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The complex Monge–Ampère equation for complex homogeneous functions in $\mathbb{C}^n$

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Abstract. We prove some existence results for the complex Monge–Ampère equation $(dd^c u)^n = g d\lambda$ in $\mathbb{C}^n$ in a certain class of homogeneous functions in $\mathbb{C}^n$, i.e. we show that for some nonnegative complex homogeneous functions g there exists a plurisubharmonic complex homogeneous solution u of the complex Monge–Ampère equation.

0. Introduction. In this paper we consider the following problem: for which nonnegative complex homogeneous functions g in $\mathbb{C}^n$ does there exist a complex homogeneous plurisubharmonic function u in $\mathbb{C}^n$ solving the *complex Monge–Ampère equation*

$$(dd^c u)^n = g d\lambda, \tag{0.1}$$

where $d\lambda$ denotes the Lebesgue measure in $\mathbb{C}^n$?

The problem of the existence of global solutions of the complex Monge–Ampère equations in $\mathbb{C}^n$ has been treated only in a few cases. In [K1] Kołodziej showed some sufficient conditions which guarantee that a finite measure μ in $\mathbb{C}^n$ admits a solution of the equation $(dd^c u)^n = d\mu$ in the class $\mathcal{L}_+$ (for definition of $\mathcal{L}_+$ see Section 1). Uniqueness, up to an additive constant, in this case was proved by Bedford and Taylor in [BT2]. In [J] Jeune proved that a perturbation of the Lebesgue measure in $\mathbb{C}^n$ by a smooth function which, together with all its derivatives, tends to 0 fast enough at infinity, admits a smooth solution of the complex Monge–Ampère equation. Monn [M] proved the existence of a solution of the complex Monge–Ampère equation in the class of radial functions in $\mathbb{C}^n$, i.e. for every nonnegative radial function g in $\mathbb{C}^n$ there exists a radial, entire plurisubharmonic function satisfying (0.1). Kołodziej [K3] showed that for given two entire locally bounded plurisubharmonic functions v and w satisfying $w \le v$, $(dd^c v)^n \le (dd^c w)^n$

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and $\lim_{|z|\to\infty}(v(z)-w(z))=0$, one can solve the Monge–Ampère equation for any measure μ such that

$$(dd^c v)^n \le d\mu \le (dd^c w)^n.$$

Furthermore, the solution u is unique among functions satisfying $w \le u \le v$.

In this paper we prove the existence of a solution of the complex Monge–Ampère equation for a certain class of homogeneous functions in $\mathbb{C}^n$. In the complex plane every complex homogeneous function is of the form $c|z|^\alpha$ and a simple computation shows that the function $u(z)=\frac{2}{(\alpha+2)^2}|z|^{\alpha+2}$ is a solution of $dd^c u = |z|^\alpha d\lambda$, where $\alpha > 0$. For this reason in this paper we always assume that $n \ge 2$.

In the first section we prove that for any nonnegative, smooth (outside the origin), complex homogeneous function g of order of homogeneity $n(t-2)$, where $0 < t < 1/(n-1)$, there exists a smooth (outside the origin) solution u of the equation (0.1). We also establish a connection, which plays a major role in proving the theorem mentioned before, between the existence of a solution of an equation of complex Monge–Ampère type in the complex projective space $\mathbb{P}^{n-1}$ and the existence of a solution of the Monge–Ampère equation in the class of homogeneous functions in $\mathbb{C}^n$. Namely, we show that a solution in $\mathbb{P}^{n-1}$ allows us to construct a corresponding solution in $\mathbb{C}^n$. The existence of a solution for some equations of Monge–Ampère type on special compact Kähler manifolds was proved by Ben Abdesselem [BA].

At the end of Section 1 we prove that, under an additional assumption on g, it is possible to solve (0.1) with a weaker restriction on the order of homogeneity.

In the second section we prove the existence of a solution of (0.1) for g locally bounded. To prove this we need a generalization of Tian's theorem from [T]. Tian solved the following equation on compact Kähler manifolds (M,ω) with a positive first Chern class:

$$(dd^c\varphi+\omega)^n = e^{-t\varphi+f}\omega^n, \tag{0.2}$$

where $dd^c\varphi+\omega \ge 0$, f is $\mathcal{C}^\infty$ smooth and $0 \le t \le 1$. For $t=1$ this equation provides the existence of a Kähler–Einstein metric on M. We prove that (0.2) has a solution for every bounded function f and $0 \le t \le \alpha(M)$, where $\alpha(M)$ is a global holomorphic invariant on M introduced by Tian.

1. Existence of a solution for smooth data

Definition 1.1. We say that a function $f:\mathbb{C}^n\to\mathbb{R}$ is *complex homogeneous of order* α where $\alpha>0$ if

$$f(\lambda z) = |\lambda|^\alpha f(z) \quad \text{for all } \lambda\in\mathbb{C} \text{ and } z\in\mathbb{C}^n.$$

We denote by $H^\alpha_{\mathbb{C}}(\mathbb{C}^n)$ the space of all complex homogeneous functions of order α in $\mathbb{C}^n$.

Sometimes we call a complex homogeneous function simply a homogeneous function.

We denote (see [Kl]) by $\mathcal{L}_+$ the set of all entire plurisubharmonic functions u in $\mathbb{C}^n$ for which there exist constants C_1 and C_2 (depending on u) such that

$$C_1 + \log(1+|z|) \le u(z) \le C_2 + \log(1+|z|).$$

We denote by $\mathcal{H}_+$ the set of all entire plurisubharmonic functions u in $\mathbb{C}^n$ which satisfy

$$u(\lambda z) = \log|\lambda| + u(z) \quad \text{for all } \lambda \in \mathbb{C} \text{ and } z \in \mathbb{C}^n.$$

It is well known that $\mathcal{H}_+ \subset \mathcal{L}_+$ and

$$\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n \quad \text{for all } u \in \mathcal{L}_+.$$

Now we recall that for a function from $\mathcal{H}_+$ much more is known about its Monge–Ampère measure.

PROPOSITION 1.2. *If $u \in \mathcal{H}_+$ then $(dd^c u)^n = (2\pi)^n \delta_0$, where δ_0 is the Dirac measure at zero.*

Proof. First we prove our proposition for smooth functions. Suppose that $u \in \mathcal{H}_+ \cap \mathcal{C}^\infty(\mathbb{C}^n \setminus \{0\})$. Then taking the $\partial^2/\partial z_j \partial \overline{z}_k$ derivative of the equation $u(\lambda z) = \log|\lambda| + u(z)$ for $z \neq 0$ we obtain

$$u_{j\overline{k}}(z) = |\lambda|^2 u_{j\overline{k}}(\lambda z) \quad \text{for } \lambda \neq 0 \text{ and } z \neq 0,$$

where $u_{j\overline{k}}(z) := \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)$. Taking $z = z/|z|$ and $\lambda = |z|$ we have

$$u_{j\overline{k}}(z) = |z|^{-2} u_{j\overline{k}}(z/|z|).$$

Recall that if a plurisubharmonic function u is $\mathcal{C}^2$ smooth then

$$(dd^c u)^n = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) d\lambda.$$

Using this equation we can show that for any $R > 0$,

$$\begin{aligned}\int_{B(0,R)\setminus\{0\}} (dd^c u)^n &= n! 4^n \int_{B(0,R)\setminus\{0\}} \det(u_{j\overline{k}}(z))\, d\lambda \\ &= n! 4^n \int_{B(0,R)\setminus\{0\}} |z|^{-2n} \det(u_{j\overline{k}}(z/|z|))\, d\lambda\end{aligned}$$

$$= \lim_{\varepsilon\to 0} n!4^n \int_{\varepsilon}^{R} r^{-1}\,dr \int_{\partial B(0,1)} \det(u_{j\bar k}(z/|z|))\,d\sigma$$

$$= \begin{cases} 0 & \text{if } \int_{\partial B(0,1)} \det(u_{j\bar k}(z/|z|))\,d\sigma = 0,\\ \infty & \text{if } \int_{\partial B(0,1)} \det(u_{j\bar k}(z/|z|))\,d\sigma \neq 0.\end{cases}$$

However we know that $\int_{\mathbb{C}^n} (dd^c u)^n = (2\pi)^n < \infty$, so $\det(u_{j\bar k}(z))$ must vanish on $\partial B(0,1)$. From that we conclude that $(dd^c u)^n = 0$ in $\mathbb{C}^n \setminus \{0\}$. This implies that the measure $(dd^c u)^n$ is supported at the origin, so $(dd^c u)^n = (2\pi)^n \delta_0$.

To finish the proof of Proposition 1.2 it is enough to find for every $u \in \mathcal{H}_+$ a sequence of smooth functions from $\mathcal{H}_+$ decreasing to u. First we recall the standard way of regularization of u.

Define a function $h : \mathbb{R} \to \mathbb{R}$ by the formula

$$h(t) = \begin{cases} \exp(-1/t) & \text{for } t > 0,\\ 0 & \text{for } t \le 0.\end{cases}$$

Set $\theta(x) = Ah(1 - |x|^2)$ for $x \in \mathbb{R}^m$, where $A = (\int_{B(0,1)} h(1-|x|^2)\,d\lambda)^{-1}$. Obviously $\theta \in \mathcal{C}^\infty(\mathbb{R}^m)$, $\operatorname{supp}\theta = \overline{B(0,1)}$ and $\int_{\mathbb{R}^m} \theta(x)\,d\lambda = 1$. For $\delta > 0$ we define $\theta_\delta(x) = (1/\delta^m)\theta(x/\delta)$. Note that $\int_{\mathbb{R}^m} \theta_\delta(x)\,d\lambda = 1$ and $\operatorname{supp}\theta_\delta = \overline{B(0,\delta)}$. It is well known that $v_\delta := u * \theta_\delta \in \mathrm{PSH} \cap \mathcal{C}^\infty$ and v_δ is decreasing to u as $\delta \searrow 0$. We call the sequence $\{v_\delta\}$ the *standard regularization* of u.

Now we define another regularization of u which preserves homogeneity. Set

$$u_\delta(z) := |z|^{-2n} \int u(w)\theta_\delta\biggl(\frac{z-w}{|z|}\biggr)\,d\lambda(w) = \int u(z - |z|w)\theta_\delta(w)\,d\lambda(w).$$

We claim that u_δ is the desired sequence. Obviously $u_\delta \in \mathcal{C}^\infty(\mathbb{C}^n \setminus \{0\})$.

First we show that if u satisfies $u(\mu z) = \log|\mu| + u(z)$ for all $\mu \in \mathbb{C}$ and $z \in \mathbb{C}^n$ then also the functions u_δ satisfy this equation. To see this observe that

$$\begin{aligned} u_\delta(\mu z) &= \int u(\mu z - |\mu z| w)\theta_\delta(w)\,d\lambda(w)\\ &= \log|\mu| + \int u\biggl(z - \frac{|\mu|}{\mu}|z|w\biggr)\theta_\delta(w)\,d\lambda(w)\\ &= \log|\mu| + \int u\biggl(z - \frac{|\mu|}{\mu}|z|w\biggr)\theta_\delta\biggl(\frac{|\mu|}{\mu}w\biggr)\biggl|\frac{\mu}{|\mu|}\biggr|^{2n} d\lambda\biggl(\frac{|\mu|}{\mu}w\biggr)\\ &= \log|\mu| + u_\delta(z). \end{aligned}$$

Now we show that $u_\delta \searrow u$ as $\delta \searrow 0$. From the above equation it is enough to check this for $|z| = 1$. But for such z our regularization is the standard regularization $u_\delta = v_\delta$. For the standard regularization we know that v_δ decreases to u, so also u_δ decreases to u.

To end the proof it is enough to check that $u_\delta \in \mathrm{PSH}(\mathbb{C}^n)$. To see this, note that we can write u_δ as

$$u_\delta(z) = \log|z| + u_\delta(z/|z|) = \log|z| + v_\delta(z/|z|),$$

where v_δ is the standard regularization of u.

We denote by $\mathbb{P}^{n-1}$ the $(n-1)$-dimensional complex projective space, i.e. the set of all one-dimensional linear subspaces of $\mathbb{C}^n$. Set $U_k = \{[Z_1,\dots,Z_n] : Z_k \neq 0\}$. Then we have $\mathbb{P}^{n-1} = \bigcup_{k=1}^n U_k$. In U_k we have local coordinates $(z_1,\dots,\widehat{z_k},\dots,z_n)$, where $z_j = Z_j/Z_k$. The Kähler metric h on $\mathbb{P}^{n-1}$ is given by

$$h_{\lambda\bar{\mu}}(z) = n\partial_\lambda\partial_{\bar{\mu}}\log\Big(1+\sum_{j\neq k}|z_j|^2\Big) \quad \text{on } U_k.$$

We denote by ω the form given by the formula

$$\omega = \frac{n}{2}dd^c\log\Big(1+\sum_{j\neq k}|z_j|^2\Big) \quad \text{on } U_k.$$

We define a mapping $\Pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ by $\Pi(z) = [z_1,\dots,z_n]$.

LEMMA 1.3. *Let $g : \mathbb{C}^n \to \mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha-2)$, where $\alpha > 0$ and suppose that there exists a solution v of the following Monge–Ampère equation on $\mathbb{P}^{n-1}$:*

$$(1.1)\qquad (dd^cv+\omega)^{n-1} = G(v,\cdot)\omega^{n-1} \quad \textit{and} \quad dd^cv+\omega \geq 0,$$

where $G : \mathbb{R}\times\mathbb{P}^{n-1} \to \mathbb{R}_+$ and

$$G(t,z) = C(n,\alpha)\widetilde{g}(\Pi^{-1}(z))e^{-\alpha t},$$

with

$$\widetilde{g}(z) = |z|^{-n(\alpha-2)}g(z) \quad \textit{and} \quad C(n,\alpha) = \frac{1}{n!2^{n+1}\alpha^{n+1}}.$$

Then there exists a solution $u \in \mathrm{PSH}(\mathbb{C}^n)\cap H^\alpha_{\mathbb{C}}$ of the complex Monge–Ampère equation on $\mathbb{C}^n$:

$$(1.2)\qquad (dd^cu)^n = g\,d\lambda.$$

Proof. First we define $w(z) := \log|z| + \frac{1}{n}v(\Pi(z))$. Observe that $w \in \mathrm{PSH}(\mathbb{C}^n_*)$ and

$$\begin{aligned} w(\lambda z) &= \log|\lambda z| + \frac{1}{n}v(\Pi(\lambda z)) = \log|\lambda| + \log|z| + \frac{1}{n}v(\Pi(z)) \\ &= \log|\lambda| + w(z), \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$. So we have checked that $w \in \mathcal{H}_+$ and Proposition 1.2 gives $(dd^cw)^n = (2\pi)^n\delta_0$. Now we can define $u(z) := \exp(\alpha w(z))$

for $z \neq 0$ and $u(0) = 0$. Then $u \in \mathrm{PSH}(\mathbb{C}^n)$ and for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$,

$$u(\lambda z) = e^{\alpha w(\lambda z)} = e^{\alpha(\log|\lambda z| + n^{-1} v(\Pi(\lambda z)))}$$
$$= e^{\alpha \log|\lambda|} e^{\alpha \log|z| + \alpha n^{-1} v(\Pi(z))} = |\lambda|^\alpha u(z).$$

Now we compute the Monge–Ampère measure for u:

$$\begin{aligned}(dd^c u)^n &= (dd^c e^{\alpha w})^n = (\alpha^2 e^{\alpha w} dw \wedge d^c w + \alpha e^{\alpha w} dd^c w)^n \\ &= \alpha^n e^{n\alpha w}(\alpha dw \wedge d^c w + dd^c w)^n \\ &= \alpha^n e^{n\alpha w}((dd^c w)^n + n\alpha dw \wedge d^c w \wedge (dd^c w)^{n-1}).\end{aligned}$$

Note that from the fact that $w \in \mathcal{H}_+$ we obtain

$$\alpha^n e^{n\alpha w}(dd^c w)^n = \alpha^n |z|^{n\alpha} e^{\alpha v(\Pi(z))} \cdot (2\pi)^n \delta_0 \equiv 0.$$

So

$$(dd^c u)^n = n\alpha^{n+1} e^{n\alpha w} dw \wedge d^c w \wedge (dd^c w)^{n-1}.$$

Denote by T the current $T = e^{n\alpha w}(dd^c w)^{n-1}$ and $z = (z_1, z') = (z_1, z_2, \ldots, z_n)$. Now fix a point $z \in \mathbb{C}^n \setminus \{0\}$. We can assume (applying rotation if necessary) that $z = (a, 0, \ldots, 0)$ and $|a| = |z|$.

Recall that Π denotes the canonical projection from $\mathbb{C}^n \setminus \{0\}$ to $\mathbb{P}^{n-1}$; we denote by Π_a the restriction of Π to $\{z_1 = a\}$. Then it is easy to see that

$$\Pi_a(a, z_2, \ldots, z_n) = (z_2/a, \ldots, z_n/a) \in U_1,$$
$$\Pi_a^{-1}(z_2, \ldots, z_n) = (a, az_2, \ldots, az_n) \in \{z_1 = a\},$$
$$\Pi_a \circ \Pi_a^{-1} = \mathrm{id}_{U_1} \quad \text{and} \quad \Pi_a^{-1} \circ \Pi_a = \mathrm{id}_{\{z_1 = a\}}.$$

Now we express the current $(dd^c w)^{n-1}$ on the set $\{z_1 = a\}$ using our assumptions:

$$\begin{aligned}(1.3)\qquad (dd^c w)^{n-1} &= (\Pi_a^{-1} \circ \Pi_a)^*(dd^c w)^{n-1} = \Pi_a^*(dd^c(w \circ \Pi_a^{-1}))^{n-1} \\ &= \Pi_a^*\bigg(dd^c\bigg(\frac{1}{2}\log|\Pi_a^{-1}(z')|^2 + \frac{1}{n} v(\Pi_a \circ \Pi_a^{-1}(z'))\bigg)\bigg)^{n-1} \\ &= \Pi_a^*\bigg(dd^c\bigg(\frac{1}{2}\log(|a|^2(1+|z'|^2)) + \frac{1}{n} v(z')\bigg)\bigg)^{n-1} \\ &= \Pi_a^*\bigg(\frac{1}{2} dd^c \log(1+|z'|^2) + \frac{1}{n} dd^c v\bigg)^{n-1} \\ &= \frac{1}{n^{n-1}} \Pi_a^*(\omega + dd^c v)^{n-1} \\ &= \frac{1}{n^{n-1}} \Pi_a^*(G\omega^{n-1}) = \frac{1}{n^{n-1}} (G \circ \Pi_a)(\Pi_a^* \omega)^{n-1} \\ &= \frac{1}{n^{n-1}} (G \circ \Pi_a)\bigg(\frac{n}{2} dd^c \log(1 + |z'/a|^2)\bigg)^{n-1}\end{aligned}$$

$$
\begin{aligned}
&= 2^{1-n}(G\circ \Pi_a)(dd^c \log(|a|^2+|z'|^2))^{n-1}\\
&= 2^{n-1}(n-1)!|a|^2(G\circ \Pi)(|a|^2+|z'|^2)^{-n}\\
&\quad \cdot \frac{i}{2}dz_2\wedge d\overline{z}_2\wedge\ldots\wedge \frac{i}{2}dz_n\wedge d\overline{z}_n.
\end{aligned}
$$

Note that

$$
dw\wedge d^c w = 4\frac{\partial w}{\partial z_1}\frac{\partial w}{\partial \overline{z}_1}\frac{i}{2}dz_1\wedge d\overline{z}_1+\ldots \tag{1.4}
$$

Since $v(\Pi(z))$ is constant on the set $\{(\zeta,0,\ldots,0):\zeta\in\mathbb{C}\}$ we conclude that

$$
\frac{\partial w}{\partial z_1}\frac{\partial w}{\partial \overline{z}_1}(z)=\frac{|z_1|^2}{|z|^4}.
$$

According to (1.3) and (1.4), on the set $\{z_1=a\}$ we obtain (remembering that $|a|=|z|=|z_1|$)

$$
\begin{aligned}
&dw\wedge d^c w\wedge (dd^c w)^{n-1}\\
&\quad = 4\frac{|z_1|^2}{|z|^4}\frac{i}{2}dz_1\wedge d\overline{z}_1\wedge 2^{n-1}(n-1)!|a|^2(G\circ\Pi)(|a|^2+|z'|^2)^{-n}\\
&\qquad \cdot\frac{i}{2}dz_2\wedge d\overline{z}_2\wedge\ldots\wedge\frac{i}{2}dz_n\wedge d\overline{z}_n\\
&\quad = (G\circ\Pi)2^{n+1}(n-1)!|z|^{-2n}d\lambda.
\end{aligned}
$$

So at our fixed point z we have checked that

$$
\begin{aligned}
(dd^c u)^n &= n2^{n+1}(n-1)!\alpha^{n+1}e^{n\alpha w}G\circ\Pi|z|^{-2n}d\lambda\\
&= n!2^{n+1}\alpha^{n+1}|z|^{n(\alpha-2)}e^{\alpha v(\Pi(z))}G\circ\Pi(z)d\lambda = g(z)d\lambda.
\end{aligned}
$$

This completes the proof of Lemma 1.3.

Our main theorem is

THEOREM 1.4. *Let $g:\mathbb{C}^n\to\mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha-2)$, where $0<\alpha<1/(n-1)$, such that $g\in\mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$. Then there exists a solution $u\in \mathrm{PSH}(\mathbb{C}^n)\cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n)\cap\mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$ of the complex Monge–Ampère equation $(dd^c u)^n = g d\lambda$ on $\mathbb{C}^n$. Moreover if g is only $\mathcal{C}^{r+\beta}$ smooth for some $r\ge 1$ and $0<\beta<1$, then u is $\mathcal{C}^{2+r+\beta}$ smooth.*

To prove Theorem 1.4 we need some facts about existence of solutions of the complex Monge–Ampère equation on $\mathbb{P}^{n-1}$ (for more details see [A1]–[A3], [BA], [T], [R]).

Let (M,h) be a compact complex Kähler manifold of dimension n. We denote by ω its first fundamental form. Consider the equation (see [A1])

$$
(dd^c\varphi+\omega)^n = e^{-t\varphi+f}\omega^n \quad\text{and}\quad dd^c\varphi+\omega\ge 0, \tag{1.5}
$$

where f is a given $\mathcal{C}^\infty$ smooth function and $t \in \mathbb{R}$. The following inequality plays an important role in solving the above equation:

$$\int_M e^{-\alpha\varphi}\omega^n \le C \exp\left(\frac{-\alpha}{\mathrm{vol}(M)} \int_M \varphi\omega^n\right), \tag{1.6}$$

for any φ such that $dd^c\varphi + \omega \ge 0$, where $C, \alpha > 0$. We also recall an invariant $\alpha(M)$ for M:

$$\alpha(M) = \sup\{\alpha > 0 : \text{there exists a constant } C \text{ such that } (1.6) \text{ is satisfied for all } \varphi \text{ with } dd^c\varphi + \omega \ge 0\}.$$

The following theorems give partial answers to the question: for which t does the equation (1.5) have a solution?

THEOREM 1.5 [BA]. *Let (M, h) be a compact complex Kähler manifold of dimension n with the first Chern class positive. Then the equation* (1.5) *has a solution for $0 \le t < \frac{n+1}{n}\alpha(M)$.*

THEOREM 1.6 [R]. *We have*

$$\alpha(\mathbb{P}^n) = \frac{1}{n+1}.$$

In particular on $\mathbb{P}^{n-1}$ the equation (1.5) *has a solution for $0 \le t < 1/(n-1)$.*

The following theorem tells us about the regularity of the solution.

THEOREM 1.7 [A1]. *Let (M, h) be as in Theorem* 1.5. *Consider the following equation on M*:

$$(dd^c\varphi + \omega)^n = e^{F(\varphi,\cdot)}\omega^n, \tag{1.7}$$

where $F : \mathbb{R} \times M \to \mathbb{R}_+$. If F is $\mathcal{C}^\infty$ smooth, then every solution of (1.7) *is $\mathcal{C}^\infty$ smooth. Moreover, if F is only $\mathcal{C}^{r+\beta}$ smooth with $r \ge 1$ and $0 < \beta < 1$, then every solution of* (1.7) *is $\mathcal{C}^{2+r+\beta}$ smooth.*

Now we can prove Theorem 1.4.

Proof of Theorem 1.4. Observe that the smoothness of g implies the smoothness of the function G from Lemma 1.3. Thus Theorems 1.5 and 1.6 yield the existence of a solution of the equation (1.1), which implies the existence of a solution of (1.2). For the regularity of the solution u, observe that if g is $\mathcal{C}^\infty$ (resp. $\mathcal{C}^{r+\beta}$) smooth then G is also $\mathcal{C}^\infty$ (resp. $\mathcal{C}^{r+\beta}$) smooth (recall that $g > 0$); then Theorem 1.7 shows that the solution v of the equation (1.1) is $\mathcal{C}^\infty$ (resp. $\mathcal{C}^{2+r+\beta}$) and by the definition so is u. This completes the proof of Theorem 1.4.

The statement of Theorem 1.4 can be strengthened if we assume additional symmetries of the function g. Suppose that $g : \mathbb{C}^n \to \mathbb{R}_+$ satisfies the following conditions:

$$(1.8)\qquad \begin{aligned} &g(z_1,\dots,z_j,\dots,z_n) = g(z_1,\dots,e^{i2\pi/p}z_j,\dots,z_n) \\ &\qquad\qquad\qquad\qquad \text{for } 1\le j\le n \text{ and some } p\in\mathbb{N}, \\ &g(z_1,\dots,z_j,\dots,z_k,\dots,z_n) = g(z_1,\dots,z_k,\dots,z_j,\dots,z_n) \\ &\qquad\qquad\qquad\qquad \text{for } 1\le j,k\le n. \end{aligned}$$

THEOREM 1.8. *Let $g : \mathbb{C}^n \to \mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha-2)$, where $0<\alpha<\min(n/(n-1), p/(n-1))$, satisfying conditions* (1.8) *and such that $g\in\mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$. Then there exists a solution $u\in \mathrm{PSH}\cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n)\cap\mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$ of $(dd^cu)^n = gd\lambda$ on $\mathbb{C}^n$ satisfying also conditions* (1.8). *Moreover if g is only $\mathcal{C}^{r+\beta}$ smooth for some $r\ge 1$ and $0<\beta<1$, then u is $\mathcal{C}^{2+r+\beta}$ smooth.*

To prove Theorem 1.8 we recall another invariant for M. Suppose that the manifold M has a nontrivial group of automorphisms. Then for any compact subgroup G of $\mathrm{Aut}(M)$ we can define the following invariant:

$$\alpha_G(M) = \sup\{\alpha>0 : \text{there exists a constant } C \text{ such that (1.6) is satisfied for all } G\text{-invariant } \varphi \text{ with } dd^c\varphi+\omega\ge 0\}.$$

For the invariant $\alpha_G(M)$ we have theorems analogous to Theorems 1.5 and 1.6.

THEOREM 1.9 [BA]. *Let (M,h) be a compact complex Kähler manifold of dimension n with the first Chern class positive and let G be a compact subgroup of* $\mathrm{Aut}(M)$. *Then the equation*

$$(dd^c\varphi+\omega)^n = e^{-t\varphi+f}\omega^n,$$

where $dd^c\varphi+\omega\ge 0$ and f is $\mathcal{C}^\infty$ smooth and G-invariant, has a $\mathcal{C}^\infty$ smooth, G-invariant solution for $0\le t<\frac{n+1}{n}\alpha_G(M)$.

For $k,j\in\{0,\dots,n\}$ and $\theta\in[0,2\pi]$ we define a class of automorphisms on $\mathbb{P}^n$:

$$\gamma_{j,\theta}([Z_0,\dots,Z_j,\dots,Z_n]) = [Z_0,\dots,Z_je^{i\theta},\dots,Z_n],$$
$$\sigma_{k,j}([Z_0,\dots,Z_j,\dots,Z_k,\dots,Z_n]) = [Z_0,\dots,Z_k,\dots,Z_j,\dots,Z_n].$$

We denote by $\mathcal{G}$ the compact subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ generated by $\gamma_{j,\theta},\sigma_{j,k}$ for $k,j\in\{0,\dots,n\}$ and $\theta\in[0,2\pi]$, and by $\mathcal{G}_p$ the compact subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ generated by $\gamma_{j,\theta},\sigma_{j,k}$ for $k,j\in\{0,\dots,n\}$ and $\theta=2\pi/p$.

THEOREM 1.10 [R]. *We have*

$$\alpha_{\mathcal{G}_p}(\mathbb{P}^n)\ge\min\left(1,\frac{p}{n+1}\right)\quad\textit{and}\quad\alpha_{\mathcal{G}}(\mathbb{P}^n)=1,$$

where $\mathcal{G}_p$ and $\mathcal{G}$ are as above. In particular on $\mathbb{P}^{n-1}$ the equation

$$(dd^c\varphi+\omega)^n = e^{-t\varphi+f}\omega^n,$$

where $dd^c\varphi+\omega \geq 0$ and f is $\mathcal{C}^\infty$ smooth and $\mathcal{G}_p$-invariant, has a $\mathcal{C}^\infty$ smooth, $\mathcal{G}_p$-invariant solution for $0 \leq t < \min(n/(n-1), p/(n-1))$.

Now we can prove Theorem 1.8.

Proof of Theorem 1.8. First observe that from the assumptions on g, the function $\widetilde{g}\circ \Pi^{-1}$ is $\mathcal{G}$-invariant, where $\widetilde{g}(z) = |z|^{-n(\alpha-2)}g(z)$ for $z \neq 0$. Now the proof of Theorem 1.8 is analogous to that of Theorem 1.4.

From the above theorems we have the following corollary.

Corollary 1.11. *Suppose that $g : \mathbb{C}^n \to \mathbb{R}_+$ is a complex homogeneous function of order $n(\alpha-2)$, where $0 < \alpha < n/(n-1)$, which satisfies the following conditions*:

$$\begin{aligned} &g(z_1,\ldots,z_j,\ldots,z_n) = g(z_1,\ldots,|z_j|,\ldots,z_n) \quad \text{for } 1 \leq j \leq n, \\ &g(z_1,\ldots,z_j,\ldots,z_k,\ldots,z_n) = g(z_1,\ldots,z_k,\ldots,z_j,\ldots,z_n) \\ &\hspace{20em} \text{for } 1 \leq j,k \leq n, \end{aligned} \tag{1.9}$$

and such that $g \in \mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$. Then there exists a solution $u \in \mathrm{PSH}\cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n)\cap\mathcal{C}^\infty(\mathbb{C}^n\setminus\{0\})$ of $(dd^cu)^n = gd\lambda$ on $\mathbb{C}^n$ satisfying also conditions (1.9). *Moreover if g is only $\mathcal{C}^{r+\beta}$ smooth for some $r \geq 1$ and $0 < \beta < 1$, then u is $\mathcal{C}^{2+r+\beta}$ smooth.*

2. Existence of a solution for bounded data. The main purpose of this section is to prove the existence of a solution of $(dd^cu)^n = gd\lambda$ in the class of homogeneous functions for bounded data, but with a stronger restriction on the order of homogeneity.

Theorem 2.1. *Let $g : \mathbb{C}^n \to \mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha-2)$, where $0 < \alpha < 1/n$, such that $g \in L^\infty(\partial B(0,1))$. Then there exists a solution $u \in \mathrm{PSH}\cap L^\infty_{\mathrm{loc}}\cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n)$ of $\mathbb{C}^n(dd^cu)^n = gd\lambda$ on $\mathbb{C}^n$.*

First we need to prove the existence of a solution of $(dd^c\varphi+\omega)^n = e^{-t\varphi+f}\omega^n$ for bounded data f on a compact Kähler manifold. This is a generalization of Tian's theorem [T] for bounded data, but with a stronger assumption on the parameter t.

Theorem 2.2. *Let (M,h) be a compact complex Kähler manifold of dimension n with the first Chern class positive and let $f \in L^\infty(M)$ be nonnegative with $\int_M f\omega^n = \mathrm{vol}(M)$. Then the equation*

$$(dd^c\varphi+\omega)^n = fe^{-t\varphi}\omega^n$$

has a solution φ with $dd^c\varphi+\omega \geq 0$ for $0 \leq t < \alpha(M)$.

If $f \in L^\infty(M)$ then there exists an approximating sequence $\{f_j\}$ such that $f_j \in \mathcal{C}^\infty(M)$, $f_j > 0$, $\{f_j\}$ is uniformly bounded and $f_j \to f$ in $L^1(M)$ as $j \to \infty$. Multiplying f_j by constants which tend to 1 as $j \to \infty$ we can get

$$\int_M f_j \omega^n = \mathrm{vol}(M).$$

Let $\phi_{t,j}$ denote a solution of

$$(dd^c\phi_{t,j} + \omega)^n = f_j e^{-t\phi_{t,j}}\omega^n \quad \text{and} \quad dd^c\phi_{t,j} + \omega > 0 \tag{2.1}$$

for $0 \le t < \alpha(M)$, provided by Theorem 1.5.

Now we show that for fixed t the sequence $\phi_{t,j}$ is uniformly bounded.

LEMMA 2.3. *For fixed $0 \le t < \alpha(M)$ the sequence $\{\phi_{t,j}\}$ is uniformly bounded.*

To prove Lemma 2.3 we need some results from [K1] and [T].

THEOREM 2.4 [T]. *Let (M, ω) be a compact complex Kähler manifold of dimension n with the first Chern class positive. Then for all $0 < t < \alpha(M)$ there exists a constant C, depending only on M, such that*

$$\int_M e^{-t\varphi}\omega^n \le C \tag{2.2}$$

for any functions $\varphi \in \mathcal{C}^2$ with $dd^c\varphi + \omega \ge 0$ and $\sup_M \varphi = 0$.

We also need a theorem which gives us a lower bound for the infimum of the solution $\phi_{t,j}$.

THEOREM 2.5 [K1]. *Let Ω be a strictly pseudoconvex subset of $\mathbb{C}^n$ and let u be a smooth solution of*

$$(dd^c u)^n = f d\lambda$$

on Ω with $\|f\|_{L^p(\Omega)} \le A$ for some $p > 1$. Suppose that $u < 0$ and $u(0) > C$ $(0 \in \Omega)$. If the sets $U(s) := \{z : u(z) < s\} \cap \Omega''$ are nonempty and relatively compact in $\Omega'' \subset \Omega' \subset\subset \Omega$ for $s \in [S, S + D]$ then $\inf_\Omega u$ is bounded from below by a constant depending on A, C, D, p, Ω', Ω, but independent of u, Ω''.

Now we can prove Lemma 2.3.

Proof of Lemma 2.3. First we recall that the functions f_j satisfy $\int_M f_j\omega^n = \mathrm{vol}(M)$ and note that by Stokes' theorem,

$$\int_M f_j e^{-t\phi_{t,j}}\omega^n = \int_M (dd^c\phi_{t,j} + \omega)^n = \mathrm{vol}(M) = \int_M f_j\omega^n.$$

Hence

$$\sup_M \phi_{t,j} \ge 0. \tag{2.3}$$

We define $\psi_{t,j} := \phi_{t,j} - \sup_M \phi_{t,j}$. Then the functions $\psi_{t,j}$ satisfy the Monge–Ampère equation

$$(dd^c\psi_{t,j} + \omega)^n = f_j e^{-t\psi_{t,j} - t\sup\phi_{t,j}}\omega^n.$$

We also set $F_j(x,z) = f_j(z)e^{-tx}$. Fix $0 < t < \alpha(M)$ and choose $\varepsilon > 0$ such that $(1+\varepsilon)t < \alpha(M)$. Now we show that the sequence $\{F_j(\psi_{t,j} - \sup\phi_{t,j}, z)\}$ is uniformly bounded in $L^{1+\varepsilon}(M)$, and from that we conclude that $\psi_{t,j}$ satisfies the assumptions of Theorem 2.5. Indeed, from (2.3) and Theorem 2.4 we obtain

$$\int_M (f_j e^{-t\psi_{t,j} - t\sup\phi_{t,j}})^{1+\varepsilon}\omega^n \le \int_M (f_j e^{-t\psi_{t,j}})^{1+\varepsilon}\omega^n$$
$$\le (\sup f_j)^{1+\varepsilon}\int_M e^{-(1+\varepsilon)t\psi_{t,j}}\omega^n \le C_1,$$

where C_1 does not depend on j.

Now fix a covering of M by strictly pseudoconvex coordinate patches V_ν'', and another two coverings of M: V_ν', V_ν such that $V_\nu \subset V_\nu' \subset\subset V_\nu''$.

Fix j and take $z \in V_\nu$ such that $\psi_{t,j}(z) = \inf_M \psi_{t,j}$. We may assume that there is a smooth, bounded function v such that $dd^c v = \omega$ in V_ν'', $v \le 0$ and $v(z) \le \inf_{\partial V_\nu} v - c_0$ for some positive $c_0 > 0$. Hence,

$$v(z) + \psi_{t,j}(z) \le \inf_{\partial V_\nu}(v + \psi_{t,j}) - c_0.$$

So if we take $D = c_0$, $S = v(z) + \psi_{t,j}(z)$ and $u = \psi_{t,j} + v$ in Theorem 2.5 the set $U(s) = \{v + \psi_{t,j} - s < 0\}$ is nonempty and relatively compact in V_ν for $s \in [S, S+D]$. Hence from Theorem 2.5 we have $\inf_M(v + \psi_{t,j}) \ge \text{const}$, but v is bounded so $\inf_M \psi_{t,j} \ge -C_2$ and $C_2 > 0$ does not depend on j. Then by the definition of $\psi_{t,j}$,

$$\sup_M \phi_{t,j} - \inf_M \phi_{t,j} \le C_2. \tag{2.4}$$

To finish the proof note that

$$\lim_{x\to+\infty}\int_M F_j(x,z)\omega^n < \int_M \omega^n < \lim_{x\to-\infty}\int_M F_j(x,z)\omega^n. \tag{2.5}$$

Hence, by (2.4), (2.5) and the equality $\int_M F_j(\phi_{t,j}, z)\omega^n = \int_M \omega^n$ we conclude that there is a constant $C_3 > 0$ such that

$$\sup_M \phi_{t,j} < C_3 \quad \text{and} \quad \inf_M \phi_{t,j} > -C_3,$$

for $j \ge j_0$. This means that the sequence $\{\phi_{t,j}\}$ is uniformly bounded, which completes the proof of Lemma 2.3.

Now we can prove Theorem 2.2. The proof is based on [K2].

Proof of Theorem 2.2. First we recall that by $\{f_j\}$ we have denoted an approximating sequence such that $f_j \in \mathcal{C}^\infty(M)$, $f_j > 0$, $\int_M f_j\omega^n = \mathrm{vol}(M)$,

$\{f_j\}$ is uniformly bounded and $f_j \to f$ in $L^1(M)$ as $j \to \infty$. Furthermore, $\phi_{t,j}$ denotes the solution of

$$(dd^c\phi_{t,j} + \omega)^n = f_j e^{-t\phi_{t,j}}\omega^n \quad \text{and} \quad dd^c\phi_{t,j} + \omega \geq 0.$$

By Lemma 2.3 we know that the sequence $\phi_{t,j}$ is uniformly bounded for any $0 \leq t < \alpha(M)$.

Fix $0 \leq t < \alpha(M)$. We may take a subsequence of $\phi_{t,j}$ (denoted also by $\phi_{t,j}$) such that $\phi_{t,j} \to \phi$ in $L^1(M)$ as $j \to \infty$, where $\phi = (\limsup_j \phi_{t,j})^*$. We show that ϕ is the desired solution.

First we prove that $f_j e^{-t\phi_{t,j}} \to f e^{-t\phi}$ in $L^1(M)$. Note that

$$\Big|\int_M (f_j e^{-t\phi_{t,j}} - f e^{-t\phi})\omega^n\Big| \leq \Big|\int_M f_j(e^{-t\phi_{t,j}} - e^{-t\phi})\omega^n\Big| + \Big|\int_M e^{-t\phi}(f_j - f)\omega^n\Big| = I_1 + I_2.$$

Then

$$\begin{aligned} I_1 &\leq \sup f_j \int_M e^{-t\phi_{t,j}}|1 - e^{t(\phi_{t,j}-\phi)}|\omega^n \\ &\leq \sup f_j e^{-t\phi_{t,j}} \int_M t|\phi_{t,j} - \phi|e^{t|\phi_{t,j}-\phi|}\omega^n \\ &\leq \sup f_j e^{-t\phi_{t,j}+t|\phi_{t,j}-\phi|} \int_M t|\phi_{t,j} - \phi|\omega^n \to 0 \quad \text{as } j \to \infty. \end{aligned}$$

Similarly

$$I_2 \leq \sup e^{-t\phi} \int_M |f_j - f|\omega^n \to 0 \quad \text{as } j \to \infty.$$

We have proved that $f_j e^{-t\phi_{t,j}} \to f e^{-t\phi}$ in $L^1(M)$, so we may choose a subsequence (denoted also by $\phi_{t,j}$) such that

$$\|f_j e^{-t\phi_{t,j}} - f e^{-t\phi}\|_{L^1(M)} \leq 2^{-j-1}. \tag{2.6}$$

Let us introduce some auxiliary functions:

$$\nu_{kl} = \max_{k\leq j\leq l} \phi_{t,j}, \qquad \nu_k = (\lim_{j\to\infty} \uparrow \nu_{kl})^*,$$

$$R_{kl} = \min_{k\leq j\leq l} f_j e^{-t\phi_{t,j}}, \qquad R_k = \lim_{l\to\infty} \downarrow R_{kl}.$$

Since, locally, ω is representable by $dd^c v$, where v is a plurisubharmonic function, we can apply [BT1, Proposition 2.8] to get

$$(\omega + dd^c\nu_{kl})^n \geq R_{kl}\omega^n.$$

Hence by the convergence theorem [BT3],

$$R_k \leq \lim_{l\to\infty} (\omega + dd^c\nu_{kl})^n = (\omega + dd^c\nu_k)^n. \tag{2.7}$$

Note that $\phi = \lim_{k\to\infty} \downarrow \nu_k$. We can apply the convergence theorem once more to get

$$(\omega + dd^c\nu_k)^n \to (\omega + dd^c\phi)^n. \tag{2.8}$$

Now we show that $R_k \to fe^{-t\phi}$ in $L^1(M)$. To prove this we shall use (2.6) and the simple fact that

$$fe^{-t\phi} - R_k = fe^{-t\phi} - f_{k+1}e^{-t\phi_{t,k+1}} + (f_{k+1}e^{-t\phi_{t,k+1}} - f_{k+2}e^{-t\phi_{t,k+2}}) + \ldots,$$

and then

$$\begin{aligned} \|fe^{-t\phi} - R_k\|_{L^1} &\le \|fe^{-t\phi} - f_{k+1}e^{-t\phi_{t,k+1}}\|_{L^1} \\ &\quad + \|f_{k+1}e^{-t\phi_{t,k+1}} - f_{k+2}e^{-t\phi_{t,k+2}}\|_{L^1} + \ldots \\ &\le 2^{-k+2} + (2^{-k+2} + 2^{-k+3}) + \ldots \\ &= 2^{-k}. \end{aligned} \tag{2.9}$$

So $R_k \to fe^{-t\phi}$ in $L^1(M)$. Combining (2.9) with (2.7) and (2.8) we obtain

$$fe^{-t\phi}\omega^n \le (\omega + dd^c\phi)^n.$$

Since the integrals over M of both currents in the above inequality are equal to $\mathrm{vol}(M)$ we get

$$fe^{-t\phi}\omega^n = (\omega + dd^c\phi)^n.$$

This completes the proof of Theorem 2.2.

Proof of Theorem 2.1. Let $\widetilde{g}(z) = |z|^{-n(\alpha-2)}g(z)$. Then $\widetilde{g}$ is a complex homogeneous function of order 0 and also $\widetilde{g} \in L^\infty$. Let

$$g_j(z) := |z|^{-2n} \int \widetilde{g}(w)\theta_{1/j}\left(\frac{z-w}{|z|}\right) d\lambda(w)$$

be the regularization of $\widetilde{g}$ defined in Proposition 1.2. Hence we know that g_j are complex homogeneous functions of order 0 and we can also assume that $g_j > 0$ by adding, if necessary, positive constants tending to zero. Moreover $\{g_j\}$ is uniformly bounded and $g_j \to \widetilde{g}$ in L^1.

Define the following functions on $\mathbb{P}^{n-1}$:

$$\begin{aligned} f(z) &= \frac{1}{n!2^{n+1}\alpha^{n+1}}\widetilde{g}(\Pi^{-1}(z)), \\ f_j(z) &= \frac{1}{n!2^{n+1}\alpha^{n+1}}g_j(\Pi^{-1}(z)). \end{aligned} \tag{2.10}$$

Multiplying g and g_j by constants which tend to 1, we can assume that

$$\int_{\mathbb{P}^{n-1}} f\omega^{n-1} = \int_{\mathbb{P}^{n-1}} f_j\omega^{n-1} = \mathrm{vol}(\mathbb{P}^{n-1}).$$

Moreover $f \in L^\infty(\mathbb{P}^{n-1})$, $\{f_j\}$ is uniformly bounded and $f_j \to f$ in $L^1(\mathbb{P}^{n-1})$. So we can apply Theorems 1.4 and 2.2 to get a function φ on $\mathbb{P}^{n-1}$ such that $dd^c\varphi + \omega \geq 0$ and $(dd^c\varphi + \omega)^{n-1} = fe^{-t\varphi}\omega^{n-1}$. Then we know from the proof of Lemma 1.3 that the function

$$u(z) = |z|^\alpha e^{(\alpha/n)\varphi(\Pi(z))}$$

is plurisubharmonic and $(dd^cu)^n = gd\lambda$. This completes the proof.

As a direct consequence of Theorem 2.1 we obtain the following corollaries.

COROLLARY 2.6. *Let $p \in \mathbb{N}$ and let $g : \mathbb{C}^n \to \mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha - 2)$, where $0 < \alpha < \min(1, p/n)$, satisfying conditions* (1.8) *and such that $g \in L^\infty(\partial B(0,1))$. Then there exists a solution $u \in \mathrm{PSH} \cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n) \cap \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ of $(dd^cu)^n = gd\lambda$ on $\mathbb{C}^n$ satisfying also conditions* (1.8).

Proof. It is enough to note that, if g satisfies conditions (1.8), then the functions (2.10) are $\mathcal{G}_p$-invariant. Then the Corollary follows from the proof of Theorems 2.2 and the proofs of Theorems 2.1 and 1.10.

COROLLARY 2.7. *Let $p \in \mathbb{N}$ and let $g : \mathbb{C}^n \to \mathbb{R}_+$ be a complex homogeneous function of order $n(\alpha - 2)$, where $0 < \alpha < 1$, satisfying conditions* (1.9) *and such that $g \in L^\infty(\partial B(0,1))$. Then there exists a solution $u \in \mathrm{PSH} \cap H^\alpha_{\mathbb{C}}(\mathbb{C}^n) \cap \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ of $(dd^cu)^n = gd\lambda$ on $\mathbb{C}^n$ satisfying also conditions* (1.9).

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