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Stochastic differential equation driven by a pure-birth process

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Abstract. A generalization of the Poisson driven stochastic differential equation is considered. A sufficient condition for asymptotic stability of a discrete time-nonhomogeneous Markov process is proved.

1. Introduction. In this paper we consider a stochastic differential equation of the type

$$d\xi(t) = a(\xi(t))dt + b(\xi(t))dN(t) \tag{1.1}$$

with the initial condition $\xi(0) = \xi_0$, where $a, b : X \to X$ are deterministic functions defined on a separable Banach space $(X, \|\cdot\|)$ and $N(t)$ is a pure jump process with values in $\{0, 1, 2, \ldots\}$ (a so-called pure-birth process) and birth rates $(\lambda_n)_{n\geq 1}$. Denote by T_n the time when the process jumps from $n-1$ to n (birth time) and set $T_0 \equiv 0$. The sequence of random variables $\xi_n = \xi(T_n)$, where ξ is the solution of equation (1.1), is a time-nonhomogeneous Markov process because its one-step transition function may depend on n. It can be described by a stochastically perturbed dynamical system

$$\xi_n = S(\tau_n, \xi_{n-1})$$

where S is a suitable transformation and $\tau_n = T_n - T_{n-1}$ is an exponential random variable with parameter λ_n. The details are given in Section 3.

We are interested in the asymptotic behaviour of the sequence of distributions

$$\mu_n = \mathrm{prob}(\xi_n \in \cdot) \quad \text{for } n = 0, 1, \ldots$$

It was shown in [6] that if $N(t)$ is a Poisson process, so that (τ_n) is a sequence of independent exponential random variables with parameter λ,

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and the transformation S satisfies

$$\|S(t,x)-S(t,y)\|\le Le^{\beta t}\|x-y\| \quad \text{for } x,y\in X \text{ and } t\ge 0$$

with constant L and β such that $\lambda(L-1)+\beta<0$, then the sequence (μ_n) is weakly convergent to a unique μ_* which is independent of the initial measure μ_0.

Our main result strengthens the last condition to $\lambda\ln L+\beta<0$ and extends this statement to pure-birth processes.

Stochastically perturbed dynamical systems were studied by many authors under the assumption that (τ_n) are sequences of independent and identically distributed random variables. However, this assumption leads to time-homogeneous Markov processes. For an account of this subject we refer the reader to [8]. What we will need from this theory is a result from [8] (Theorem 1) which states, roughly speaking, that if such a system contracts on average then it has a stationary measure with finite first moment.

The outline of the paper is as follows. After preliminaries given in Section 2, in the next section we describe the solution of (1.1) by means of a transformation S. In Section 4 we derive a recurrence relation between the measures μ_n in terms of Markov operators. In the last section we state and prove our main result.

We denote by $\mathbb{Z}_+$ the set of nonnegative integers and set $\mathbb{R}_+=[0,\infty)$.

2. Preliminaries. Let $(\Omega,\Sigma,\mathrm{prob})$ be a probability space. We assume that $(\tau_n)_{n\ge1}$ is a sequence of independent exponential random variables with parameters $(\lambda_n)_{n\ge1}$ such that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty. \tag{2.1}$$

We set $T_0\equiv 0$, $T_n=T_{n-1}+\tau_n$ and define a pure-birth process by setting $N(t)=\max\{n\in\mathbb{Z}_+ : T_n\le t\}$. The random variables T_n are called the *jump points* and the λ_n the *birth rates* of $\{N(t)\}_{t\ge0}$. Note that condition (2.1) guarantees that $T_n\to\infty$ as $n\to\infty$ (see [2]). Hence for every $s>0$ and $\omega\in\Omega$ there is $n\in\mathbb{Z}_+$ such that $T_n(\omega)\le s<T_{n+1}(\omega)$.

Throughout the paper we assume that $(X,\|\cdot\|)$ is a separable Banach space. We denote by $\mathcal{B}_X$ the σ-algebra of Borel subsets of X and by $\mathcal{M}$ the family of all finite Borel measures on X. By $\mathcal{M}_1$ we denote the family of all $\mu\in\mathcal{M}$ such that $\mu(X)=1$. We call elements of the set $\mathcal{M}_1$ *distributions.* Further

$$\mathcal{M}_{\mathrm{s}}=\{\mu_1-\mu_2 : \mu_1,\mu_2\in\mathcal{M}\}$$

is the space of all finite signed measures.

3. Stochastic differential equation. Let $a, b : X \to X$ be continuous functions. Consider the stochastic differential equation

$$\xi(t) = \xi_0 + \int_0^t a(\xi(s-))\, ds + \int_0^t b(\xi(s-))\, dN(s), \tag{3.1}$$

where $\xi_0 : \Omega \to X$ is a random variable independent of the sequence (τ_n) and the second integral is the Lebesgue–Stieltjes integral equal to

$$\int_0^t b(\xi(s-))\, dN(s) = \sum_{n=1}^{\infty} b(\xi(T_n-))1_{(0,t]}(T_n).$$

By a *solution* of (3.1) we mean an X-valued right continuous process $\xi(t)$, $t \ge 0$, with left-hand limits, defined on the probability space $(\Omega, \mathcal{F}, \mathrm{prob})$ and such that for every $t \ge 0$ equation (3.1) is satisfied a.e.

From now on we assume that a is a Lipschitz map. We denote by $\pi : \mathbb{R}_+ \times X \to X$ the semigroup generated by the Cauchy problem

$$v'(t) = a(v(t)) \quad \text{for } t > 0 \tag{3.2}$$

with the initial condition

$$v(0) = y, \tag{3.3}$$

i.e. for every $y \in X$ the unique solution of (3.2), (3.3) is given by $v(t) = \pi(t, y)$ for $t \ge 0$.

As a result, the solution of (3.1) is given by

$$\xi(t) = \pi(t - T_{n-1}, \xi(T_{n-1})) \quad \text{for } t \in [T_{n-1}, T_n) \text{ and } n \in \mathbb{N},$$

where the random variables $\xi_n = \xi(T_n)$ satisfy the recurrence formula

$$\xi_n = q(\pi(T_n - T_{n-1}, \xi_{n-1})) \quad \text{for } n \in \mathbb{N}, \tag{3.4}$$

and the map $q : X \to X$ is given by

$$q(x) = x + b(x) \quad \text{for } x \in X. \tag{3.5}$$

We define the transformation $S : \mathbb{R}_+ \times X \to X$ by

$$S(t, x) = q(\pi(t, x)) \quad \text{for } x \in X \text{ and } t \ge 0.$$

Hence formula (3.4) can be rewritten as

$$\xi_n = S(\tau_n, \xi_{n-1}) \quad \text{for } n \in \mathbb{N}. \tag{3.6}$$

Note that the random variables T_n and τ_n are such that $T_n - T_{n-1} = \tau_n$. Since ξ_0 is independent of the sequence (τ_n), the random variables τ_n and ξ_{n-1} are independent for every $n \in \mathbb{N}$. Hence (ξ_n) is a Markov process.

4. Time-nonhomogeneous Markov process. In this section we derive a recurrence relation between μ_n and μ_{n-1} where

$$\mu_n(A) = \mathrm{prob}(\xi_n \in A) \quad \text{for } A \in \mathcal{B}_X,\ n \in \mathbb{Z}_+.$$

Fix $n \in \mathbb{N}$. Let $f : X \to \mathbb{R}$ be an arbitrary bounded Borel measurable function. The mathematical expectation of $f(\xi_n)$ is given by

$$\mathbf{E}(f(\xi_n)) = \int_X f(x)\,\mu_n(dx).$$

On the other hand, from (3.6) and the independence of the random variables τ_n and ξ_{n-1} it follows that

$$\mathbf{E}(f(\xi_n)) = \mathbf{E}(f(S(\tau_n, \xi_{n-1}))) = \int_X \Big[\int_0^\infty f(S(t,x))\lambda_n e^{-\lambda_n t}\,dt\Big]\,\mu_{n-1}(dx).$$

Hence for $A \in \mathcal{B}_X$ and $f = 1_A$, we obtain

$$\mu_n(A) = \int_X \Big[\int_0^\infty 1_A(S(t,x))\lambda_n e^{-\lambda_n t}\,dt\Big]\,\mu_{n-1}(dx), \tag{4.1}$$

which is the desired recurrence relation between μ_n and μ_{n-1}. Define an operator P_n by

$$P_n\mu(A) = \int_X \Big[\int_0^\infty 1_A(S(t,x))\lambda_n e^{-\lambda_n t}\,dt\Big]\,\mu(dx).$$

Then (4.1) may be rewritten as

$$\mu_n = P_n\mu_{n-1} \quad \text{for } n \in \mathbb{N}. \tag{4.2}$$

Clearly, P_n is a Markov operator in the space $\mathcal{M}_{\mathrm{s}}$: it is linear and maps each distribution to a distribution ([7]).

Define a linear operator $U_n : C(X) \to C(X)$ by

$$U_n f(x) = \int_0^\infty \lambda_n f(S(t,x))e^{-\lambda_n t}\,dt \quad \text{for } f \in C(X),$$

where $C(X)$ denotes the space of all bounded continuous functions on $(X, \|\cdot\|)$ with the supremum norm $\|\cdot\|_C$. Note that U_n is a contraction on $C(X)$, i.e. $\|U_n f\|_C \le \|f\|_C$ for every $f \in C(X)$.

For $f \in C(X)$ and $\mu \in \mathcal{M}_{\mathrm{s}}$ we adopt the scalar product notation

$$\langle f, \mu\rangle = \int_X f(x)\,\mu(dx).$$

It can be easily shown that P_n is the unique Markov operator satisfying

$$\langle U_n f, \mu\rangle = \langle f, P_n\mu\rangle \quad \text{for } f \in C(X) \text{ and } \mu \in \mathcal{M}_{\mathrm{s}},$$

so we call U_n the *dual operator* to P_n.

Denoting by $P(n,m)$ the composition of the Markov operators

$$P(n,m) = P_n \circ \ldots \circ P_{m+1} \quad \text{for } n > m,\ n, m \in \mathbb{Z}_+,$$

and letting $P(n,m)$ be the identity for $n=m$, we obtain the *chain rule*

$$P(n,m) = P(n,k)P(k,m) \quad \text{for } n \ge k \ge m. \tag{4.3}$$

From (4.2) it now follows that

$$\mu_n = P(n,m)\mu_m \quad \text{for } n \ge m, n,\ m \in \mathbb{Z}_+.$$

Note that $P(n,m)$ is a Markov operator with dual $U(m,n) = U_{m+1}\circ\ldots\circ U_n$.

Recall that a sequence (μ_n) of distributions is *weakly convergent* to a distribution μ if $\lim_{n\to\infty}\langle f,\mu_n\rangle = \langle f,\mu\rangle$ for all $f \in C(X)$.

Since the sequence $(\mu_n)_{n\ge1}$ is completely determined by $(P_n)_{n\ge1}$ we say that the Markov process (3.6), or equivalently (P_n), is *asymptotically stable* if there exists a unique measure $\mu_* \in \mathcal{M}_1$ such that for every $\mu \in \mathcal{M}_1$ and $m \in \mathbb{Z}_+$ the sequence $(P(n,m)\mu)_{n\ge m}$ is weakly convergent to μ_*.

REMARK 1. From the chain rule (4.3) it follows that if the sequence $(P(n,m)\mu)_{n\ge m}$ is weakly convergent to μ_* for all $\mu \in \mathcal{M}_1$ and for all but finitely many $m \in \mathbb{Z}_+$, say $m \ge k$, then the Markov process (P_n) is asymptotically stable. In fact, for $m < k$ and $\mu \in \mathcal{M}_1$ we have $P(n,m)\mu = P(n,k)\mu_0$ for $n \ge k$, where $\mu_0 = P(k,m)\mu$ and $\mu_0 \in \mathcal{M}_1$. Since the sequence $(P(n,k)\mu_0)_{n\ge k}$ is weakly convergent to μ_*, the sequence $(P(n,m)\mu)_{n\ge m}$ is also convergent to μ_*.

5. Asymptotic stability. In our study of asymptotic stability of (3.6) we make the following assumptions:

(a_1) The map q defined by (3.5) is such that

$$\|q(x) - q(y)\| \le L\|x-y\| \quad \text{for } x,y \in X,$$

where $L \ge 0$ is a constant.

(a_2) The solution π of the Cauchy problem (3.2) is such that

$$\|\pi(t,x) - \pi(t,y)\| \le e^{\beta t}\|x-y\| \quad \text{for } t \ge 0,\ x,y \in X,$$

where $\beta \in \mathbb{R}$.

(a_3) For some initial value $x_0 \in X$ the solution $\pi(t,x_0)$ of (3.2) is uniformly bounded, i.e.

$$\sup_{t\ge0}\|\pi(t,x_0)\| < \infty.$$

(a_4) The sequence $(\lambda_n)_{n\ge1}$ of birth rates converges to $\lambda \in [0,\infty]$ such that

$$\ln L + \beta/\lambda < 0. \tag{5.1}$$

Condition (5.1) can be rewritten in the form $\lambda \ln L + \beta < 0$. Thus, for $\lambda = 0$ we have $\beta < 0$, whereas for $\lambda = \infty$ we have $\beta/\infty = 0$ and $\ln L < 0$.

We are now ready to state the main result of the paper.

THEOREM 1. *Assume* (a_1) *to* (a_4). *Then the Markov process* (P_n) *is asymptotically stable.*

REMARK 2. Under assumptions (a_1) to (a_3) condition (5.1) is optimal. Consider the following example. Let $a(x) = 0$ and $b(x) = -2x$ for $x \in X$. Then $L = 1$ and $\beta = 0$, so (5.1) becomes an equality. For every initial random variable ξ_0 we obtain $\xi_n = -\xi_{n-1}$ for every $n \in \mathbb{N}$. Thus this process is not asymptotically stable.

In order to prepare the proof of the theorem, we will look more closely at the concept of weak convergence of measures and turn $\mathcal{M}_1$ as well as its subsets into metric spaces.

We start from the following characterization of weak convergence.

PROPOSITION 1. *Let* $\gamma \in (0,1]$ *and* $\mu_n \in \mathcal{M}_1$ *for each* $n \geq 1$. *Given* $\mu \in \mathcal{M}_1$, *the following are equivalent*:

(i) $\lim_{n\to\infty}\langle f, \mu_n\rangle = \langle f, \mu\rangle$ *for every* $f \in C(X)$.

(ii) $\lim_{n\to\infty}\langle f, \mu_n\rangle = \langle f, \mu\rangle$ *for every* $f \in \mathcal{F}_\gamma$ *where* $\mathcal{F}_\gamma$ *is the set of all* $f \in C(X)$ *such that* $\|f\|_C \leq 1$ *and* $|f(x) - f(y)| \leq \|x - y\|^\gamma$ *for* $x, y \in X$.

Proof. Let C be a closed subset of X. For $k \in \mathbb{N}$ and $x \in X$ define

$$f_k(x) = \max\{0, 1 - k^\gamma \varrho(x, C)\}$$

where $\varrho(x, C) = \inf\{\|x - z\|^\gamma : z \in C\}$. Since

$$|\varrho(x, C) - \varrho(y, C)| \leq \|x - y\|^\gamma \quad \text{for } x, y \in X,$$

we see that $k^{-\gamma} f_k \in \mathcal{F}_\gamma$ for each $k \in \mathbb{N}$. Moreover, each f_k has the value 1 on C and has the value 0 at points whose distance from C is greater than $1/k$. Hence for each $x \in X$, $f_k(x) \downarrow 1_C(x)$ as $k \to \infty$. By the Lebesgue bounded convergence theorem and (ii),

$$\mu(C) = \lim_{k\to\infty}\langle f_k, \mu\rangle = \lim_{k\to\infty}\lim_{n\to\infty}\langle f_k, \mu_n\rangle \geq \limsup_{n\to\infty}\mu_n(C).$$

Since C was an arbitrary closed subset of X, we conclude that (i) holds by the standard characterization of weak convergence (see [9], Theorem 1.1.1). ∎

Let $\gamma \in (0,1]$. The equivalence of (i) and (ii) implies that for any $\mu_1, \mu_2 \in \mathcal{M}_1$, if $\langle f, \mu_1\rangle = \langle f, \mu_2\rangle$ for $f \in \mathcal{F}_\gamma$, then $\langle f, \mu_1\rangle = \langle f, \mu_2\rangle$ for all $f \in C(X)$ and consequently $\mu_1 = \mu_2$. This allows us to define a metric on $\mathcal{M}_1$ by

$$d_\gamma^{\mathcal{F}}(\mu_1, \mu_2) = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{F}_\gamma\} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1.$$

Note that we always have $d_\gamma^{\mathcal{F}}(\mu_1, \mu_2) \leq 2$. This metric has the property that convergence in the metric space $(\mathcal{M}_1, d_\gamma^{\mathcal{F}})$ is equivalent to weak convergence of distributions, the converse implication of this equivalence being a consequence of Corollary 1.1.2 from [9]. For $\gamma = 1$ it is the so-called *Fortet–Mourier metric* (see [3, 6]).

We introduce another distance on $\mathcal{M}_1$ by

$$d_\gamma(\mu_1, \mu_2) = \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{K}_\gamma\} \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1,$$

where $\mathcal{K}_\gamma$ is the set of all $f \in C(X)$ such that $|f(x) - f(y)| \le \|x - y\|^\gamma$ for $x, y \in X$. This quantity is always defined but for some measures it may be infinite. It is easy to check that the function d_γ is finite for elements of

$$\mathcal{M}_1^\gamma = \Big\{\mu \in \mathcal{M}_1 : \int_X \|x - x_0\|^\gamma \, \mu(dx) < \infty\Big\},$$

where x_0 is as in (a_3), and defines a metric on this set. For $\gamma = 1$ it is the so-called *Vasershtein metric* ([11]) and is frequently used in the theory of fractals (see [1, 5]). Note that the definition of the set $\mathcal{M}_1^\gamma$ is independent of the particular choice of the point x_0.

Observe that for all $\mu_1, \mu_2 \in \mathcal{M}_1$ we have

$$d_\gamma^{\mathcal{F}}(\mu_1, \mu_2) \le d_\gamma(\mu_1, \mu_2). \tag{5.2}$$

Thus, convergence in the metric space $(\mathcal{M}_1, d_\gamma)$ implies weak convergence of distributions. In general, however, the converse is not true. For a deeper discussion of this problem for the case $\gamma = 1$ we refer the reader to [4]. In particular, our Proposition 1 is an analogue of Theorem 3.8 of [4].

The following proposition provides a criterion for the asymptotic stability of (P_n) in terms of the metric space $(\mathcal{M}_1^\gamma, d_\gamma)$.

PROPOSITION 2. *Suppose that there exist* $\gamma \in (0, 1]$, $\mu_* \in \mathcal{M}_1^\gamma$ *and* $k \in \mathbb{N}$ *such that* $P_n(\mathcal{M}_1^\gamma) \subseteq \mathcal{M}_1^\gamma$ *and* $U_n(\mathcal{K}_\gamma) \subseteq \mathcal{K}_\gamma$ *for* $n \ge k$, *and*

$$\lim_{n\to\infty} d_\gamma(P(n, m)\mu, \mu_*) = 0 \quad \textit{for } \mu \in \mathcal{M}_1^\gamma,\ m \ge k. \tag{5.3}$$

Then the Markov process (P_n) *is asymptotically stable.*

Proof. We first show that $\mathcal{M}_1^\gamma$ is dense in $(\mathcal{M}_1, d_\gamma^{\mathcal{F}})$. Let $\mu \in \mathcal{M}_1$. Every distribution with bounded support belongs to $\mathcal{M}_1^\gamma$ and $\mu(B(x_0, n)) \to 1$ as $n \to \infty$. Therefore we can define measures from $\mathcal{M}_1^\gamma$ by

$$\mu_n(A) = \frac{\mu(A \cap B(x_0, n))}{\mu(B(x_0, n))} \quad \text{for } A \in \mathcal{B}_X$$

for sufficiently large $n \in \mathbb{N}$. Since $\mu_n(A) = 0$ for $A \subseteq X \setminus B(x_0, n)$ and $|f| \le 1$ for $f \in \mathcal{F}_\gamma$, we have

$$\begin{aligned}
d_\gamma^{\mathcal{F}}(\mu_n, \mu) &\le \sup\{|\langle f 1_{B(x_0,n)}, \mu_n - \mu\rangle| + |\langle f 1_{X\setminus B(x_0,n)}, \mu\rangle| : f \in \mathcal{F}_\gamma\} \\
&\le \sup\Big\{\Big(\frac{1}{\mu(B(x_0, n))} - 1\Big)|\langle f 1_{B(x_0,n)}, \mu\rangle| + \langle |f| 1_{X\setminus B(x_0,n)}, \mu\rangle : f \in \mathcal{F}_\gamma\Big\} \\
&\le 1 - \mu(B(x_0, n)) + \mu(X \setminus B(x_0, n)).
\end{aligned}$$

Consequently, $d_\gamma^{\mathcal{F}}(\mu_n, \mu) \to 0$ as $n \to \infty$ and the desired conclusion holds.

Fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$. There exists a measure $\mu_\varepsilon \in \mathcal{M}_1^\gamma$ such that $d_\gamma^{\mathcal{F}}(\mu, \mu_\varepsilon) < \varepsilon$. Let $n \geq m \geq k$. By hypothesis, $U(m,n)(\mathcal{K}_\gamma) \subseteq \mathcal{K}_\gamma$. Since the dual operator $U(m,n)$ is a contraction, we also have $U(m,n)(\mathcal{F}_\gamma) \subseteq \mathcal{F}_\gamma$. Hence

$$\begin{aligned} d_\gamma^{\mathcal{F}}(P(n,m)\mu, P(n,m)\mu_\varepsilon) &= \sup\{|\langle U(m,n)f, \mu - \mu_\varepsilon\rangle| : f \in \mathcal{F}_\gamma\} \\ &\leq \sup\{|\langle f, \mu - \mu_\varepsilon\rangle| : f \in \mathcal{F}_\gamma\} = d_\gamma^{\mathcal{F}}(\mu, \mu_\varepsilon) < \varepsilon. \end{aligned}$$

From this and (5.2) it follows that

$$d_\gamma^{\mathcal{F}}(P(n,m)\mu, \mu_*) \leq \varepsilon + d_\gamma(P(n,m)\mu_\varepsilon, \mu_*).$$

We conclude from (5.3) that

$$\limsup_{n\to\infty} d_\gamma^{\mathcal{F}}(P(n,m)\mu, \mu_*) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof is complete. ■

The next proposition provides a criterion for checking condition (5.3). In what follows, a measure $\mu_* \in \mathcal{M}_1^\gamma$ is called d_γ-*attractive* if it satisfies (5.3) for some k.

PROPOSITION 3. *Suppose that there exist $\gamma \in (0,1]$ and $\alpha \in (0,1)$ such that $P_n(\mathcal{M}_1^\gamma) \subseteq \mathcal{M}_1^\gamma$ and*

$$d_\gamma(P_n\mu_1, P_n\mu_2) \leq \alpha d_\gamma(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1^\gamma \tag{5.4}$$

for all but finitely many $n \in \mathbb{N}$. Then a measure $\mu_* \in \mathcal{M}_1^\gamma$ is d_γ-attractive if and only if

$$\lim_{n\to\infty} d_\gamma(P_n\mu_*, \mu_*) = 0. \tag{5.5}$$

Proof. Let k be such that $P_n(\mathcal{M}_1^\gamma) \subseteq \mathcal{M}_1^\gamma$ for $n \geq k$ and (5.4) is satisfied for $n \geq k$.

Suppose first that (5.5) holds. From (5.4) it follows that

$$\lim_{n\to\infty} d_\gamma(P(n,m)\mu_1, P(n,m)\mu_2) = 0 \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1^\gamma$$

for any $m \geq k$. Since $P(n,k) = P(n,m) \circ P(m,k)$, it remains to verify that $d_\gamma(P(n,k)\mu_*, \mu_*) \to 0$ as $n \to \infty$. By (5.4) and the triangle inequality

$$d_\gamma(P(n+1,k)\mu_*, \mu_*) \leq \alpha d_\gamma(P(n,k)\mu_*, \mu_*) + d_\gamma(P_{n+1}\mu_*, \mu_*)$$

for sufficiently large n. Hence and from (5.5) it follows that the sequence $(d_\gamma(P(n,k)\mu_*, \mu_*))$ is bounded and that

$$\limsup_{n\to\infty} d_\gamma(P(n+1,k)\mu_*, \mu_*) \leq \alpha \limsup_{n\to\infty} d_\gamma(P(n,k)\mu_*, \mu_*).$$

Since $\alpha < 1$, $\limsup_{n\to\infty} d_\gamma(P(n,k)\mu_*, \mu_*) = 0$, as required.

For the converse note that

$$d_\gamma(P_{n+1}\mu_*, \mu_*) \leq \alpha d_\gamma(\mu_*, P(n,k)\mu_*) + d_\gamma(P(n+1,k)\mu_*, \mu_*) \quad \text{for } n \geq k.$$

Hence condition (5.5) holds. ■

Before we start the proof of Theorem 1, we need the following lemma which will be extensively used in what follows.

Assumptions (a_1) and (a_2) imply that the transformation $S = q \circ \pi$ satisfies

$$\|S(t,x) - S(t,y)\| \le Le^{\beta t}\|x-y\| \quad \text{for } x,y \in X \text{ and } t \ge 0. \tag{5.6}$$

LEMMA 1. *Let $\gamma \in (0,1]$ and $n \in \mathbb{N}$. Suppose that $\lambda_n > \gamma\beta$. Then for every $f \in \mathcal{K}_\gamma$,*

$$|U_n f(x) - U_n f(y)| \le \frac{L^\gamma \lambda_n}{\lambda_n - \gamma\beta}\,\|x-y\|^\gamma \quad \textit{for } x,y \in X. \tag{5.7}$$

Moreover, $P_n(\mathcal{M}_1^\gamma) \subseteq \mathcal{M}_1^\gamma$ and

$$d_\gamma(P_n\mu_1, P_n\mu_2) \le \frac{L^\gamma \lambda_n}{\lambda_n - \gamma\beta}\, d_\gamma(\mu_1,\mu_2) \quad \textit{for } \mu_1,\mu_2 \in \mathcal{M}_1^\gamma. \tag{5.8}$$

Proof. Condition (5.7) and the P_n-invariance of $\mathcal{M}_1^\gamma$ follow immediately from (5.6) and the definition of U_n. For the proof of (5.8) set $c_n = L^\gamma\lambda_n/(\lambda_n - \gamma\beta)$. From (5.7) it follows that $U_n(f/c_n) \in \mathcal{K}_\gamma$ for every $f \in \mathcal{K}_\gamma$. Hence

$$\begin{aligned} d_\gamma(P_n\mu_1, P_n\mu_2) &= c_n \sup\left\{\left|\left\langle \frac{1}{c_n}U_n f, \mu_1 - \mu_2\right\rangle\right| : f \in \mathcal{K}_\gamma\right\} \\ &\le c_n \sup\{|\langle f, \mu_1 - \mu_2\rangle| : f \in \mathcal{K}_\gamma\} = c_n d_\gamma(\mu_1,\mu_2). \ \blacksquare \end{aligned}$$

REMARK 3. Under the assumptions of Lemma 1, the set $\mathcal{K}_\gamma$ is U_n-invariant whenever $L^\gamma\lambda_n/(\lambda_n - \gamma\beta) \le 1$.

We now turn to the proof of the theorem. In the following three propositions we consider the cases $\lambda = 0$, $\lambda \in (0,\infty)$, and $\lambda = \infty$ respectively. In each case we show that all assumptions of Proposition 2 are satisfied, which proves the theorem.

PROPOSITION 4. *Let $\lambda = 0$. Then there is a point $z \in X$ such that $z = \pi(t,z)$ for all $t \ge 0$ and the point measure $\delta_{q(z)}$ is d_1-attractive.*

Proof. For $\lambda = 0$ assumption (5.1) reduces to $\beta < 0$, in which case the semigroup $\pi(t,\cdot)$ has a fixed point z. Hence for all $n \in \mathbb{N}$ and $f \in C(X)$ we have $U_n f(z) = f(q(z))$. For $\gamma = 1$ and $\beta < 0$ we always have $\lambda_n > \gamma\beta$. Further by Lemma 1 it follows that

$$|U_n f(q(z)) - f(q(z))| = |U_n f(q(z)) - U_n f(z)| \le \frac{L\lambda_n}{\lambda_n - \beta}\,\|q(z) - z\|$$

for $f \in \mathcal{K}_1$ and all $n \in \mathbb{N}$. Since $\lambda_n \to 0$ as $n \to \infty$, this implies that $L\lambda_n/(\lambda_n - \beta)$ and $d_1(P_n\delta_{q(z)}, \delta_{q(z)})$ converge to 0 as $n \to \infty$. Consequently, by Proposition 3 the measure $\delta_{q(z)}$ is d_1-attractive. $\blacksquare$

Proposition 5. *Let $\lambda \in (0,\infty)$. Then there exist $\gamma \in (0,1)$ and a d_γ-attractive measure μ_* such that*

$$\int_X \int_0^\infty \lambda f(S(t,x))e^{-\lambda t}\,dt\,\mu_*(dx) = \int_X f(x)\,\mu_*(dx) \quad \textit{for } f \in C(X). \tag{5.9}$$

Proof. First observe that

$$\lim_{r\to 0}\left(\frac{L^r\lambda}{\lambda - r\beta}\right)^{1/r} = Le^{\beta/\lambda}$$

and by assumption (5.1) this value is strictly less than 1. Therefore, there exists $\gamma \in (0,1)$ such that

$$\lambda > \gamma\beta \quad \text{and} \quad \frac{L^\gamma\lambda}{\lambda-\gamma\beta} < 1.$$

Since λ_n tends to λ as $n\to\infty$, there are $k\in\mathbb{N}$ and $\alpha<1$ such that

$$\lambda_n > \gamma\beta \quad \text{and} \quad \frac{L^\gamma\lambda_n}{\lambda_n-\gamma\beta} \le \alpha \quad \text{for } n\ge k.$$

Let P be a Markov operator of the form

$$P\mu(A) = \int_X \Big[\int_0^\infty \lambda 1_A(S(t,x))e^{-\lambda t}\,dt\Big]\,\mu(dx) \quad \text{for } A\in\mathcal{B}_X,\ \mu\in\mathcal{M}_{\mathrm{s}}.$$

As in Lemma 1 we obtain $P(\mathcal{M}_1^\gamma)\subseteq\mathcal{M}_1^\gamma$.

Fix $\mu\in\mathcal{M}_1^\gamma$ and $n\ge k$. Since $P_n\mu(X) = P\mu(X) = 1$, for any constant c and any $f\in\mathcal{K}_\gamma$ we have

$$\langle f, P_n\mu - P\mu\rangle = \langle f-c, P_n\mu - P\mu\rangle.$$

As a result,

$$d_\gamma(P_n\mu, P\mu) = \sup\{|\langle f, P_n\mu - P\mu\rangle| : f(x_0) = 0,\ f\in\mathcal{K}_\gamma\}.$$

Let $f\in\mathcal{K}_\gamma$ be such that $f(x_0)=0$. Put $h_n(t) = |\lambda_n e^{-\lambda_n t} - \lambda e^{-\lambda t}|$ for $t\ge 0$. Then

$$|U_nf(x) - Uf(x)| \le \int_0^\infty |f(S(t,x)) - f(x_0)|h_n(t)\,dt,$$

where U is the dual to P. From (5.6) it follows that

$$\begin{aligned}|U_nf(x) - Uf(x)| \le \|x-x_0\|^\gamma \int_0^\infty L^\gamma e^{\gamma\beta t}h_n(t)\,dt \\ + \int_0^\infty \|S(t,x_0)-x_0\|^\gamma h_n(t)\,dt.\end{aligned}$$

Since $\mu\in\mathcal{M}_1^\gamma$, q is a Lipschitz map, and $\pi(t,x_0)$ is uniformly bounded, there are constants c_1 and c_2 such that

$$d_\gamma(P_n\mu, P\mu) \le \int_0^\infty (c_1 e^{\gamma\beta t} + c_2) h_n(t)\, dt.$$

By the Lebesgue bounded convergence theorem the right-hand side tends to 0 as $n \to \infty$. Therefore

$$\lim_{n\to\infty} d_\gamma(P_n\mu, P\mu) = 0.$$

In view of Proposition 3, it remains to verify that the operator P has a fixed point in $\mathcal{M}_1^\gamma$. To this end, we make use of a result from [8].

Let $(\eta_n)_{n\ge1}$ be an i.i.d. sequence of exponentially distributed random variables with parameter λ defined on a common probability space. Consider the following stochastically perturbed dynamical system:

$$\zeta_n = S(\eta_n, \zeta_{n-1}) \quad \text{for } n \in \mathbb{N}$$

with $\zeta_0 \equiv x_0$. Let $\varrho(x,y) = \|x-y\|^\gamma$ for $x, y \in X$. Note that (X, ϱ) is a complete separable metric space with the same family of Borel sets as in $(X, \|\cdot\|)$ and the same family of continuous and bounded functions. By (5.6),

$$\varrho(S(t,x), S(t,y)) \le L(t)\varrho(x,y) \quad \text{for } x, y \in X,\ t \ge 0$$

where $L(t) = Le^{\gamma\beta t}$. Since $\mathbf{E}L(\eta_1) < 1$ and $\mathbf{E}\varrho(x_0, S(x_0, \eta_1)) < \infty$, we infer from Theorem 1 and Remark 1 of [8] that there exists a measure $\mu_* \in \mathcal{M}_1$ such that

$$\int_X f(x)\, \mu_*(dx) = \int_X \int_{\mathbb{R}_+} f(S(t,x))\lambda e^{-\lambda t}\, dt\, \mu_*(dx)$$

for every bounded continuous function f on (X, ϱ) and

$$\int_X \varrho(x_0, x)\, \mu_*(dx) < \infty.$$

Consequently, $\mu_* \in \mathcal{M}_1^\gamma$ and μ_* satisfies (5.9). The duality between U and P yields

$$\langle f, \mu_*\rangle = \langle Uf, \mu_*\rangle = \langle f, P\mu_*\rangle$$

for $f \in C(X)$, which completes the proof. ∎

PROPOSITION 6. *Let $\lambda = \infty$. Then there is a point $z \in X$ such that $z = q(z)$ and the point measure δ_z is d_1-attractive.*

Proof. In this case we have $L < 1$ and consequently the transformation q has a fixed point z. Moreover, since $\lambda_n \to \infty$ as $n \to \infty$, there are $k \in \mathbb{N}$ and $\alpha < 1$ such that

$$\lambda_n > \beta \quad \text{and} \quad \frac{L\lambda_n}{\lambda_n - \beta} \le \alpha \quad \text{for } n \ge k.$$

In view of Proposition 3 it remains to verify that $d_1(P_n\delta_z, \delta_z) \to 0$ as $n \to \infty$.

For every $f \in \mathcal{K}_1$ we have

$$|U_n f(z) - f(z)| \le \int_0^\infty |f(S(t,z)) - f(z)|\lambda_n e^{-\lambda_n t}\,dt \le \int_0^\infty \|S(t,z) - z\|\lambda_n e^{-\lambda_n t}\,dt.$$

The last inequality implies

$$d_1(P_n\delta_z, \delta_z) \le \int_0^\infty \|S(t/\lambda_n, z) - z\| e^{-t}\,dt.$$

From (5.6) and the fact that $\pi(t, x_0)$ is uniformly bounded and q is a Lipschitz map it follows that

$$\|S(t/\lambda_n, z) - z\| \le L e^{\beta t/\lambda_n}\|z - x_0\| + c,$$

where c is a constant. Since β/λ_n tends to 0 as $n \to \infty$ and for every positive t the sequence $(S(t/\lambda_n, z))_{n\ge1}$ converges to $q(z)$, the conclusion follows from the Lebesgue bounded convergence theorem. ∎

REMARK 4. The paper [10] is devoted to the problem of asymptotic stability of discrete time-nonhomogeneous processes, but methods developed there are not applicable to the case presented here.

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