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On generalized absolute Cesàro summability factors

by A. Nihal Tuncer (Kayseri)

Abstract. Using δ-quasi-monotone and any almost increasing sequences we prove a theorem on $|C,\alpha,\beta;\delta|_k$ summability factors of infinite series, which generalizes a theorem of Mazhar [7] on $|C,1|_k$ summability factors.

1. Introduction. A sequence (b_n) of positive numbers is said to be *quasi-monotone* if $n\Delta b_n \geq -\gamma b_n$ for some γ, and is said to be δ-*quasi-monotone* if $b_n \to 0, b_n > 0$ ultimately (that is, $b_n > 0$ for $n > n_1$ and n_1 depends on the sequence (b_n)) and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [2]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α and t_n^α the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, i.e.,

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu a_\nu, \tag{2}$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = O(n^\alpha),\ \alpha > -1, \quad A_0^\alpha = 1,\ A_{-n}^\alpha = 0 \ \text{ for } n > 0. \tag{3}$$

The series $\sum a_n$ is said to be *summable* $|C,\alpha|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k < \infty. \tag{4}$$

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [6]) condition (4) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty. \tag{5}$$

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The series $\sum a_n$ is said to be *summable* $|C,\alpha,\beta;\delta|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{\beta(\delta k+k-1)-k}|t_n^\alpha|^k < \infty, \tag{6}$$

where $\delta \geq 0$ and β is a real number.

Mazhar [7] proved the following theorem for $|C,1|_k$ summability factors.

THEOREM A. *Let* $\lambda_n \to 0$ *as* $n \to \infty$. *Suppose that there exists a sequence* (B_n) *of numbers which is* δ*-quasi-monotone with* $\sum n\delta_n \log n < \infty$, $\sum B_n \log n$ *is convergent and* $|\Delta\lambda_n| \leq |B_n|$ *for all* n. *If*

$$\sum_{n=1}^{m} \frac{1}{n}|t_n^1|^k = O(\log m) \qquad \textit{as } m \to \infty, \tag{7}$$

where (t_n^1) *is the* n*th* $(C,1)$ *mean of the sequence* (na_n), *then the series* $\sum a_n\lambda_n$ *is summable* $|C,1|_k$, $k \geq 1$.

REMARK. Note that in this theorem, the condition "$\sum nB_n \log n$ is convergent" could replace the conditions "$\sum n\delta_n \log n < \infty$ and $\sum B_n \log n$ is convergent."

2. The aim of this paper is to generalize Theorem A to $|C,\alpha,\beta;\delta|_k$ summability factors under weaker conditions by using almost increasing sequences. We now define this concept. A positive sequence (d_n) is said to be *almost increasing* if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq d_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $d_n = ne^{(-1)^n}$. Since $\log n$ is increasing in Theorem A, we are weakening the hypotheses of the theorem by replacing that increasing sequence by any almost increasing sequence.

Now, we shall prove the following theorem.

THEOREM. *Let* (X_n) *be an almost increasing sequence and* $\lambda_n \to 0$ *as* $n \to \infty$. *Let* $\delta \geq 0$, $k \geq 1$, $0 < \alpha \leq 1$, *and* β *be a real number such that* $-\beta(\delta k + k - 1) + k + \alpha k > 1$. *Suppose that there exists a sequence* (B_n) *of numbers which is* δ*-quasi-monotone*, $\sum nB_nX_n$ *is convergent and* $|\Delta\lambda_n| \leq |B_n|$ *for all* n, *where* $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. *If the sequence* (w_n^α) *defined by*

$$w_n^\alpha = \max[|t_\nu^\alpha| : 1 \leq \nu \leq n] \tag{8}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}(w_n^\alpha)^k = O(X_m) \qquad \textit{as } m \to \infty, \tag{9}$$

then the series $\sum a_n\lambda_n$ *is summable* $|C,\alpha,\beta;\delta|_k$.

If we take $X_n = \log n$, $\beta = 1$, $\delta = 0$ and $\alpha = 1$ in this Theorem, then we get Theorem A.

We need the following lemma.

LEMMA 1 ([3]). *If* $0 < \alpha \le 1$ *and* $1 \le \nu \le n$, *then*

$$\Big|\sum_{p=0}^{\nu} A_{n-p}^{\alpha-1} a_p\Big| \le \max_{1\le m\le \nu}\Big|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p\Big|, \tag{10}$$

where A_n^α *is as in* (3).

Proof of the Theorem. Let (T_n^α) be the nth (C,α) mean of the sequence $(na_n\lambda_n)$, where $0 < \alpha \le 1$. Then, by (2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\nu a_\nu \lambda_\nu.$$

By Abel's transformation, we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1}\Delta\lambda_\nu \sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\nu a_\nu,$$

so that, making use of Lemma 1, we get

$$\begin{aligned} |T_n^\alpha| &\le \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1}|\Delta\lambda_\nu|\Big|\sum_{p=1}^{\nu} A_{n-p}^{\alpha-1} p a_p\Big| + \frac{|\lambda_n|}{A_n^\alpha}\Big|\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\nu a_\nu\Big| \\ &\le \frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1} A_\nu^\alpha w_\nu^\alpha |\Delta\lambda_\nu| + |\lambda_n| w_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha, \qquad \text{say.} \end{aligned}$$

Since $|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \le 2^k(|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k)$, to complete the proof it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{\beta(\delta k+k-1)-k}|T_{n,r}^\alpha|^k < \infty \qquad \text{for } r = 1,2,$$

by (5). Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we have

$$\begin{aligned} \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}|T_{n,1}^\alpha|^k &\le \sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k}\Big\{\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n-1} A_\nu^\alpha w_\nu^\alpha|\Delta\lambda_\nu|\Big\}^k \\ &= O(1)\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k-\alpha k}\Big\{\sum_{\nu=1}^{n-1}(A_\nu^\alpha)^k (w_\nu^\alpha)^k |B_\nu|\Big\}\Big\{\sum_{\nu=1}^{n-1}|B_\nu|\Big\}^{k-1} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1} n^{\beta(\delta k+k-1)-k-\alpha k}\sum_{\nu=1}^{n-1}\nu^{\alpha k}(w_\nu^\alpha)^k|B_\nu| \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}(w_\nu^\alpha)^k|B_\nu|\sum_{n=\nu+1}^{m+1}\frac{1}{n^{-\beta(\delta k+k-1)+k+\alpha k}} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\alpha k}(w_\nu^\alpha)^k|B_\nu|\int_{\nu}^{\infty}\frac{dx}{x^{-\beta(\delta k+k-1)+k+\alpha k}} \\
&= O(1)\sum_{\nu=1}^{m}\nu|B_\nu|\nu^{\beta(\delta k+k-1)-k}(w_\nu^\alpha)^k \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu|B_\nu|)|\sum_{r=1}^{\nu}r^{\beta(\delta k+k-1)-k}(w_r^\alpha)^k \\
&\quad + O(1)m|B_m|\sum_{\nu=1}^{m}\nu^{\beta(\delta k+k-1)-k}(w_\nu^\alpha)^k \\
&= O(1)\sum_{\nu=1}^{m-1}|\Delta(\nu|B_\nu|)|X_\nu + O(1)m|B_m|X_m \\
&= O(1)\sum_{\nu=1}^{m-1}\nu|B_\nu|X_\nu + O(1)\sum_{\nu=1}^{m-1}(\nu+1)|B_{\nu+1}|X_{\nu+1} + O(1)m|B_m|X_m \\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}
$$

by the hypotheses of the Theorem. Again, we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}|T_{n,2}^\alpha|^k &= \sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}(|\lambda_n|w_n^\alpha)^k \\
&= \sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}|\lambda_n||\lambda_n|^{k-1}(w_n^\alpha)^k \\
&= O(1)\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}(w_n^\alpha)^k\sum_{\nu=n}^{\infty}|\Delta\lambda_\nu| \\
&= O(1)\sum_{\nu=1}^{\infty}|\Delta\lambda_\nu|\sum_{n=1}^{\nu} n^{\beta(\delta k+k-1)-k}(w_n^\alpha)^k \\
&= O(1)\sum_{\nu=1}^{\infty}|B_\nu|\sum_{n=1}^{\nu} n^{\beta(\delta k+k-1)-k}(w_n^\alpha)^k \\
&= O(1)\sum_{\nu=1}^{\infty}|B_\nu|X_\nu < \infty,
\end{aligned}
$$

by the hypotheses of the Theorem. Therefore, we get

$$\sum_{n=1}^{m} n^{\beta(\delta k+k-1)-k}|T_{n,r}^{\alpha}|^{k} = O(1) \qquad \text{as } m \to \infty \text{ for } r = 1, 2.$$

This completes the proof of the Theorem.

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Department of Mathematics
Erciyes University
38039 Kayseri, Turkey
E-mail: ntuncer@erciyes.edu.tr

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