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The Łojasiewicz exponent at infinity for overdetermined polynomial mappings

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Abstract. We prove that the study of the Łojasiewicz exponent at infinity of overdetermined polynomial mappings $\mathbb{C}^n \to \mathbb{C}^m$, $m > n$, can be reduced to the one when $m = n$.

Introduction. In the paper we study the Łojasiewicz exponent at infinity for *overdetermined* polynomial mappings, i.e. polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$, where $m > n$. In the case $m = n$ this exponent is well known (see [C], [CK_1]–[CK_4], [P_1], [P_2], [PT]). It is strongly related to the properties of properness and injectivity of polynomial mappings (see [H], [C], [CK_1], [CK_3], [CK_4], [P_1], [P_2], [PT]). Numerous papers have been devoted to the estimation of this exponent from below and to the effective Nullstellensatz (see [C], [B_1], [B_2], [JKS], [K], [S], [BY], [CK_5]). The deepest result in this direction is the Kollár inequality [K]. We investigate it in Corollary 3.2.

We reduce the computation of the exponent of $f : \mathbb{C}^n \to \mathbb{C}^m$, $m > n$, to the case $m = n$ (see Theorem 2.1 and Corollary 3.1). The key point of the proof is the reduction of the study of the fibres of a polynomial mapping to the case of $m \le n$ (Proposition 1.1). We obtain it by composing f with a linear mapping. This method can be applied to obtain a characterization of the Łojasiewicz exponent of a proper polynomial mapping (Corollary 3.3, cf. [C], [P_1]). Corollary 3.3 also gives a criterion for injectivity of polynomial mappings (cf. [P_1]).

Additionally, using the Łojasiewicz exponent at infinity we prove a criterion of properness of polynomial mappings (Corollary 3.4, cf. [C], [P_1]).

1. Fibres of polynomial mappings. In what follows we write "the generic $x \in A$" instead of "there exists an algebraic set V such that $A \setminus V$ is a dense subset of A and $x \in A \setminus V$".

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For any $m, k \in \mathbb{N}$ we denote by $\mathbf{L}(m,k)$ the set of all nonsingular linear mappings $\mathbb{C}^m \to \mathbb{C}^k$, where for $k = 0$ we put $\mathbb{C}^k = \{0\}$. Let $m \geq k$. Denote by $\Delta(m,k)$ the set of all linear mappings $L = (L_1, \ldots, L_k) \in \mathbf{L}(m,k)$ of the form

$$L_i(y_1, \ldots, y_m) = y_i + \sum_{j=k+1}^{m} \alpha_{i,j} y_j, \qquad i = 1, \ldots, k,$$

where $\alpha_{i,j} \in \mathbb{C}$; $\Delta^0(m,k)$ is the set of all $L = (L_1, \ldots, L_k) \in \Delta(m,k)$ such that

$$L_1(y_1, \ldots, y_m) = y_1.$$

Proposition 1.1. *Let $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping with $\deg f_j > 0$ for $j = 1, \ldots, m$, where $m \geq n \geq 1$.*

(i) *For the generic $L \in \mathbf{L}(m,n)$,*

$$\#[(L \circ f)^{-1}(0) \setminus f^{-1}(0)] < \infty. \tag{1}$$

(ii) *For the generic $L \in \Delta^0(m,n)$, (1) holds.*

The proof will be preceded by an easy lemma. We denote by $G_k(\mathbb{C}^m)$ the Grassmann space of k-dimensional linear subspaces of $\mathbb{C}^m$, $k < m$.

Lemma 1.1. *Let $V \subset \mathbb{C}^m$ be an algebraic set of dimension s.*

(i) *If $s + k < m$, then for the generic $H \in G_k(\mathbb{C}^m)$,*

$$H \cap V \subset \{0\}.$$

(ii) *If $s + k = m$, then for the generic $H \in G_k(\mathbb{C}^m)$,*

$$\#(H \cap V) < \infty.$$

Proof of Proposition 1.1. In the proof of (ii) we will need a version of (i) in the case of regular mappings. So, we prove (i) in the slightly general case of regular mappings $f = (f_1, \ldots, f_m) : X \to \mathbb{C}^m$, where X is an irreducible algebraic set, $\dim X \leq n$ and $f_j \neq \text{const}$. Let $W \subset \mathbb{C}^m$ be the closure of $f(X)$ and $k = \dim W$. Obviously $k \leq n$. We have two cases:

1°. $k < n$. By Lemma 1.1(i) there exists a Zariski open and dense subset $U \subset G_{m-n}(\mathbb{C}^m)$ such that for any $H \in U$ we have $W \cap H \subset \{0\}$. Hence, the set $\mathcal{U} = \{L \in \mathbf{L}(m,n) : \ker L \in U\}$ is a Zariski open and dense subset of $\mathbf{L}(m,n)$. Moreover, for any $L \in \mathcal{U}$ we have $f^{-1}(0) = (L \circ f)^{-1}(0)$. This gives (1) in this case.

2°. $k = n$. Let

$$\Gamma = \overline{\{w \in W : \dim f^{-1}(w) > 0\}}.$$

By Corollary 3.16 and Proposition 2.31 of [M], Γ is an algebraic set. Moreover $\dim \Gamma < n$, since in the opposite case, by the definition of Γ and

Corollary 3.15 of [M], we have

$$n+1 \le \dim f^{-1}(\Gamma) \le \dim f^{-1}(W) = n,$$

which is impossible. Thus, by Lemma 1.1, there exists a Zariski open and dense subset $U \subset G_{m-n}(\mathbb{C}^m)$ such that for any $H \in U$ we have

$$\#(W \cap H) < \infty \tag{2}$$

and

$$\Gamma \cap H \subset \{0\}. \tag{3}$$

Let, by (2), $W \cap H = \{w^1, \ldots, w^p\}$ and $L \in \mathbf{L}(m,n)$ be such that $H = \ker L$. Then

$$(L \circ f)^{-1}(0) = f^{-1}(w^1) \cup \ldots \cup f^{-1}(w^p).$$

From (3) it follows that $w^i \notin \Gamma$ if $w^i \neq 0$. In consequence for $w^i \neq 0$ we have $\# f^{-1}(w^i) < \infty$. This gives (1). Since $\{L \in \mathbf{L}(m,n) : \ker L \in U\}$ is a Zariski open and dense subset of $\mathbf{L}(m,n)$, we have the assertion in this case. This gives (i).

To prove (ii), let $X' = f^{-1}(\{0\} \times \mathbb{C}^{m-1})$ and $g : X' \to \mathbb{C}^{m-1}$ be a regular mapping of the form

$$g(x) = (f_2(x), \ldots, f_m(x)), \qquad x \in X'.$$

Since $f_1 \neq 0$, $\dim X' \le n-1$. From the first part of the proof, we now see that for the generic $M \in \mathbf{L}(m-1, n-1)$,

$$\#[(M \circ g)^{-1}(0) \setminus g^{-1}(0)] < \infty. \tag{4}$$

Obviously the set $\mathcal{U}$ of all linear mappings $M = (L_2, \ldots, L_n) \in \mathbf{L}(m-1, n-1)$ of the form $L_i(w_2, \ldots, w_m) = L'_i(w_2, \ldots, w_n) + L''_i(w_{n+1}, \ldots, w_m)$, $i = 2, \ldots, n$, such that $\mathrm{Jac}[L'_2, \ldots, L'_n] \neq 0$ is a Zariski open and dense subset of $\mathbf{L}(m-1, n-1)$. Moreover, $(L'_2, \ldots, L'_n)^{-1} \circ M \in \Delta(m-1, n-1)$. So, for the generic $M \in \Delta(m-1, n-1)$ we have (4). Since $L = (w_1, M) \in \Delta^0(m,n)$ for any $M \in \mathcal{U}$, we obtain (ii).

2. The Łojasiewicz exponent at infinity. In this section we prove Theorem 2.1 on reduction of calculations of the Łojasiewicz exponent at infinity of a mapping $\mathbb{C}^n \to \mathbb{C}^m$ to the case $\mathbb{C}^n \to \mathbb{C}^n$.

Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping such that $\# f^{-1}(0) < \infty$. Let

$$N_\infty(f) = \{\nu \in \mathbb{R} : \exists_{C>0, R>0}\ \forall_{x \in \mathbb{C}^n}\ |x| > R \ \Rightarrow\ |f(x)| \ge C|x|^\nu\},$$

where $|\cdot|$ denotes the policylindric norm. We define the *Łojasiewicz exponent at infinity* of the mapping f as $\sup N_\infty(f)$ and denote it by $\mathcal{L}_\infty(f)$.

THEOREM 2.1. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \ge n$, be a polynomial mapping such that $\# f^{-1}(0) < \infty$. Then for any $L \in \mathbf{L}(m,n)$ such that $\#(L \circ f)^{-1}(0) < \infty$*

we have

$$\mathcal{L}_\infty(f) \geq \mathcal{L}_\infty(L \circ f). \tag{5}$$

Moreover, for the generic $L \in \mathbf{L}(m,n)$,

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(L \circ f). \tag{6}$$

The proof of this theorem will be preceded by two lemmas.

LEMMA 2.1. *Let* $V \subset \mathbb{C}^m$ *be an algebraic set of dimension* s. *If* $s < n$, *then there exists a Zariski open and dense subset* $U \subset \mathbf{L}(m,n)$ *such that for any* $L \in U$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $y \in V$,

$$|L(y)| < \delta \;\Rightarrow\; |y| < \varepsilon.$$

Proof. There exists a Zariski open and dense subset $U_1 \subset \mathbf{L}(m,n)$ such that for any $L \in U_1$,

$$V \cap \ker L \subset \{0\} \tag{7}$$

(see Lemma 1.1(i)). By Sadullaev's Theorem ([Ł], VII,7.1), there exists a Zariski open and dense subset $U \subset U_1$ such that for any $L \in U$ there exists $C_L > 0$ such that $V \subset \{y \in \mathbb{C}^m : |y| \leq C_L(1 + |L(y)|)\}$, which implies

$$y \in V \wedge |y| > 2C_L \;\Rightarrow\; |L(y)| > 1. \tag{8}$$

Let $L \in U$ and $\varepsilon > 0$. If $0 \notin V$, then either $A_1 = \{|L(y)| : y \in V \wedge |y| \leq 2C_L\}$ is an empty set and we put $\delta_1 = 1$, or $A_1 \neq \emptyset$ and we put $\delta_1 = \min A_1$. By (7) we have $\delta_1 > 0$. Putting $\delta = \min(\delta_1, 1)$, by (8) and the definition of A_1 we obtain $|L(y)| \geq \delta$ for any $y \in V$. This gives the assertion in this case. If $0 \in V$, then either $A_2 = \{|L(y)| : y \in V \wedge \varepsilon \leq |y| \leq 2C_L\}$ is an empty set and we put $\delta_2 = 1$, or $A_2 \neq \emptyset$ and we put $\delta_2 = \min A_2$. By (7) we have $\delta_2 > 0$. Putting $\delta = \min(\delta_2, 1)$, by (8) and the definition of A_2 we obtain the assertion in this case. This ends the proof.

LEMMA 2.2. *Let* $f : \mathbb{C}^n \to \mathbb{C}^m$ *with* $m \geq n$ *be a polynomial mapping. Then there exists a Zariski open and dense subset* $U \subset \mathbf{L}(m,n)$ *such that for any* $L \in U$ *and any* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $r > 0$ *such that for any* $x \in \mathbb{C}^n$,

$$|x| > r \wedge |L \circ f(x)| < \delta \;\Rightarrow\; |f(x)| < \varepsilon.$$

Proof. Let $W = \overline{f(\mathbb{C}^n)}$. Then $\dim W \leq n$. Assume first that $\dim W < n$. Then, by Lemma 2.1, there exists a Zariski open and dense subset $U \subset \mathbf{L}(m,n)$ such that for any $L \in U$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in W$,

$$|L(y)| < \delta \;\Rightarrow\; |y| < \varepsilon.$$

Then for any $x \in \mathbb{C}^n$,

$$|L \circ f(x)| < \delta \;\Rightarrow\; |f(x)| < \varepsilon.$$

Thus we have the assertion in this case.

Let now $\dim W = n$. Then, by Proposition 3.15 of [M], we easily see that there exists an algebraic set $V \subset W$ such that $\dim V \leq n-1$ and the mapping

$$f|_{\mathbb{C}^n \setminus f^{-1}(V)} : \mathbb{C}^n \setminus f^{-1}(V) \to W \setminus V \tag{9}$$

is a finite covering. Thus it is a proper mapping. By Lemma 2.1, there exists a Zariski open and dense subset $U_1 \subset \mathbf{L}(m,n)$ such that for any $L \in U_1$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in V$,

$$|L(y)| < \delta \;\Rightarrow\; |y| < \varepsilon. \tag{10}$$

By Sadullaev's Theorem there exists a Zariski open and dense subset $U_2 \subset \mathbf{L}(m,n)$ such that for any $L \in U_2$ there exists $C_L > 0$ such that

$$W \subset \{y \in \mathbb{C}^m : |y| \leq C_L(1 + |L(y)|)\}. \tag{11}$$

Take any $L \in U_1 \cap U_2$ and $\varepsilon > 0$. Assume to the contrary that there exists a sequence $\{x_n\}$ such that $|x_n| \to \infty$, $|L(f(x_n))| \to 0$ and

$$|f(x_n)| \geq \varepsilon.$$

Without loss of generality, by (11), we may assume that $f(x_n) \to y_0$, where $y_0 \in \mathbb{C}^m$. Since the mapping (9) is proper, we see that $y_0 \in V$. So, $|y_0| \geq \varepsilon$ and $L(y_0) = 0$. This contradicts (10) and ends the proof.

Proof of Theorem 2.1. Let

$$U = \{L \in \mathbf{L}(m,n) : \#[(L \circ f)^{-1}(0)] < \infty\}.$$

Let $L \in U$. Then there exists $M \in \mathbf{L}(m, m-n)$ such that $\widetilde{L} = (L, M) \in \mathbf{L}(m,m)$ and $\mathcal{L}_\infty(f) = \mathcal{L}_\infty(\widetilde{L} \circ f)$. Obviously for $x \in \mathbb{C}^n$ we have $|\widetilde{L} \circ f(x)| \geq |L \circ f(x)|$, so $\mathcal{L}_\infty(\widetilde{L} \circ f) \geq \mathcal{L}_\infty(L \circ f)$. This gives (5).

To prove the "moreover" part, let $W = \overline{f(\mathbb{C}^n)}$. Consider two cases: $\mathcal{L}_\infty(f) > 0$ and $\mathcal{L}_\infty(f) \leq 0$.

First, assume that $\mathcal{L}_\infty(f) > 0$. Since $\dim W \leq n$, Sadullaev's Theorem yields a Zariski open and dense subset $U_1 \subset \mathbf{L}(m,n)$ such that for any $L \in U_1$ there exist $r > 0$ and $M \in \mathbf{L}(m, m-n)$ such that $(L, M) \in \mathbf{L}(m,n)$ and for any $y \in W$,

$$|y| \geq r \;\Rightarrow\; |M(y)| \leq |L(y)|.$$

So,

$$|y| > r \;\Rightarrow\; |(L,M)(y)| = |L(y)|. \tag{12}$$

By Proposition 1.1(i), $U \cap U_1$ contains a Zariski open and dense subset of $\mathbf{L}(m,n)$. Let $L \in U \cap U_1$ and $M \in \mathbf{L}(m, m-n)$ be as above. Since $\mathcal{L}_\infty(f) > 0$, there exists $R_1 > 0$ such that $|f(x)| > r$ for any $x \in \mathbb{C}^n$ with $|x| > R_1$. Then, from (12),

$$|x| > R_1 \;\Rightarrow\; |(L,M) \circ f(x)| = |L \circ f(x)|.$$

Thus, $\mathcal{L}_\infty(L\circ f)=\mathcal{L}_\infty((L,M)\circ f)$. Since (L,M) is a linear automorphism, we have $\mathcal{L}_\infty((L,M)\circ f)=\mathcal{L}_\infty(f)$, so we obtain (6) in this case.

Consider the case $\mathcal{L}_\infty(f)\le 0$. If $0\notin W$, then $\mathcal{L}_\infty(f)=0$ and, by Lemma 2.2, for the generic $L\in\mathbf{L}(m,n)$, there exist $\delta>0$ and $r>0$ such that $|L\circ f(x)|\ge\delta$ for any $x\in\mathbb{C}^n$ with $|x|>r$. Thus $\mathcal{L}_\infty(L\circ f)=0$. This gives (6) in this case. Now, let $0\in W$ and let $C_0(W)$ be the tangent cone to W at $0\in\mathbb{C}^m$ (see [W], p. 510). By Sadullaev's Theorem, there exists a Zariski open and dense subset $U_2\subset\mathbf{L}(m,n)$ such that for any $L\in U_2$ there exists $M\in\mathbf{L}(m,m-n)$ such that $(L,M)\in\mathbf{L}(m,m)$ and for any $y\in C_0(W)$,

$$|M(y)|\le\tfrac{1}{2}|L(y)|.$$

Thus there exists $\varepsilon>0$ such that for any $y\in W$ with $|y|<\varepsilon$,

$$|M(y)|\le|L(y)|.$$

Hence, by Lemma 2.2, for $L\in U_2\cap U$ there exist $\delta>0$ and $r>0$ such that for any $x\in\mathbb{C}^n$ with $|x|>r$ we have

$$|L\circ f(x)|<\delta\ \Rightarrow\ |f(x)|<\varepsilon,$$

so, there exists $M\in\mathbf{L}(m,m-n)$ such that $(L,M)\in\mathbf{L}(m,m)$ and

$$|L\circ f(x)|<\delta\ \Rightarrow\ |(L,M)\circ f(x)|=|L\circ f(x)|.$$

Thus, since $\mathcal{L}_\infty(f)\le 0$, we have $\mathcal{L}_\infty(L\circ f)=\mathcal{L}_\infty((L,M)\circ f)$. Since (L,M) is a linear automorphism, it follows that $\mathcal{L}_\infty(f)=\mathcal{L}_\infty((L,M)\circ f)$, so we have (6) in this case.

This ends the proof.

3. Corollaries. From Theorem 2.1 we easily obtain the following corollary.

COROLLARY 3.1. *Let* $f=(f_1,\ldots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ *be a polynomial mapping, where* $m\ge n\ge 1$ *and* $\#f^{-1}(0)<\infty$, $d_j=\deg f_j>0$,

$$d_2\ge\ldots\ge d_m\ge d_1.$$

Then for the generic $L=(L_1,\ldots,L_n)\in\Delta^0(m,n)$ *we have*

$$\deg L_j\circ f=d_j\quad\textit{for } j=1,\ldots,n,\tag{13}$$

$$\#(L\circ f)^{-1}(0)<\infty,\tag{14}$$

$$\mathcal{L}_\infty(f)\ge\mathcal{L}_\infty(L\circ f).\tag{15}$$

Proof. By Proposition 1.1(ii) and Theorem 2.1, for the generic $L\in\Delta^0(m,n)$ we have (14) and (15). By the assumption on the degrees d_j and the definition of $\Delta^0(m,n)$, for the generic $L\in\Delta^0(m,n)$ we have (13). This ends the proof.

For $(d_1, \ldots, d_m) \in \mathbb{Z}^m$, define

$$B(n; d_1, \ldots, d_m) = \begin{cases} d_1 \ldots d_m & \text{if } m \le n, \\ d_1 \ldots d_{n-1} d_m & \text{if } m > n. \end{cases}$$

Proposition 1.10 of [K] gives immediately

KOLLÁR INEQUALITY. *Let $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping such that $\#(f^{-1}(0)) < \infty$. Let $d_j = \deg f_j$, $d_1 \ge \ldots \ge d_m > 0$. Then*

$$(*) \qquad \mathcal{L}_\infty(f) \ge d_m - B(n; d_1, \ldots, d_m).$$

For $m = n = 2$ this inequality was obtained by Chądzyński [C]. The proof of Proposition 1.10 in [K] is based on Proposition 4.1 of [K], where it is assumed that $m \le n$. Reduction of the case $m > n$ to the case $m \le n$ is not clearly explained. Corollary 3.1 gives us such a reduction:

COROLLARY 3.2. *Under the assumptions of the Kollár inequality, if $(*)$ holds for $m \le n$, then it also holds for $m > n$.*

Proof. For $m > n$ the inequality $(*)$ follows from Corollary 3.1 and from $(*)$ for $m \le n$.

Let us give some corollaries on proper polynomial mappings.

COROLLARY 3.3 ([P$_1$], for $m = n$). *Let $f = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping with $d_i = \deg f_i$, $d_1 \ge \ldots \ge d_m > 0$. If f is a proper mapping, then*

$$(16) \qquad \mathcal{L}_\infty(f) \ge d_m / B(n; d_1, \ldots, d_m).$$

If

$$(17) \qquad \mathcal{L}_\infty(f) = d_m / B(n; d_1, \ldots, d_m),$$

then f is injective, and so $\mathbb{C}[f_1, \ldots, f_m] = \mathbb{C}[x_1, \ldots, x_n]$.

Proof. From the assumption we have $m \ge n$. Since f is a proper mapping, it is well known that $\mathcal{L}_\infty(f) > 0$ (see [CK$_1$], Corollary 2). From Theorem 2.1 it follows that there exists $L = (L_1, \ldots, L_n) \in \Delta(m, n)$ such that

$$(18) \qquad \deg L_j \circ f = d_j, \quad j = 1, \ldots, n,$$

$$(19) \qquad \mathcal{L}_\infty(f) = \mathcal{L}_\infty(L \circ f).$$

Then $\mathcal{L}_\infty(L \circ f) > 0$, and so $L \circ f$ is a proper mapping. Thus, by (18) and Corollary 1.13 of [P$_1$] we have

$$\mathcal{L}_\infty(L \circ f) \ge \frac{1}{d_1 \ldots d_{n-1}} = \frac{d_m}{B(n; d_1, \ldots, d_m)}.$$

Now (19) gives (16).

Assume that (17) holds. Then, by (19),

$$\mathcal{L}_\infty(L\circ f)=\frac{d_n}{d_1\dots d_n},$$

so, by Corollary 1.13 of [P$_1$], $L\circ f:\mathbb{C}^n\to\mathbb{C}^n$ is a polynomial automorphism. This shows that f is injective, completing the proof.

Remark 3.1. Let $f=(f_1,\dots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ be a polynomial mapping with $d_i=\deg f_i$, $d_1\ge\dots\ge d_m>0$. If

$$\mathcal{L}_\infty(f)<\frac{d_m}{B(n;d_1,\dots,d_m)},$$

then $\mathcal{L}_\infty(f)\le 0$. Indeed, by Corollary 3.3, f is not a proper mapping. So, by Corollary 2 of [CK$_1$], $\mathcal{L}_\infty(f)\le 0$.

Corollary 3.4 ([C] for $n=m=2$, [P$_1$] for $m=n$). *Let $f=(f_1,\dots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ be a polynomial mapping with $d_i=\deg f_i$, $d_1\ge\dots\ge d_m>0$. If*

$$B(n;d_1,\dots,d_m)-d_m<\#f^{-1}(0)<\infty,$$

then f is a proper mapping. In particular the ring extension $\mathbb{C}[f_1,\dots,f_m]\subset\mathbb{C}[x_1,\dots,x_n]$ is integral.

Proof. From the assumption we have $m\ge n$. By Corollary 3.1, there exists $L=(L_1,\dots,L_n)\in\mathbf{L}(m,n)$ such that

$$\deg L_j\circ f=d_j,\quad j=1,\dots,n-1,\qquad \deg L_n\circ f=d_m, \tag{20}$$

$$\#f^{-1}(0)\le\#(L\circ f)^{-1}(0)<\infty, \tag{21}$$

$$\mathcal{L}_\infty(f)\ge\mathcal{L}_\infty(L\circ f). \tag{22}$$

By (21) and the assumption, $B(n;d_1,\dots,d_m)-d_m<\#(L\circ f)^{-1}(0)<\infty$, so $L\circ f:\mathbb{C}^n\to\mathbb{C}^n$ is a dominating mapping and for the generic $y\in\mathbb{C}^n$,

$$B(n;d_1,\dots,d_m)-d_m<\#(L\circ f)^{-1}(y)<\infty.$$

Hence, from (20) and Proposition 1.3 of [P$_1$] it follows that the mapping $L\circ f$ is proper. Thus, from Corollary 3.3, $\mathcal{L}_\infty(L\circ f)>0$. So, by (22), $\mathcal{L}_\infty(f)>0$. In consequence f is a proper mapping. The second assertion is an algebraic equivalent of the first one.

Remark 3.2. Let $f=(f_1,\dots,f_m):\mathbb{C}^n\to\mathbb{C}^m$ be a polynomial mapping with $d_i=\deg f_i$, $d_1\ge\dots\ge d_m>0$. From Corollaries 3.3 and 3.4 we see that if $\max_{j=1,\dots,m}d_j>1$ and

$$B(n;d_1,\dots,d_m)-d_m<\#f^{-1}(0)<\infty,$$

then

$$\mathcal{L}_\infty(f)>d_m/B(n;d_1,\dots,d_m).$$

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