ldots k_1}(y, s), \overline{\Pi}^{\bar t\ldots\bar t}_{k_p\ldots k_1}(y, s)) < \beta/6 \tag{3.26}$$

for $t_i \in [\bar t, \bar t + \overline{\gamma}]$, $i = 1, \ldots, p$.

We show that (3.20) holds with

$$\Delta = \frac{1}{2h}[\sigma\overline{\gamma}ae^{-a(\bar t+\overline{\gamma})}]^p.$$

Fix $\mu_1, \mu_2 \in \mathcal{M}_1(\overline{Y})$. Set $\overline{\mu} = (\mu_1 + \mu_2)/2$. Lemma 2.1 gives $\omega(\overline{\mu}) \ne \emptyset$. Let $\{n_j\}_{j\ge1}$ be a sequence of integers such that $\{P^{n_j}\overline{\mu}\}_{j\ge1}$ converges to some measure $\mu_* \in \mathcal{M}_1(\overline{Y})$. By the Alexandrov theorem we have

$$\liminf_{j\to\infty} P^{n_j}\overline{\mu}(G) \ge \mu_*(G) \ge \mu_*(H) \ge 4/5.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$P^{n_0}\overline{\mu}(G) \ge 3/4 \tag{3.27}$$

and consequently

$$P^{n_0}\mu_i(G) \geq 1/2 \quad \text{for } i = 1,2.$$

Thus we get

$$P^{n_0}\mu_1(O_{(y_l,s_l)}) \geq \frac{1}{2h} \quad \text{and} \quad P^{n_0}\mu_2(O_{(y_k,s_k)}) \geq \frac{1}{2h}$$

for some $l, k \in \{1, \ldots, h\}$. Write for simplicity $O_1 = O_{(y_l,s_l)}$, $O_2 = O_{(y_k,s_k)}$. Now, (3.1) and (3.22) imply

$$\overline{\varrho}(\overline{\Pi}^{\overline{t}\ldots\overline{t}}_{i_p\ldots i_1}(y_l,s_l), \overline{\Pi}^{\overline{t}\ldots\overline{t}}_{i_p\ldots i_1}(y_k,s_k)) \leq r_0^p \operatorname{diam}_{\overline{\varrho}} H \leq \beta/3 \tag{3.28}$$

for some $(i_p,\ldots,i_1) \in \{1,\ldots,N\}^p$.

Define $A = A_1 \cup A_2$, where

$$A_i = \operatorname{cl}\{\overline{\Pi}^{t_p\ldots t_1}_{i_p\ldots i_1}(y,s) : (y,s) \in O_i,\ t_j \in [\overline{t}, \overline{t} + \overline{\gamma}] \text{ for } j \in \{1,\ldots,p\}\}$$

for $i = 1,2$ (here cl denotes closure in the space $\overline{Y}$). Using (3.23) and (3.26) we check at once that $\operatorname{diam}_{\overline{\varrho}} A \leq \varepsilon$. Proceeding analogously to the proof of the estimates (3.16) and (3.19) we get

$$P^{n_0+p}\mu_i(A) \geq \frac{1}{2h}[\sigma\overline{\gamma}ae^{-a(\overline{t}+\overline{\gamma})}]^p \quad \text{for } i = 1,2,$$

which finishes the proof of Claim III.

Claim IV. *The sequence* $\{\mu_n\}_{n\geq 1}$ *defined by* (2.2) *is weakly convergent to a distribution* μ_*.

Since by Claim II the operator P given by (2.4) has an invariant measure, Lemma 2.2 yields

$$\lim_{n\to\infty} \|P^n\mu_1 - P^n\mu_2\|_{\mathcal{F}} = 0 \quad \text{for } \mu_1,\mu_2 \in \mathcal{M}_1(\overline{Y}).$$

Thus the operator P is asymptotically stable. Hence there exists an invariant measure $\overline{\mu}_*$ such that

$$\lim_{n\to\infty} \langle \overline{f}, \overline{\mu}_n\rangle = \langle \overline{f}, \overline{\mu}_*\rangle \quad \text{for } \overline{f} \in C(\overline{Y}),$$

where $\overline{\mu}_{n+1} = P\overline{\mu}_n$. Hence

$$\lim_{n\to\infty} \int_{\overline{Y}} \overline{f}(x,i)\, d\overline{\mu}_n(x,i) = \int_{\overline{Y}} \overline{f}(x,i)\, d\overline{\mu}_*(x,i) \quad \text{for } \overline{f} \in C(\overline{Y}). \tag{3.29}$$

Further, for every $f \in C(Y)$ we define the sequence of functions $\overline{f}_j : \overline{Y} \to \overline{Y}$, $j = 1,\ldots,N$, by the formula

$$\overline{f}_j(x,i) = \begin{cases} f(x) & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

It is evident that each $\bar{f}_j$ belongs to $C(\overline{Y})$. From (3.29) it follows that

$$\lim_{n\to\infty}\sum_{j=1}^{N}\int_{\overline{Y}}\bar{f}_j(x,i)\,d\bar{\mu}_n(x,i)=\sum_{j=1}^{N}\int_{\overline{Y}}\bar{f}_j(x,i)\,d\bar{\mu}_*(x,i),$$

and consequently

$$\lim_{n\to\infty}\sum_{j=1}^{N}\int_{Y}f(x)\,\bar{\mu}_n(dx\times\{j\})=\sum_{j=1}^{N}\int_{Y}f(x)\,\bar{\mu}_*(dx\times\{j\}).$$

Setting

$$\mu_*(A)=\sum_{j=1}^{N}\bar{\mu}_*(A\times\{j\})\quad\text{ for } A\in\mathcal{B}(Y)$$

and using the definitions of μ_n and $\bar{\mu}_n$ we finally obtain

$$\lim_{n\to\infty}\int_{Y}f(x)\,\mu_n(dx)=\int_{Y}f(x)\,\mu_*(dx)\quad\text{ for } f\in C(Y).\ \blacksquare$$

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