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A local characterization of affine holomorphic immersions with an anti-complex and ∇-parallel shape operator

by Maria Robaszewska (Kraków)

Abstract. We study the complex hypersurfaces $f : M^{(n)} \to \mathbb{C}^{n+1}$ which together with their transversal bundles have the property that around any point of M there exists a local section of the transversal bundle inducing a ∇-parallel anti-complex shape operator S. We give a class of examples of such hypersurfaces with an arbitrary rank of S from 1 to $[n/2]$ and show that every such hypersurface with positive type number and $S \neq 0$ is locally of this kind, modulo an affine isomorphism of $\mathbb{C}^{n+1}$.

1. Introduction. Among the connections induced on complex hypersurfaces $f : M^{(n)} \to \mathbb{C}^{n+1}$ by $\mathcal{C}^\infty$ complex transversal bundles there are two particular kinds of great interest: holomorphic connections and affine Kähler connections. The latter are meant to be a generalization of Kähler connections. In terms of the curvature tensor, a holomorphic affine connection is characterized by the condition

$$R(JX, Y) = JR(X, Y) \quad \text{for all vector fields } X, Y,$$

while for an affine Kähler connection we have, by definition,

$$R(JX, JY) = R(X, Y) \quad \text{for all } X, Y$$

(see [NS]). Since, provided the affine fundamental form h does not vanish on M, a holomorphic connection is induced by a holomorphic transversal bundle, it is possible to adapt the ideas from the real affine hypersurface geometry to this case. For instance, having a non-degenerate hypersurface one can construct a holomorphic analogue of affine normal vector field [DVV]. Then one can consider the condition $S = \lambda I$, $\lambda = \text{const}$, which describes the affine spheres [DVV].

On the contrary, the non-flat affine Kähler connections induced on hypersurfaces (in particular, the non-flat Kähler ones) cannot be treated in this

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way. Instead of the holomorphic transversal bundles, with the complex shape operator, one has to consider the transversal bundles $\mathcal{N}$ having the property that the shape operator corresponding to sections of $\mathcal{N}$ is anti-complex. This property of $\mathcal{N}$ implies the desired condition for the curvature tensor R of ∇ and is necessary for ∇ to be affine Kähler if $tf > 1$ at some point of M (see [O]).

Clearly, no section ξ of such a bundle can induce S proportional to the identity, except for the case $S = 0$. Being ∇-parallel is a weaker condition on S than $S = \lambda I$, $\lambda = \mathrm{const}$. This condition is shown to have some non-trivial realizations even if we require S to be anti-complex. It is worth noting that we need to consider degenerate hypersurfaces, because the non-degeneracy implies $S = 0$ (Lemma 2).

2. Preliminaries. Let M be an n-dimensional connected complex manifold. We shall consider a holomorphic immersion $f : M \to \mathbb{C}^{n+1}$ together with a $\mathcal{C}^\infty$ complex transversal bundle $\mathcal{N}$. If $\xi : U \to \mathbb{C}^{n+1}$ is a local section of $\mathcal{N}$, then the induced connection ∇ on M, the second fundamental form h, the shape operator S and the transversal forms μ and ν are defined by the following Gauss and Weingarten formulas [NS]:

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi - h(JX,Y)J\xi,$$
$$D_X \xi = -f_* SX + \mu(X)\xi + \nu(X)J\xi.$$

Here D denotes the standard connection on $\mathbb{C}^{n+1}$, and J the complex structure on M and on $\mathbb{C}^{n+1}$ as well.

Let $m \in M$. The complex rank of the $\mathbb{C}$-bilinear form $h_m^c(\cdot,\cdot) = h_m(\cdot,\cdot) - ih_m(J\cdot,\cdot)$ depends on f only. It is called the *type number* of f at m and denoted by tf_m (see [O]). We shall assume that it is positive everywhere on M.

Our first requirement on the transversal bundle is that the induced shape operator S is *anti-complex*, i.e. $SJ = -JS$ (see [O]). The fundamental equations satisfied by ∇, h, μ, ν and such an S are the following:

$$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY + h(JY,Z)SJX - h(JX,Z)SJY \qquad (\textit{Gauss}),$$

$$(\nabla_X h)(Y,Z) + \mu(X)h(Y,Z) + \nu(X)h(JY,Z)$$
$$= (\nabla_Y h)(X,Z) + \mu(Y)h(X,Z) + \nu(Y)h(JX,Z) \qquad (\textit{Codazzi I}),$$

$$(\nabla_X S)Y - \mu(X)SY + \nu(X)SJY = (\nabla_Y S)X - \mu(Y)SX + \nu(Y)SJX \qquad (\textit{Codazzi II}),$$

$$h(X,SY) - h(Y,SX) = 2d\mu(X,Y) \qquad (\textit{Ricci I}),$$

$$h(SX,JY) - h(SY,JX) = 2d\nu(X,Y) \qquad (\textit{Ricci II}).$$

Furthermore, we shall assume that for every point $m \in M$ there exists a local section $\xi : U \to \mathbb{C}^{n+1}$ of $\mathcal{N}$ with $U \ni m$ such that $\nabla S^\xi = 0$, where S^ξ denotes the shape operator induced by ξ. If the points $m_1, m_2 \in U$ can be joined by a curve lying in U, and $\mathcal{B}_1$ is a basis of $T_{m_1}M$, then by parallel displacement we can obtain a basis $\mathcal{B}_2$ of $T_{m_2}M$ with respect to which $S^\xi_{m_2}$ has the same matrix as $S^\xi_{m_1}$ with respect to $\mathcal{B}_1$. Hence $\operatorname{rank} S^\xi_{m_1} = \operatorname{rank} S^\xi_{m_2}$, where $\operatorname{rank} S^\xi_m := \dim_{\mathbb{C}} \operatorname{im} S^\xi_m$. The assumed connectedness of M and the independence of $\operatorname{rank} S^\xi_m$ of ξ at a fixed m imply that $q := \operatorname{rank} S$ is well defined for the whole M.

When studying immersions with ∇-parallel shape operator we shall make use of the following remarks:

REMARK 1. *If $\nabla S = 0$, then for every X, Y, Z,*

$$R(X,Y)SZ = S(R(X,Y)Z).$$

Proof. This is an obvious consequence of the commutativity of S and ∇_W for any W.

REMARK 2. *If $\nabla S = 0$ and $SJ = -JS$, then $\ker S \subset \ker \mu \cap \ker \nu$ or $S = 0$.*

Proof. Let $SX = 0$. By the second Codazzi equation we have $-\mu(X)SY + \nu(X)SJY = 0$ for any Y. If $S \neq 0$, then there exists Y such that $SY \neq 0$. Since SY and $SJY = -JSY$ are linearly independent over $\mathbb{R}$, it follows that $\mu(X) = 0$ and $\nu(X) = 0$.

REMARK 3. *If $\nabla S = 0$, $SJ = -JS$ and $S \neq 0$, then the section ξ inducing S is anti-holomorphic, i.e. $\nu(X) = \mu(JX)$ for any X.*

Proof. We may assume that $SX \neq 0$. The assertion follows easily from the second Codazzi equation, written for $Y = JX$.

For an anti-holomorphic ξ we can rewrite the first Codazzi equation as

$$(\nabla_X h)(Y,Z) + \mu(X)h(Y,Z) + \mu(JX)h(JY,Z) = (\nabla_Y h)(X,Z) + \mu(Y)h(X,Z) + \mu(JY)h(JX,Z).$$

3. Theorem. We can now formulate our main result.

THEOREM. *Let M be an n-dimensional connected complex manifold, $n > 1$, $f : M \to \mathbb{C}^{n+1}$ a holomorphic immersion and ∇ a linear connection induced on M by a transversal bundle $\mathcal{N}$. Let f and M satisfy the following assumptions*:

(1) *$tf > 0$ everywhere on M,*

(2) *for every $m \in M$ there exists a neighbourhood U of m and a local section $\xi : U \to \mathbb{C}^{n+1}$ of $\mathcal{N}$ inducing an anti-complex and ∇-parallel shape operator S,*

(3) $q := \operatorname{rank} S > 0$.

Under the conditions stated above,

(i) $q \leq n/2$;

(ii) *for every* $m \in M$ *there exists a neighbourhood* V *of* m, *a complex chart* $\widetilde{\phi} : V \to \mathbb{C}^n \cong \mathbb{C}^q \times \mathbb{C}^{n-2q} \times \mathbb{C}^q$, *a complex affine isomorphism* $\widetilde{A} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, *and a holomorphic function* $\widetilde{\mathcal{F}}$ *such that*

$$\begin{aligned}&\widetilde{A} \circ f \circ \widetilde{\phi}^{-1}(\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q)\\ &\quad = (\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q, \widetilde{\mathcal{F}}(\widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q));\end{aligned}$$

(iii) *if* $q > 1$, *then the local section* $\vec{\widetilde{A}} \circ \xi : V \to \mathbb{C}^{n+1}$ *of* $\vec{\widetilde{A}}\mathcal{N}$ (*where* $\vec{\widetilde{A}}$ *denotes the linear part of* $\widetilde{A}$) *inducing the* ∇-*parallel shape operator is described in this chart by the following formula*:

$$\vec{\widetilde{A}} \circ \xi \circ \widetilde{\phi}^{-1}(\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q) = (\overline{\widetilde{z}^1}, \ldots, \overline{\widetilde{z}^q}, \underbrace{0, \ldots, 0}_{n-q \text{ times}}, 1);$$

(iv) *if* $q = 1$, *then*

$$\vec{\widetilde{A}} \circ \xi \circ \widetilde{\phi}^{-1}(\widetilde{x}, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2}, \widetilde{z}) = (\overline{\widetilde{\mathcal{G}}(\widetilde{z})}, \underbrace{0, \ldots, 0}_{n-1 \text{ times}}, \overline{e^{\widetilde{\mathcal{M}}(\widetilde{z})}})$$

where $\widetilde{\mathcal{G}}$ *and* $\widetilde{\mathcal{M}}$ *are holomorphic functions such that*

$$\widetilde{\mathcal{G}}'(\widetilde{z}) - \widetilde{\mathcal{M}}'(\widetilde{z})\widetilde{\mathcal{G}}(\widetilde{z}) \equiv 1.$$

In the real representation, setting $\widetilde{x}^k = x^{2k-1} + ix^{2k}$, $\widetilde{y}^l = y^{2l-1} + iy^{2l}$, $\widetilde{z}^j = z^{2j-1} + iz^{2j}$ *for* $j, k = 1, \ldots, q$; $l = 1, \ldots, n-2q$; $\widetilde{\mathcal{F}} = \mathcal{F}^1 + i\mathcal{F}^2$, $\widetilde{\mathcal{G}} = \mathcal{G}^1 + i\mathcal{G}^2$, $\widetilde{\mathcal{M}} = \mathcal{M}^1 + i\mathcal{M}^2$, *we have*

$$\begin{aligned}&A \circ f \circ \phi^{-1}(x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q})\\ &\qquad = (x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q}, \mathcal{F}^1(y, z), \mathcal{F}^2(y, z)),\\ &\vec{A} \circ \xi \circ \phi^{-1}(x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q})\\ &\qquad\qquad = (z^1, -z^2, \ldots, z^{2q-1}, -z^{2q}, \underbrace{0, \ldots, 0}_{2n-2q \text{ times}}, 1, 0)\end{aligned}$$

if $q > 1$, *and*

$$\begin{aligned}&\vec{A} \circ \xi \circ \phi^{-1}(x^1, x^2, y^1, \ldots, y^{2n-4}, z^1, z^2)\\ &\qquad (\mathcal{G}^1(z), -\mathcal{G}^2(z), \underbrace{0, \ldots, 0}_{2n-2 \text{ times}}, e^{\mathcal{M}^1(z)} \cos \mathcal{M}^2(z), -e^{\mathcal{M}^1(z)} \sin \mathcal{M}^2(z))\end{aligned}$$

if $q = 1$.

REMARK 4. An easy computation shows that the converse is also true:

(a) *For any holomorphic function* $\widetilde{\mathcal{F}}$ *of* $n-q$ *variables, where* $q \le n/2$, *the shape operator* S *induced on the hypersurface*

$$\begin{aligned} f : \mathbb{C}^n \supset U \ni (&\widetilde{x}^1, \dots, \widetilde{x}^q, \widetilde{y}^1, \dots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \dots, \widetilde{z}^q) \\ &\mapsto (\widetilde{x}^1, \dots, \widetilde{x}^q, \widetilde{y}^1, \dots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \dots, \widetilde{z}^q, \widetilde{\mathcal{F}}(\widetilde{y}, \widetilde{z})) \in \mathbb{C}^{n+1} \end{aligned}$$

endowed with the transversal field

$$\xi(\widetilde{x}^1, \dots, \widetilde{x}^q, \widetilde{y}^1, \dots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \dots, \widetilde{z}^q) = (\overline{\widetilde{z}^1}, \dots, \overline{\widetilde{z}^q}, \underbrace{0, \dots, 0}_{n-q \text{ times}}, 1)$$

is parallel with respect to the induced connection and $\operatorname{rank} S = q$.

(b) *For any holomorphic function* $\widetilde{\mathcal{F}}$ *of* $n-1$ *variables and for any holomorphic functions* $\widetilde{\mathcal{G}}$ *and* $\widetilde{\mathcal{M}}$ *of one variable satisfying the equation*

$$\widetilde{\mathcal{G}}'(\widetilde{z}) - \widetilde{\mathcal{M}}'(\widetilde{z})\widetilde{\mathcal{G}}(\widetilde{z}) \equiv 1,$$

the shape operator S *induced on the hypersurface*

$$f : \mathbb{C}^n \supset U \ni (\widetilde{x}, \widetilde{y}^1, \dots, \widetilde{y}^{n-2}, \widetilde{z}) \mapsto (\widetilde{x}, \widetilde{y}^1, \dots, \widetilde{y}^{n-2}, \widetilde{z}, \widetilde{\mathcal{F}}(\widetilde{y}, \widetilde{z})) \in \mathbb{C}^{n+1}$$

endowed with the transversal field

$$\xi(\widetilde{x}, \widetilde{y}^1, \dots, \widetilde{y}^{n-2}, \widetilde{z}) = (\overline{\widetilde{\mathcal{G}}(\widetilde{z})}, \underbrace{0, \dots, 0}_{n-1 \text{ times}}, \overline{e^{\widetilde{\mathcal{M}}(\widetilde{z})}})$$

is parallel with respect to the induced connection and has $\operatorname{rank} S = 1$.

Proof of the Theorem. We begin by proving two lemmas in which we establish some inclusions between $\operatorname{im} S$, $\ker S$ and $\ker h$.

LEMMA 1. *If* $tf > 0$, $SJ = -JS$, *and* $\nabla S = 0$, *then the following conditions are equivalent*:

(1) $d\mu = 0$ *and* $d\nu = 0$,
(2) $\operatorname{im} S \subset \ker h$,
(3) $\operatorname{im} S \subset \ker S$.

Proof. (1)⇔(2). Suppose that $d\mu = 0$ and $d\nu = 0$. Let $m' \in M$ and $X, Y \in T_{m'}M$. Applying the Ricci equations we have

$$\begin{aligned} 0 &= 2d\mu(X,Y) + 2d\nu(JX,Y) \\ &= h(X,SY) - h(Y,SX) - h(J^2X,SY) + h(JY,SJX) \\ &= h(X,SY) - h(Y,SX) + h(X,SY) + h(JY,-JSX) \\ &= h(X,SY) - h(Y,SX) + h(X,SY) + h(Y,SX) = 2h(X,SY), \end{aligned}$$

hence for any $X, Y \in T_{m'}M$ we have $h(X,SY) = 0$. The Ricci equations make it obvious that (2) implies (1).

(2)⇒(3). Suppose that $\operatorname{im} S \subset \ker h$, which yields $R(X,Y)SZ = 0$ by the Gauss equation. Since $tf_{m'} > 0$, there exist $X_0, Y_0 \in T_{m'}M$ such that $h(X_0, Y_0) \neq 0$. We first show that $S^2 X_0 = 0$. Indeed, making use of Remark 1 we have

$$0 = S(R(X_0, JX_0)Y_0) = 2(h(JX_0, Y_0)S^2 X_0 - h(X_0, Y_0)S^2 JX_0)$$

with $h(X_0, Y_0) \neq 0$, which means that $S^2 X_0$ and $S^2 JX_0$ are linearly dependent over $\mathbb{R}$. This is possible only when $S^2 X_0 = S^2 JX_0 = 0$.

Now we take an arbitrary $Z \in T_{m'}M$. Then

$$0 = S(R(X_0, Z)Y_0) = -h(X_0, Y_0)S^2 Z - h(JX_0, Y_0)S^2 JZ,$$

and by a similar argument $S^2 Z = 0$.

(3)⇒(2). If $S^2 = 0$, then the right-hand side of the equality $R(X,Y)SZ = S(R(X,Y)Z)$ vanishes for every X, Y, Z. For $S = 0$, (2) holds, therefore we can assume that $S \neq 0$. Take X_0 such that $SX_0 \neq 0$. Since

$$0 = R(X_0, JX_0)SZ = 2(h(JX_0, SZ)SX_0 - h(X_0, SZ)SJX_0)$$

and $SX_0,\ SJX_0$ are linearly independent over $\mathbb{R}$, we have

$$h(X_0, SZ) = h(JX_0, SZ) = 0 \quad \text{for any } Z.$$

Now we can write for any Y, Z,

$$0 = R(X_0, Y)SZ = h(Y, SZ)SX_0 + h(JY, SZ)SJX_0.$$

Hence $h(Y, SZ) = 0$ for any Y, Z. ∎

Lemma 2. *Under the assumptions of Lemma* 1, *the equivalent conditions* (1), (2) *and* (3) *are satisfied.*

Proof. If $\nabla S = 0$, then $\operatorname{rank} S$ is constant on the domain of S. We have to consider three cases.

Case 1: $\operatorname{rank} S = 0$. Then, of course, (3) holds.

Case 2: $\operatorname{rank} S = 1$. Suppose, contrary to our claim, that $S^2 \neq 0$. Let $m' \in M$. We fix $X_0 \in T_{m'}M$ such that $S^2 X_0 \neq 0$. We shall obtain a contradiction with the assumption $tf > 0$.

Step 1. $\ker S \subset \ker h$.

Let $Z \in \ker S$. Then

$$\begin{aligned} 0 &= R(X_0, JX_0)SZ = S(R(X_0, JX_0)Z) \\ &= 2(h(JX_0, Z)S^2 X_0 - h(X_0, Z)S^2 JX_0), \end{aligned}$$

hence $h(X_0, Z) = h(JX_0, Z) = 0$. For any Y we now have

$$0 = R(X_0, Y)SZ = S(R(X_0, Y)Z) = h(Y, Z)S^2 X_0 + h(JY, Z)S^2 JX_0,$$

which implies $h(Y, Z) = 0$ for any $Z \in \ker S$ and for any Y.

Step 2. (a) $T_{m'}M = \ker S \oplus \mathbb{C}X_0$ *and* (b) $T_{m'}M = \ker S \oplus \mathbb{C}SX_0$.

Since $\dim_{\mathbb{C}} \ker S = n-1$, it is sufficient and easy to check that $\ker S \cap \mathbb{C}X_0 = \{0\}$ and $\ker S \cap \mathbb{C}SX_0 = \{0\}$.

STEP 3. $h(X_0, SX_0) = h(JX_0, SJX_0) = 0$.

If $h(X_0, X_0) = 0$ and $h(X_0, JX_0) = 0$ then $X_0 \in \ker h$ by Steps 1 and 2(a), and so the claimed equality holds.

Assume now that $h(X_0, X_0) \neq 0$ or $h(X_0, JX_0) \neq 0$. We have

$$\begin{aligned} 0 &= R(X_0, JX_0)SX_0 - S(R(X_0, JX_0)X_0 \\ &= 2S(h(JX_0, SX_0)X_0 - h(X_0, SX_0)JX_0 \\ &\quad - h(JX_0, X_0)SX_0 + h(X_0, X_0)SJX_0), \end{aligned}$$

therefore $Z_0 \in \ker S$, where

$$\begin{aligned} Z_0 := h(JX_0, SX_0)X_0 - h(X_0, SX_0)JX_0 \\ - h(JX_0, X_0)SX_0 + h(X_0, X_0)SJX_0. \end{aligned}$$

According to Step 1, we have $h(Z_0, X_0) = 0$ and $h(Z_0, JX_0) = 0$, hence

$$\begin{aligned} &h(JX_0, SX_0)h(X_0, X_0) - h(X_0, SX_0)h(JX_0, X_0) \\ &\qquad - h(JX_0, X_0)h(SX_0, X_0) + h(X_0, X_0)h(SJX_0, X_0) = 0 \end{aligned}$$

and

$$\begin{aligned} &h(JX_0, SX_0)h(X_0, JX_0) - h(X_0, SX_0)h(JX_0, JX_0) \\ &\qquad - h(JX_0, X_0)h(SX_0, JX_0) + h(X_0, X_0)h(SJX_0, JX_0) = 0. \end{aligned}$$

Thus we obtain

$$h(SX_0, X_0)h(JX_0, X_0) = 0 \quad \text{and} \quad h(SX_0, X_0)h(X_0, X_0) = 0,$$

which implies $h(SX_0, X_0) = 0$, and consequently, by the anti-complexity of S and the properties of $h(\cdot, \cdot)$, $h(SJX_0, JX_0) = 0$.

STEP 4. $h(Z, SW) + h(W, SZ) = 0$ *for any* $Z, W \in T_{m'}M$.

By Step 2(a) we have $Z = Z_1 + \alpha X_0 + \beta JX_0$, $W = W_1 + \gamma X_0 + \delta JX_0$ with $Z_1, W_1 \in \ker S$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. An easy computation gives

$$\begin{aligned} &h(Z, SW) + h(W, SZ) \\ &= h(\alpha X_0 + \beta JX_0, S(\gamma X_0 + \delta JX_0)) + h(\gamma X_0 + \delta JX_0, S(\alpha X_0 + \beta JX_0)) \\ &= 2\alpha\gamma h(X_0, SX_0) + 2\beta\delta h(JX_0, SJX_0) + (\alpha\delta + \beta\gamma)h(X_0, (SJ + JS)X_0), \end{aligned}$$

which vanishes by the anti-complexity of S and by Step 3.

STEP 5. $(\nabla_W h)(X_0, SX_0) = (\nabla_W h)(JX_0, SJX_0) = 0$ *for any* W.

We extend X_0 to a local vector field X_0 such that $S^2X_0 \neq 0$ at any point of the domain of X_0. We have

$$(\nabla_W h)(X_0, SX_0)$$
$$= W(h(X_0, SX_0)) - h(\nabla_W X_0, SX_0) - h(X_0, \nabla_W (SX_0))$$
$$= W(h(X_0, SX_0)) - (h(\nabla_W X_0, SX_0) + h(X_0, S(\nabla_W X_0))) = 0.$$

The same is true for JX_0 in place of X_0.

STEP 6. $\nabla h = -2\mu \otimes h$.

If $Y \in \ker S_{m'}$, then we can extend Y to a local section Y of $\ker S$. For any $X, Z, \nabla_X Y \in \ker S$ and

$$(\nabla_X h)(Y, Z) = X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0$$
$$= -2\mu(X)h(Y, Z).$$

Let $X \in \ker S$. Then for any Y, Z we have

$$(\nabla_X h)(Y, Z) \underset{\text{Codazzi I}}{=} -\mu(X)h(Y, Z) - \mu(JX)h(JY, Z)$$
$$+ (\nabla_Y h)(X, Z) + \mu(Y)h(X, Z) + \mu(JY)h(JX, Z)$$
$$= 0 = -2\mu(X)h(Y, Z),$$

since $\ker S \subset \ker \mu \cap \ker \nu$ and $\ker S \subset \ker h$.

It follows that $(\nabla_X h)(Y, \cdot) = -2\mu(X)h(Y, \cdot)$ if $X \in \ker S$ or $Y \in \ker S$.

From the first Codazzi equation we obtain

$$(\nabla_{X_0} h)(JX_0, SX_0) + \mu(X_0)h(JX_0, SX_0) + \mu(JX_0)h(J^2X_0, SX_0)$$
$$= (\nabla_{JX_0} h)(X_0, SX_0) + \mu(JX_0)h(X_0, SX_0) + \mu(J^2X_0)h(JX_0, SX_0).$$

Hence

$$(\nabla_{X_0} h)(JX_0, SX_0) = -2\mu(X_0)h(JX_0, SX_0),$$

by Steps 3 and 5.

Similarly,

$$(\nabla_{X_0} h)(JX_0, SJX_0) + \mu(X_0)h(JX_0, SJX_0) + \mu(JX_0)h(J^2X_0, SJX_0)$$
$$= (\nabla_{JX_0} h)(X_0, SJX_0) + \mu(JX_0)h(X_0, SJX_0) + \mu(J^2X_0)h(JX_0, SJX_0),$$

which gives

$$(\nabla_{JX_0} h)(X_0, SJX_0) = -2\mu(JX_0)h(X_0, SJX_0).$$

We now have

$$(\nabla_{X_0} h)(X_0, SX_0) = 0 = -2\mu(X_0)h(X_0, SX_0),$$
$$(\nabla_{X_0} h)(X_0, SJX_0) = -(\nabla_{X_0} h)(JX_0, SX_0) = 2\mu(X_0)h(JX_0, SX_0)$$
$$= -2\mu(X_0)h(X_0, SJX_0)$$

and

$$(\nabla_{X_0} h)(X_0, Z) = -2\mu(X_0)h(X_0, Z)$$

for $Z \in \ker S$. Therefore

$$(\nabla_{X_0} h)(X_0, \cdot\ ) = -2\mu(X_0)h(X_0, \cdot\ )$$

by Step 2(b).

In the same manner we can see that

$$\begin{aligned}(\nabla_{X_0} h)(JX_0, \cdot) &= -2\mu(X_0)h(JX_0, \cdot),\\ (\nabla_{JX_0} h)(X_0, \cdot) &= -2\mu(JX_0)h(X_0, \cdot),\\ (\nabla_{JX_0} h)(JX_0, \cdot) &= -2\mu(JX_0)h(JX_0, \cdot),\end{aligned}$$

which completes the proof of Step 6.

As a consequence of Step 6 we obtain

STEP 7. *$R(X,Y) \cdot h = -4d\mu(X,Y)h$ for any X, Y.*

Applying the Ricci equation yields

STEP 8. *$R(X,Y) \cdot h = -2(h(X,SY) - h(Y,SX))h$. In particular, we have $R(X_0, JX_0) \cdot h = -4h(X_0, SJX_0)h$.*

On the other hand, a direct computation gives

$$\begin{aligned}(R(X_0, JX_0) \cdot h)(X_0, X_0)& \\ &= -2h(R(X_0, JX_0)X_0, X_0)\\ &= -2h(2(h(JX_0, X_0)SX_0 - h(X_0, X_0)SJX_0), X_0)\\ &= -4h(JX_0, X_0)h(SX_0, X_0) + 4h(X_0, X_0)h(SJX_0, X_0)\\ &= 4h(X_0, X_0)h(X_0, SJX_0),\end{aligned}$$

and

$$\begin{aligned}(R(X_0, JX_0) \cdot h)(X_0, JX_0)& \\ = &- h(R(X_0, JX_0)X_0, JX_0) - h(X_0, R(X_0, JX_0)JX_0)\\ = &- h(2(h(JX_0, X_0)SX_0 - h(X_0, X_0)SJX_0), JX_0)\\ &- h(X_0, 2(h(JX_0, JX_0)SX_0 - h(X_0, JX_0)SJX_0))\\ = &- 2h(JX_0, X_0)h(SX_0, JX_0) + 2h(X_0, X_0)h(SJX_0, JX_0)\\ &- 2h(JX_0, JX_0)h(X_0, SX_0) + 2h(X_0, JX_0)h(X_0, SJX_0)\\ = &\ 4h(JX_0, X_0)h(X_0, SJX_0).\end{aligned}$$

A comparison with Step 8 gives

$$h(X_0, SJX_0) = 0 \quad \text{or} \quad h(X_0, X_0) = h(X_0, JX_0) = 0,$$

which together with Steps 1–3 leads to a contradiction with the assumption $h_{m'} \neq 0$.

CASE 3: $\operatorname{rank} S > 1$. If $SX = 0$, then, by Remark 2, $\mu(X) = \nu(X) = 0$. Let $SX \neq 0$. Then there exists Y such that SX and SY are linearly independent over $\mathbb{C}$. From Codazzi II we have

$$(\mu(X) + i\nu(X))SY - (\mu(Y) + i\nu(Y))SX = 0,$$

which implies $\mu(X) = \nu(X) = 0$. Therefore (1) of Lemma 1 holds. ∎

We now return to the proof of the theorem.

Fix $m \in M$. Let $\xi : U \to \mathbb{C}^{n+1}$ be defined on a connected neighbourhood U of m and have the property described in assumption (2) of the Theorem.

LEMMA 3. *There exists a q-dimensional complex subspace $\mathcal{W}$ of $\mathbb{C}^{n+1}$ such that $f_* \operatorname{im} S_{m'} = \mathcal{W}$ for every $m' \in U$.*

Proof. It is sufficient to show that $f_* \operatorname{im} S_m = f_* \operatorname{im} S_{m'}$. Let $\gamma : [0,1] \to U$ be a $\mathcal{C}^1$ curve joining m and m'; $\gamma(0) = m$, $\gamma(1) = m'$. We choose $X_{1m}, \ldots, X_{qm} \in T_m M$ such that $SX_{1m}, \ldots, SX_{qm}$ form a basis over $\mathbb{C}$ of $f_* \operatorname{im} S_m$. Let $\widetilde{X}_1, \ldots, \widetilde{X}_q$ be the vector fields defined along the curve γ, parallel with respect to ∇, $\widetilde{X}_i(0) = X_{im}$ for $i \in \{1, \ldots, q\}$. It is easy to check that the map

$$[0,1] \ni t \mapsto f_* S_{\gamma(t)} \widetilde{X}_i(t) \in \mathbb{C}^{n+1}$$

is constant. Indeed,

$$\begin{aligned} \frac{d}{dt}(t \mapsto f_* S_{\gamma(t)} \widetilde{X}_i(t)) &= D_{\dot\gamma(t)} f_* S\widetilde{X}_i \\ &= f_* \nabla_{\dot\gamma(t)} S\widetilde{X}_i + h(\dot\gamma(t), S\widetilde{X}_i(t))\xi_{\gamma(t)} \\ &\quad - h(J\dot\gamma(t), S\widetilde{X}_i(t))J\xi_{\gamma(t)}. \end{aligned}$$

The last two terms vanish because $\operatorname{im} S \subset \ker h$, and $\nabla_{\dot\gamma(t)} S\widetilde{X}_i = S\nabla_{\dot\gamma(t)}\widetilde{X}_i = S0 = 0$. It follows that

$$\operatorname{span}_{\mathbb{C}}\{f_* S_m \widetilde{X}_i(0) : i = 1, \ldots, q\} = \operatorname{span}_{\mathbb{C}}\{f_* S_{m'} \widetilde{X}_i(1) : i = 1, \ldots, q\},$$

that is, $f_* \operatorname{im} S_m = f_* \operatorname{im} S_{m'} =: \mathcal{W}$. ∎

Let $\widetilde{A}_1 : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a linear isomorphism such that

$$\widetilde{A}_1 \mathcal{W} = \operatorname{span}_{\mathbb{C}}\{\widetilde{e}_1, \ldots, \widetilde{e}_q\}.$$

Here and subsequently $\widetilde{e}_1, \ldots, \widetilde{e}_{n+1}$ denotes the standard basis of $\mathbb{C}^{n+1}$, whereas $e_1, \ldots, e_{2n+2}$ is the standard basis of $\mathbb{R}^{2n+2}$.

LEMMA 4. *There exists an $i_0 \in \{q+1, \ldots, n+1\}$ such that $\widetilde{e}_{i_0} \notin (\widetilde{A}_1 \circ f)_* T_m M$ and the i_0th coordinate of $\widetilde{A}_1 \xi_m$ does not vanish.*

Proof. Suppose that the assertion is false. Then $\widetilde{e}_j \in (\widetilde{A}_1 \circ f)_* T_m M$ for every $j \in \{q+1, \ldots, n+1\}$ such that the jth coordinate of $\widetilde{A}_1 \xi$ does not vanish at m. Then obviously $\widetilde{A}_1\xi_m \in (\widetilde{A}_1 \circ f)_* T_m M$, which contradicts the transversality of ξ. ∎

Let $\widetilde{A}_2^0$ be the linear isomorphism of $\mathbb{C}^{n+1}$ defined by

$$\widetilde{A}_2^0 \widetilde{e}_k := \begin{cases} \widetilde{e}_k & \text{if } k \notin \{i_0, n+1\}, \\ \widetilde{e}_{n+1} & \text{if } k = i_0, \\ \widetilde{e}_{i_0} & \text{if } k = n+1. \end{cases}$$

Let $\widetilde{A}_2 := \widetilde{A}_2^0 \circ \widetilde{A}_1$. Now $\widetilde{e}_{n+1}$ is transversal to $(\widetilde{A}_2 \circ f)_* T_m M$ and $\widetilde{A}_2\xi$ has the non-vanishing $(n+1)$th coordinate at m. Moreover, the isomorphism $\widetilde{A}_2^0$ does not change the subspace $(\widetilde{A}_1 \circ f)_* \operatorname{im} S = \operatorname{span}_{\mathbb{C}}\{\widetilde{e}_1, \ldots, \widetilde{e}_q\}$.

We denote by π the projection $\pi : \mathbb{C}^{n+1} \ni (\zeta^1, \ldots, \zeta^{n+1}) \mapsto (\zeta^1, \ldots, \zeta^n) \in \mathbb{C}^n$. It is easy to check that

$$d_m(\pi \circ \widetilde{A}_2 \circ f) : T_m M \to \mathbb{C}^n$$

is a monomorphism. Indeed, if $d_m(\pi \circ \widetilde{A}_2 \circ f).V = 0$, then $(\widetilde{A}_2 \circ f)_* V \in \mathbb{C}\widetilde{e}_{n+1}$. But $(\widetilde{A}_2 \circ f)_* T_m M \cap \mathbb{C}\widetilde{e}_{n+1} = \{0\}$ and $(\widetilde{A}_2 \circ f)_*$ is a monomorphism; therefore $V = 0$.

We can now take $\widetilde{\phi}_1 := \pi \circ \widetilde{A}_2 \circ f$ as a complex chart on some neighbourhood $U_1 \subset U$ of m. In this chart

$$\widetilde{A}_2 \circ f \circ \widetilde{\phi}_1^{-1}(\zeta^1, \ldots, \zeta^n) = (\zeta^1, \ldots, \zeta^n, \widetilde{\varphi}(\zeta)),$$

with a holomorphic function $\widetilde{\varphi}$.

In the real representation, identifying $\mathbb{C}^k$ with $\mathbb{R}^{2k}$,

$$\iota_k : \mathbb{R}^{2k} \ni (w^1, \ldots, w^{2k}) \mapsto (w^1 + iw^2, \ldots, w^{2k-1} + iw^{2k}) \in \mathbb{C}^k,$$

we can write

$$A_2 \circ f \circ \phi_1^{-1}(w^1, \ldots, w^{2n}) = (w^1, \ldots, w^{2n}, \varphi^1(w), \varphi^2(w)).$$

Here $A_2 := {\iota_{n+1}}^{-1} \circ \widetilde{A}_2$ and $\phi_1 := {\iota_n}^{-1} \circ \widetilde{\phi}_1$.

LEMMA 5. (a) $\partial\varphi^k/\partial w^s = 0$ *for* $k = 1, 2$ *and* $s = 1, \ldots, 2q$.
(b) $\operatorname{im} S = \operatorname{span}_{\mathbb{R}}\{\partial/\partial w^s : s = 1, \ldots, 2q\}$.

Proof. At any point $m' \in U_1$ we have

$$(A_2 \circ f)_* \left(\frac{\partial}{\partial w^s} \right) = e_s + \frac{\partial\varphi^1}{\partial w^s} e_{2n+1} + \frac{\partial\varphi^2}{\partial w^s} e_{2n+2}.$$

If $s \in \{1, \ldots, 2q\}$, then $e_s = (A_2 \circ f)_* SW_s$ with some $W_s \in T_{m'}M$, because $e_s \in \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_{2q}\} = (A_2 \circ f)_* \operatorname{im} S_{m'}$. Therefore we have

$$(A_2 \circ f)_* \left(\frac{\partial}{\partial w^s} - SW_s \right) = \frac{\partial\varphi^1}{\partial w^s} e_{2n+1} + \frac{\partial\varphi^2}{\partial w^s} e_{2n+2}.$$

From the transversality of e_{2n+1} and e_{2n+2} to $(A_2 \circ f)_* TM$ and from the injectivity of $(A_2 \circ f)_*$ it follows that (a) holds, and $\partial/\partial w^s - SW_s = 0$ for

$s = 1, \ldots, 2q$. Hence $\mathrm{span}_{\mathbb{R}}\{\partial/\partial w^s : s = 1, \ldots, 2q\} \subset \mathrm{im}\, S$, which implies (b), because the dimensions are equal. ■

LEMMA 6. *The transversal field $A_2\xi$ does not depend on $w^1, \ldots, w^{2q}$.*

Proof. We use the Weingarten formula

$$D_{\partial/\partial w^s} A_2\xi = -(A_2 \circ f)_* S \frac{\partial}{\partial w^s} + \mu\left(\frac{\partial}{\partial w^s}\right) A_2\xi + \nu\left(\frac{\partial}{\partial w^s}\right) J A_2\xi.$$

According to Lemmas 5(b), 2 and Remark 2, for $s = 1, \ldots, 2q$,

$$\frac{\partial}{\partial w^s} \in \mathrm{im}\, S \subset \ker S \subset \ker \mu \cap \ker \nu,$$

hence $D_{\partial/\partial w^s} A_2\xi = 0$. ■

We now introduce the functions $\Xi^1, \ldots, \Xi^{2n+2}$ by

$$(A_2\xi \circ {\phi_1}^{-1})(w) = \sum_{k=1}^{n+1} [\Xi^{2k-1}(w) e_{2k-1} - \Xi^{2k}(w) e_{2k}].$$

LEMMA 7.

$$\mathrm{rank}_{\mathbb{R}} \left[\frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,\ldots,2n+2;\, j=2q+1,\ldots,2n} = 2q$$

for $w \in \phi_1(U_1)$.

Proof. We have

$$\begin{aligned} \mathrm{rank}_{\mathbb{R}} &\left[\frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,\ldots,2n+2;\, j=2q+1,\ldots,2n} \\ &= \mathrm{rank}_{\mathbb{R}} \left[(-1)^{k-1} \frac{\partial \Xi^k}{\partial w^j}(w)\right]_{k=1,\ldots,2n+2;\, j=2q+1,\ldots,2n} \\ &= \dim_{\mathbb{R}} \mathrm{span}\{D_{\partial/\partial w^j} A_2\xi : j = 2q+1, \ldots, 2n\} \\ &= \dim_{\mathbb{R}} \mathrm{span}\{D_{\partial/\partial w^j} A_2\xi : j = 1, \ldots, 2n\} \\ &= \dim_{\mathbb{R}} \mathrm{im}\, S_{\phi_1^{-1}(w)}. \end{aligned}$$

The last equality is due to the isomorphism of $\mathrm{im}\, S$ and $\mathrm{im}\{X \mapsto D_X A_2\xi\}$.

A consequence of Lemma 7 is

COROLLARY. $n \geq 2q$.

LEMMA 8. $\widetilde{\Xi}^k := \Xi^{2k-1} + i\Xi^{2k}$ *is a holomorphic function for* $k = 1, \ldots, n+1$.

Proof. It is sufficient to show that Ξ^{2k-1} and Ξ^{2k} satisfy the Cauchy–Riemann equations. From Remark 3 it follows that

$$D_{JX}(A_2\xi) = -JD_X(A_2\xi)$$

for any $X \in T_{m'}M$. Let $X = \partial/\partial w^{2s-1}$. Then $JX = \partial/\partial w^{2s}$ and

$$D_{JX}(A_2\xi) = \sum_{l=1}^{n+1}\left[\frac{\partial \Xi^{2l-1}}{\partial w^{2s}}e_{2l-1} - \frac{\partial \Xi^{2l}}{\partial w^{2s}}e_{2l}\right],$$

$$\begin{aligned}-JD_X(A_2\xi) &= -J\sum_{l=1}^{n+1}\left[\frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}}e_{2l-1} - \frac{\partial \Xi^{2l}}{\partial w^{2s-1}}e_{2l}\right]\\ &= \sum_{l=1}^{n+1}\left[-\frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}}e_{2l} - \frac{\partial \Xi^{2l}}{\partial w^{2s-1}}e_{2l-1}\right].\end{aligned}$$

Therefore

$$\frac{\partial \Xi^{2l-1}}{\partial w^{2s}} = -\frac{\partial \Xi^{2l}}{\partial w^{2s-1}} \quad \text{and} \quad \frac{\partial \Xi^{2l-1}}{\partial w^{2s-1}} = \frac{\partial \Xi^{2l}}{\partial w^{2s}}. \ \blacksquare$$

Now $\widetilde{\Xi}^k$, $k = 1,\ldots,n+1$, are holomorphic functions of the complex variables $\zeta^s = w^{2s-1} + iw^{2s}$, $s = q+1,\ldots,n$. By Lemma 7, we have

$$\operatorname{rank}_{\mathbb{C}}\left[\frac{\partial \widetilde{\Xi}^k}{\partial \zeta^l}\right]_{k=1,\ldots,n+1;\, l=q+1,\ldots,n} = q.$$

LEMMA 9. *Let $r \le N \le M$. Let $\mathcal{U}$ be an open set in $\mathbb{K}^N$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $F : \mathcal{U} \to \mathbb{K}^M$ be a $\mathcal{C}^1$ mapping such that* $\operatorname{rank} F'(x) = r$ *for every $x \in \mathcal{U}$. Let $x_0 \in \mathcal{U}$ and let $i_1 < \ldots < i_r$ and $j_1 < \ldots < j_r$ be chosen so that*

$$\det\left[\frac{\partial F^{i_k}}{\partial x^{j_l}}(x_0)\right]_{k=1,\ldots,r;\, l=1,\ldots,r} \ne 0.$$

Then there exist a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of x_0 and a diffeomorphism $\Phi : \mathcal{U}' \to \Phi(\mathcal{U}') \subset \mathbb{K}^N$ such that

$$(F \circ \Phi^{-1})^{i_k}(y^1,\ldots,y^N) \equiv y^{j_k} \quad \text{for } k = 1,\ldots,r$$

and

$$\frac{\partial (F\circ\Phi^{-1})^k}{\partial y^l} \equiv 0 \quad \text{if } l \notin \{j_1,\ldots,j_r\},\ k \in \{1,\ldots,M\}.$$

Proof. We define

$$\widehat{\Phi}(x^1,\ldots,x^N)^k = \begin{cases} x^k & \text{if } k \notin \{j_1,\ldots,j_r\},\\ F^{i_s}(x^1,\ldots,x^N) & \text{if } k = j_s.\end{cases}$$

Since

$$\det\left[\frac{\partial \widehat{\Phi}^k}{\partial x^l}(x_0)\right]_{k=1,\ldots,N;\, l=1,\ldots,N} = \det\left[\frac{\partial F^{i_s}}{\partial x^{j_s}}(x_0)\right]_{s=1,\ldots,r} \ne 0,$$

there exists a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of x_0 such that $\Phi := \widehat{\Phi}|_{\mathcal{U}'}$ is a diffeomorphism.

It remains to prove that $F \circ \Phi^{-1}$ depends only on the variables $y^{j_1}, \ldots, y^{j_r}$.

Let $y \in \Phi(\mathcal{U}')$. Then $(F \circ \Phi^{-1})'(y) = F'(\Phi^{-1}(y)) \circ (\Phi^{-1})'(y)$ and $\operatorname{rank}(F \circ \Phi^{-1})'(y) = \operatorname{rank} F'(\Phi^{-1}(y)) = r$, because $(\Phi^{-1})'(y)$ is an isomorphism.

Suppose that for some $k_0 \in \{1, \ldots, M\}$ and $l_0 \in \{1, \ldots, N\} \setminus \{j_1, \ldots, j_r\}$,

$$\frac{\partial (F \circ \Phi^{-1})^{k_0}}{\partial y^{l_0}}(y) \neq 0.$$

Then $k_0 \notin \{i_1, \ldots, i_r\}$ and

$$\det \left[\frac{\partial (F \circ \Phi^{-1})^{k}}{\partial y^{l}}(y) \right]_{k \in \{i_1, \ldots, i_r, k_0\};\, l \in \{j_1, \ldots, j_r, l_0\}} \neq 0,$$

which contradicts the rank assumption. ■

We now restrict our attention to the case $q > 1$.

Lemma 10. *If* $q > 1$, *then*

(a) $\operatorname{rank}_{\mathbb{C}} [\partial \widetilde{\Xi}^k / \partial \zeta^l]_{k=1,\ldots,q;\, l=q+1,\ldots,n} = q$,
(b) $\widetilde{\Xi}^k(\cdot) = b^k = \mathrm{const}$ *for* $k = q+1, \ldots, n+1$,
(c) $b^{n+1} \neq 0$.

Proof. Since, for $q > 1$, $\mu = 0$ and $\nu = 0$ (see proof of Lemma 2, Case 3), we have

$$D_{\partial/\partial w^l} A_2 \xi \in (A_2 \circ f)_* \operatorname{im} S = \operatorname{span}_{\mathbb{R}}\{e_1, \ldots, e_{2q}\},$$

therefore $\partial \Xi^k / \partial w^l = 0$ for $k = 2q+1, \ldots, 2n+2$ and $l = 2q+1, \ldots, 2n$, which implies

$$\frac{\partial \widetilde{\Xi}^k}{\partial \zeta^l} = 0 \quad \text{for } k = q+1, \ldots, n+1,\ l = q+1, \ldots, n,$$

and

$$\operatorname{rank}_{\mathbb{C}} \left[\frac{\partial \widetilde{\Xi}^k}{\partial \zeta^l} \right]_{k=1,\ldots,n+1;\, l=q+1,\ldots,n} = \operatorname{rank}_{\mathbb{C}} \left[\frac{\partial \widetilde{\Xi}^k}{\partial \zeta^l} \right]_{k=1,\ldots,q;\, l=q+1,\ldots,n}.$$

Point (c) is a consequence of Lemma 4 and the definition of $\widetilde{A}_2$. ■

We may now apply Lemma 9, taking $r = q$, $N = n$, $M = n+1$, $\mathcal{U} = \widetilde{\phi}_1(U_1)$, $\mathbb{K} = \mathbb{C}$, $F = (\widetilde{\Xi}^1, \ldots, \widetilde{\Xi}^{n+1})$, $(i_1, \ldots, i_q) = (1, \ldots, q)$ and $j_1 < \ldots < j_q$ from $\{q+1, \ldots, n\}$ chosen so that

$$\det \left[\frac{\partial \widetilde{\Xi}^k}{\partial \zeta^{j_s}}(\widetilde{\phi}_1(m)) \right]_{k=1,\ldots,q;\, s=1,\ldots,q} \neq 0.$$

In this way we obtain a new chart $\widetilde{\phi}_2 := \Phi \circ \widetilde{\phi}_1$ on the neighbourhood $U_2 := \widetilde{\phi}_1^{-1}(\mathcal{U}')$ of m. Since

$$(\widetilde{\Xi}^k \circ \Phi^{-1})(\eta^1, \dots, \eta^n) = \begin{cases} \eta^{j_k} & \text{for } k = 1, \dots, q, \\ b^k & \text{for } k = q+1, \dots, n+1, \end{cases}$$

the transversal field is now described by the formula

$$(\widetilde{A}_2 \xi \circ \widetilde{\phi}_2^{-1})(\eta^1, \dots, \eta^n) = \sum_{k=1}^{q} \overline{\eta^{j_k}}\, \widetilde{e}_k + \sum_{k=q+1}^{n+1} \overline{b^k}\, \widetilde{e}_k.$$

Let $\widetilde{A}_3^0$ be the linear isomorphism of $\mathbb{C}^{n+1}$ which transforms the basis $(\widetilde{e}_1, \dots, \widetilde{e}_{n+1})$ onto the basis

$$(\widetilde{e}_1, \dots, \widetilde{e}_q, \widetilde{e}_{q+1}, \dots, \widehat{\widetilde{e}_{j_1}}, \dots, \widehat{\widetilde{e}_{j_q}}, \dots, \widetilde{e}_n, \widetilde{e}_{j_1}, \dots, \widetilde{e}_{j_q}, \widetilde{e}_{n+1}),$$

and let

$$\widetilde{\phi}_3^0(\eta^1, \dots, \eta^n) := (\eta^1, \dots, \eta^q, \eta^{q+1}, \dots, \widehat{\eta^{j_1}}, \dots, \widehat{\eta^{j_q}}, \dots, \eta^n, \eta^{j_1}, \dots, \eta^{j_q}).$$

Taking $\widetilde{A}_3 := \widetilde{A}_3^0 \circ \widetilde{A}_2$, $\widetilde{\phi}_3 := \widetilde{\phi}_3^0 \circ \widetilde{\phi}_2$, we may write

$$\begin{aligned} &\widetilde{A}_3 \circ f \circ \widetilde{\phi}_3^{-1}(\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{t}^1, \dots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \dots, \widetilde{u}^q) \\ &\qquad = (\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{t}^1, \dots, \widetilde{t}^{n-2q}, \widetilde{\chi}^1(\widetilde{t}, \widetilde{u}), \dots, \widetilde{\chi}^q(\widetilde{t}, \widetilde{u}), \widetilde{\varrho}(\widetilde{t}, \widetilde{u})) \end{aligned}$$

and

$$\widetilde{A}_3 \xi \circ \widetilde{\phi}_3^{-1}(\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{t}^1, \dots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \dots, \widetilde{u}^q) = \sum_{k=1}^{q} \overline{\widetilde{u}^k}\, \widetilde{e}_k + \sum_{k=q+1}^{n+1} \overline{a^k}\, \widetilde{e}_k.$$

Applying now the isomorphism $\widetilde{A}_4^0$, where

$$\widetilde{A}_4^0 \widetilde{e}_k := \begin{cases} \widetilde{e}_k & \text{for } k = 1, \dots, n, \\ (1/\overline{a^{n+1}})(-\sum_{k=q+1}^{n} \overline{a^k}\, \widetilde{e}_k + \widetilde{e}_{n+1}) & \text{for } k = n+1, \end{cases}$$

and $\widetilde{A}_4 := \widetilde{A}_4^0 \circ \widetilde{A}_3$, we obtain

$$\begin{aligned} &\widetilde{A}_4 \circ f \circ \widetilde{\phi}_3^{-1}(\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{t}^1, \dots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \dots, \widetilde{u}^q) \\ &\qquad = (\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{\sigma}^1(\widetilde{t}, \widetilde{u}), \dots, \widetilde{\sigma}^{n-q}(\widetilde{t}, \widetilde{u}), \widetilde{\mathcal{Q}}(\widetilde{t}, \widetilde{u})) \end{aligned}$$

and

$$\widetilde{A}_4 \xi \circ \widetilde{\phi}_3^{-1}(\widetilde{s}^1, \dots, \widetilde{s}^q, \widetilde{t}^1, \dots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \dots, \widetilde{u}^q) = \sum_{k=1}^{q} \overline{\widetilde{u}^k}\, \widetilde{e}_k + \widetilde{e}_{n+1}.$$

Here

$$\widetilde{\sigma}^k(\widetilde{t}, \widetilde{u}) = \begin{cases} \widetilde{t}^k - \dfrac{\overline{a^{q+k}}}{\overline{a^{n+1}}} \widetilde{\varrho}(\widetilde{t}, \widetilde{u}) & \text{for } k = 1, \dots, n-2q, \\[2ex] \widetilde{\chi}^{k-(n-2q)}(\widetilde{t}, \widetilde{u}) - \dfrac{\overline{a^{q+k}}}{\overline{a^{n+1}}} \widetilde{\varrho}(\widetilde{t}, \widetilde{u}) & \text{for } k = n-2q+1, \dots, n-q; \end{cases}$$

and $\widetilde{\mathcal{Q}} = (1/\overline{a^{n+1}})\widetilde{\varrho}$.

We now turn to the case $q = 1$.

By Lemmas 7–9, on a connected neighbourhood $\widetilde{\widetilde{\widetilde{U}}}_2$ of m we may define a complex chart $\widetilde{\widetilde{\widetilde{\phi}}}_2$ in which the coordinates of the transversal field are functions of one variable η^{i_0} only, with $i_0 > 1$.

Next we apply the isomorphisms

$$\widetilde{\widetilde{\widetilde{\phi}}}{}_3^0(\eta^1, \ldots, \eta^n) := (\eta^1, \eta^2, \ldots, \widehat{\eta^{i_0}}, \ldots, \eta^n, \eta^{i_0})$$

and

$$\widetilde{\widetilde{A}}{}_3^0(\theta^1, \ldots, \theta^{n+1}) := (\theta^1, \theta^2, \ldots, \widehat{\theta^{i_0}}, \ldots, \theta^n, \theta^{i_0}, \theta^{n+1}),$$

to obtain

$$\widetilde{\widetilde{A}}_3 \circ f \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_3^{-1}(\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{u}) = (\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{\widetilde{\chi}}(\widetilde{t}, \widetilde{u}), \widetilde{\widetilde{\varrho}}(\widetilde{t}, \widetilde{u}))$$

and

$$\widetilde{\widetilde{A}}_3 \xi \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_3^{-1}(\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{u}) = \sum_{k=1}^{n+1} \overline{\Theta^k(\widetilde{u})}\, \widetilde{e}_k$$

with $\Theta^{n+1}(\widetilde{\widetilde{\widetilde{\phi}}}_3(m)) \neq 0$; $\widetilde{\widetilde{\widetilde{\phi}}}_3 := \widetilde{\widetilde{\widetilde{\phi}}}{}_3^0 \circ \widetilde{\widetilde{\widetilde{\phi}}}_2$ and $\widetilde{\widetilde{A}}_3 := \widetilde{\widetilde{A}}{}_3^0 \circ \widetilde{A}_2$.

In the real representation $\widetilde{\widetilde{\chi}} = \chi^1 + i\chi^2$, $\widetilde{\widetilde{\varrho}} = \varrho^1 + i\varrho^2$, $\Theta^k = \vartheta^{2k-1} + i\vartheta^{2k}$, we have

$$\begin{aligned} &A_3 \circ f \circ \phi_3^{-1}(s^1, s^2, t^1, \ldots, t^{2n-4}, u^1, u^2) \\ &\qquad = (s^1, s^2, t^1, \ldots, t^{2n-4}, \chi^1(t, u), \chi^2(t, u), \varrho^1(t, u), \varrho^2(t, u)) \end{aligned}$$

and

$$A_3 \xi \circ \phi_3^{-1}(s^1, s^2, t^1, \ldots, t^{2n-4}, u^1, u^2) = \sum_{k=1}^{n+1} [\vartheta^{2k-1}(u) e_{2k-1} - \vartheta^{2k}(u) e_{2k}].$$

Let $\widehat{\pi}$ denote the projection

$$\mathbb{C}^n \ni (\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{u}) \mapsto \widetilde{u} \in \mathbb{C}.$$

LEMMA 11. *There exist* $c_2, \ldots, c_{n+1} \in \mathbb{C}$, $c_{n+1} \neq 0$, *a neighbourhood* $\widetilde{\widetilde{\widetilde{U}}}_3$ *of* m *and a holomorphic function* $\widetilde{\mathcal{H}}$ *such that*

$$\Theta^k(\widetilde{u}) = c_k e^{\widetilde{\mathcal{H}}(\widetilde{u})}$$

for $k = 1, \ldots, n+1$ *and* $\widetilde{u} \in \widehat{\pi}(\widetilde{\widetilde{\widetilde{\phi}}}_3(\widetilde{\widetilde{\widetilde{U}}}_3))$.

Proof. We fix $j \in \{2, \ldots, n+1\}$. Since Θ^j is a holomorphic function, and since $\widetilde{\widetilde{\widetilde{U}}}_2$ is assumed to be connected, there are two possibilities: either $\Theta^j \equiv 0$ on $\widehat{\pi} \circ \widetilde{\widetilde{\widetilde{\phi}}}_3(\widetilde{\widetilde{\widetilde{U}}}_2)$ or there exists a neighbourhood $\widetilde{W}_j$ of $\widetilde{u}_0 := \widehat{\pi}(\widetilde{\widetilde{\widetilde{\phi}}}_3(m))$ such that $\Theta^j(\widetilde{u}) \neq 0$ for any $\widetilde{u} \in \widetilde{W}_j \setminus \{\widetilde{u}_0\}$.

In the former case we take $c_j = 0$.

Suppose that $\Theta^j \not\equiv 0$. We can find $r > 0$ such that $B(\widetilde{u}_0, r) \subset \widetilde{W}_j$. Then

$$\widetilde{W}'_j := B(\widetilde{u}_0, r) \setminus \{\widetilde{u} : \operatorname{Re}\widetilde{u} = \operatorname{Re}\widetilde{u}_0,\ \operatorname{Im}\widetilde{u} \le \operatorname{Im}\widetilde{u}_0\}$$

is a simply connected domain and $\Theta^j(\widetilde{u}) \neq 0$ for any $\widetilde{u} \in \widetilde{W}'_j$. If this is the case, there exists a holomorphic function λ^j on $\widetilde{W}'_j$ such that $e^{\lambda^j} = \Theta^j|_{\widetilde{W}'_j}$ and $(\lambda^j)' = {\Theta^j}'/\Theta^j$ on $\widetilde{W}'_j$.

On the other hand, using the real representation we may write

$$(\Theta^j)' = \frac{\partial \vartheta^{2j-1}}{\partial u^1} + i\frac{\partial \vartheta^{2j}}{\partial u^1}.$$

From the Weingarten formula

$$D_{\partial/\partial u^1} A_3\xi = -(A_3 \circ f)_* S\frac{\partial}{\partial u^1} + \mu\biggl(\frac{\partial}{\partial u^1}\biggr)A_3\xi + \nu\biggl(\frac{\partial}{\partial u^1}\biggr)JA_3\xi$$

it follows that for any $j \in \{2, \ldots, n+1\}$,

$$\frac{\partial \vartheta^{2j-1}}{\partial u^1} = \mu\biggl(\frac{\partial}{\partial u^1}\biggr)\vartheta^{2j-1} + \nu\biggl(\frac{\partial}{\partial u^1}\biggr)\vartheta^{2j}$$

and

$$\frac{\partial \vartheta^{2j}}{\partial u^1} = \mu\biggl(\frac{\partial}{\partial u^1}\biggr)\vartheta^{2j} - \nu\biggl(\frac{\partial}{\partial u^1}\biggr)\vartheta^{2j-1}.$$

Hence

$$(\Theta^j)' = \biggl(\mu\biggl(\frac{\partial}{\partial u^1}\biggr) - i\nu\biggl(\frac{\partial}{\partial u^1}\biggr)\biggr)(\vartheta^{2j-1} + i\vartheta^{2j}).$$

We may also assume that $\widetilde{\widetilde{\widetilde{U}}}_2$ is simply connected, and from Lemma 2 we know that the 1-forms μ, ν are closed; therefore there exist functions $\mathcal{K}$ and $\mathcal{L}$ on $\widetilde{\widetilde{\widetilde{U}}}_2$ such that $\mu = d\mathcal{K}$, $\nu = -d\mathcal{L}$. The functions $\mathcal{K} \circ \phi_3^{-1}$ and $\mathcal{L} \circ \phi_3^{-1}$ do not depend on the variables s and t, because $\partial/\partial s^i, \partial/\partial t^j \in \ker S \subset \ker \mu \cap \ker \nu$,

$$\frac{\partial \mathcal{K} \circ \phi_3^{-1}}{\partial s^i} = d\mathcal{K}\biggl(\frac{\partial}{\partial s^i}\biggr) = \mu\biggl(\frac{\partial}{\partial s^i}\biggr) = 0,$$

and similarly for $\mathcal{L}$ in place of $\mathcal{K}$ or t in place of s. It follows that there exist functions $\mathcal{H}^1$ and $\mathcal{H}^2$ defined on some open subset of $\mathbb{R}^2$ such that $\mathcal{K} \circ \phi_3^{-1} = \mathcal{H}^1 \circ \widehat{\pi}$ and $\mathcal{L} \circ \phi_3^{-1} = \mathcal{H}^2 \circ \widehat{\pi}$. We now have

$$\mu\biggl(\frac{\partial}{\partial u^s}\biggr) = d\mathcal{K}((\phi_3^{-1})_* e_{2n-2+s}) = d\mathcal{H}^1(\widehat{\pi}_* e_{2n-2+s}) = d\mathcal{H}^1(e_s) = \frac{\partial \mathcal{H}^1}{\partial u^s}$$

and

$$\begin{aligned}\nu\biggl(\frac{\partial}{\partial u^s}\biggr) &= -d\mathcal{L}((\phi_3^{-1})_* e_{2n-2+s}) = -d\mathcal{H}^2(\widehat{\pi}_* e_{2n-2+s}) \\ &= -d\mathcal{H}^2(e_s) = -\frac{\partial \mathcal{H}^2}{\partial u^s}.\end{aligned}$$

According to Remark 3, $\mathcal{H}^1$ and $\mathcal{H}^2$ satisfy the Cauchy–Riemann equations, therefore $\widetilde{\mathcal{H}} := \mathcal{H}^1 + i\mathcal{H}^2$ is a holomorphic function.

Since

$$(\widetilde{\mathcal{H}})' = \frac{\partial \mathcal{H}^1}{\partial u^1} + i\frac{\partial \mathcal{H}^2}{\partial u^1} = \mu\left(\frac{\partial}{\partial u^1}\right) - i\nu\left(\frac{\partial}{\partial u^1}\right),$$

we may go back to the $(\Theta^j)'$ and write

$$(\Theta^j)' = \widetilde{\mathcal{H}}'\Theta^j$$

on some neighbourhood $\widehat{\pi}(\widetilde{\widetilde{\phi}}_3(\widetilde{\widetilde{U}}'_3))$ of $\widetilde{u}_0$ (including $\widetilde{u}_0$). Comparing this with $(\lambda^j)'$ we obtain $\widetilde{\mathcal{H}}' = (\lambda^j)'$ on $\widetilde{W}''_j := \widetilde{W}'_j \cap \widehat{\pi}(\widetilde{\widetilde{\phi}}_3(\widetilde{\widetilde{U}}'_3))$. Hence there exists $d_j \in \mathbb{C}$ such that $\lambda^j = \widetilde{\mathcal{H}} + d_j$ and

$$\Theta^j = e^{\widetilde{\mathcal{H}}+d_j} = c_j e^{\widetilde{\mathcal{H}}}$$

with a non-zero constant $c_j := e^{d_j}$. We may extend this equality from $\widetilde{W}''_j$ to some neighbourhood $\widehat{\pi}(\widetilde{\widetilde{\phi}}_3(\widetilde{\widetilde{U}}_3))$ of $\widetilde{u}_0$, because both sides are continuous and well defined in the neighbourhood of $\widetilde{u}_0$. Since $\Theta^{n+1} \not\equiv 0$, we have in particular $c_{n+1} \neq 0$. ■

Next we use the following isomorphism of $\mathbb{C}^{n+1}$:

$$\widetilde{\widetilde{A}}{}^0_4\widetilde{e}_k := \begin{cases} \widetilde{e}_k & \text{for } k = 1, \ldots, n, \\ (1/\overline{c}_{n+1})(\widetilde{e}_{n+1} - \sum_{s=2}^n \overline{c}_s \widetilde{e}_s) & \text{for } k = n+1. \end{cases}$$

In this way we obtain

$$\widetilde{\widetilde{A}}_4 \circ f \circ \widetilde{\widetilde{\phi}}{}_3^{-1}(\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{u}) = (\widetilde{s}, \widetilde{\widetilde{\sigma}}{}^1(\widetilde{t}, \widetilde{u}), \ldots, \widetilde{\widetilde{\sigma}}{}^{n-1}(\widetilde{t}, \widetilde{u}), \widetilde{\widetilde{\mathcal{Q}}}(\widetilde{t}, \widetilde{u}))$$

and

$$\widetilde{\widetilde{A}}_4\xi \circ \widetilde{\widetilde{\phi}}{}_3^{-1}(\widetilde{s}, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2}, \widetilde{u}) = \overline{\Theta^1(\widetilde{u})}\, \widetilde{e}_1 + \overline{e^{\widetilde{\mathcal{H}}(\widetilde{u})}}\, \widetilde{e}_{n+1},$$

with $\widetilde{\widetilde{A}}_4 := \widetilde{\widetilde{A}}{}^0_4 \circ \widetilde{\widetilde{A}}_3$, $\widetilde{\widetilde{\sigma}}{}^k(\widetilde{t}, \widetilde{u}) := \widetilde{t}^k - \overline{(c_{k+1}/c_{n+1})}\, \widetilde{\widetilde{\varrho}}(\widetilde{t}, \widetilde{u})$ for $k = 1, \ldots, n-2$, $\widetilde{\widetilde{\sigma}}{}^{n-1} := \widetilde{\widetilde{\chi}}(\widetilde{t}, \widetilde{u}) - \overline{(c_{k+1}/c_{n+1})}\, \widetilde{\widetilde{\varrho}}(\widetilde{t}, \widetilde{u})$ and $\widetilde{\widetilde{\mathcal{Q}}}(\widetilde{t}, \widetilde{u}) := \overline{(1/c_{n+1})}\, \widetilde{\widetilde{\varrho}}(\widetilde{t}, \widetilde{u})$.

Thus for any $q > 0$ it is possible to find a map $\widehat{\phi}_3$ and an isomorphism $\widehat{A}_4$ of $\mathbb{C}^{n+1}$ such that the immersion and the transversal field have the shape

$$\begin{aligned} &\widehat{A}_4 \circ f \circ \widehat{\phi}{}_3^{-1}(\widetilde{s}^1, \ldots, \widetilde{s}^q, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \ldots, \widetilde{u}^q) \\ &\qquad = (\widetilde{s}^1, \ldots, \widetilde{s}^q, \widehat{\sigma}^1(\widetilde{t}, \widetilde{u}), \ldots, \widehat{\sigma}^{n-q}(\widetilde{t}, \widetilde{u}), \widehat{\mathcal{Q}}(\widetilde{t}, \widetilde{u})) \end{aligned}$$

and

$$\begin{aligned} &\widehat{A}_4\xi \circ \widehat{\phi}{}_3^{-1}(\widetilde{s}^1, \ldots, \widetilde{s}^q, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \ldots, \widetilde{u}^q) \\ &\qquad = \sum_{k=1}^{q} \overline{\widehat{\Theta}^k(\widetilde{u})}\, \widetilde{e}_k + \overline{\widehat{\Theta}^{n+1}(\widetilde{u})}\, \widetilde{e}_{n+1}, \end{aligned}$$

where $\widehat{\sigma}^i$, $\widehat{\mathcal{Q}}$, $\widehat{\Theta}^j$ are holomorphic functions and $\widehat{\Theta}^{n+1} \neq 0$.

Since $\widehat{A}_4 \circ f$ is an immersion, and $\widehat{A}_4 \circ \xi$ is transversal to $(\widehat{A}_4 \circ f)_* TM$, we have

$$\begin{vmatrix} & I_q & & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{matrix} & \begin{matrix} \overline{\widehat{\Theta}^1} \\ \vdots \\ \overline{\widehat{\Theta}^q} \end{matrix} \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} \cdots \\ \ddots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \left[\frac{\partial \widehat{\sigma}^k}{\partial \widetilde{t}^l}\right]_{\substack{1\le k\le n-q;\\ 1\le l\le n-2q}} & \left[\frac{\partial \widehat{\sigma}^k}{\partial \widetilde{u}^l}\right]_{\substack{1\le k\le n-q;\\ 1\le l\le q}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 & \cdots & 0 & \frac{\partial \widehat{\mathcal{Q}}}{\partial \widetilde{t}^1} \ \cdots \ \frac{\partial \widehat{\mathcal{Q}}}{\partial \widetilde{t}^{n-2q}} & \frac{\partial \widehat{\mathcal{Q}}}{\partial \widetilde{u}^1} \ \cdots \ \frac{\partial \widehat{\mathcal{Q}}}{\partial \widetilde{u}^q} & \overline{\widehat{\Theta}^{n+1}} \end{vmatrix} \neq 0.$$

Therefore

$$\det\left(\left[\frac{\partial \widehat{\sigma}^k}{\partial \widetilde{t}^l}\right]_{k=1,\ldots,n-q;\, l=1,\ldots,n-2q} \left[\frac{\partial \widehat{\sigma}^k}{\partial \widetilde{u}^l}\right]_{k=1,\ldots,n-q;\, l=1,\ldots,q}\right) \neq 0$$

and there exist $i_1 < \ldots < i_{n-2q}$ such that

$$\det\left[\frac{\partial \widehat{\sigma}^{i_k}}{\partial \widetilde{t}^l}\right]_{k=1,\ldots,n-2q;\, l=1,\ldots,n-2q} \neq 0.$$

By an appropiate isomorphism $\widehat{A}^0_5$ we may vary the order of basis vectors in $\mathbb{C}^{n+1}$, putting $\widehat{\sigma}^{i_1}, \ldots, \widehat{\sigma}^{i_{n-2q}}$ at positions $q+1, \ldots, n-q$. This permutation does not affect the field $\widehat{A}_4 \xi$, because its coordinates from the $(q+1)$th to the nth vanish.

Applying now the local diffeomorphism

$$\begin{aligned} &\widehat{\phi}^0_4(\widetilde{s}^1, \ldots, \widetilde{s}^q, \widetilde{t}^1, \ldots, \widetilde{t}^{n-2q}, \widetilde{u}^1, \ldots, \widetilde{u}^q) \\ &\qquad := (\widetilde{s}^1, \ldots, \widetilde{s}^q, \widehat{\sigma}^{i_1}(\widetilde{t}, \widetilde{u}), \ldots, \widehat{\sigma}^{i_{n-2q}}(\widetilde{t}, \widetilde{u}), \widetilde{u}^1, \ldots, \widetilde{u}^q) \end{aligned}$$

gives a new chart $\widehat{\phi}_4 := \widehat{\phi}^0_4 \circ \widehat{\phi}_3$ such that $\widehat{A}_5 \circ f$ and $\widehat{A}_5 \circ \xi$ are described by the formulas

$$\begin{aligned} &\widehat{A}_5 \circ f \circ \widehat{\phi}_4^{-1}(\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q) \\ &\qquad = (\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widehat{\psi}^1(\widetilde{y}, \widetilde{z}), \ldots, \widehat{\psi}^q(\widetilde{y}, \widetilde{z}), \widehat{\mathcal{F}}(\widetilde{y}, \widetilde{z})) \end{aligned}$$

and

$$\begin{aligned} &\widehat{A}_5 \circ \xi \circ \widehat{\phi}_4^{-1}(\widetilde{x}^1, \ldots, \widetilde{x}^q, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \ldots, \widetilde{z}^q) \\ &\qquad = \sum_{k=1}^{q} \overline{\widehat{\Theta}^k(\widetilde{z})}\, \widetilde{e}_k + \overline{\widehat{\Theta}^{n+1}(\widetilde{z})}\, \widetilde{e}_{n+1}. \end{aligned}$$

LEMMA 12. *For $k = 1, \ldots, q$,*

$$\widehat{\psi}^k(\widetilde{y}, \widetilde{z}) = \sum_{s=1}^{q} C^k{}_s \widetilde{y}^s + \widehat{\Pi}^k(\widetilde{z}),$$

where ${C^k}_s$, $s = 1, \ldots, n-2q$, *are complex numbers*, *and* $\widehat{\Pi}^k$ *is a holomorphic function.*

Proof. We now use the real representation of $\widehat{A}_5 \circ f$ and $\widehat{A}_5 \circ \xi$, setting $\widehat{\psi}^l = \check{\psi}^{2l-1} + i\check{\psi}^{2l}$ for $l = 1, \ldots, q$, $\widehat{\mathcal{F}} = \check{\mathcal{F}}^1 + i\check{\mathcal{F}}^2$, and $\widehat{\Theta}^s = \check{\Theta}^{2s-1} + i\check{\Theta}^{2s}$:

$$\begin{aligned}
&\check{A}_5 \circ f \circ \check{\phi}_4^{-1}(x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q}) \\
&\quad = (x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, \check{\psi}^1(y,z), \ldots, \check{\psi}^{2q}(y,z), \check{\mathcal{F}}^1(y,z), \check{\mathcal{F}}^2(y,z)),
\end{aligned}$$

$$\begin{aligned}
&\check{A}_5 \circ \xi \circ \check{\phi}_4^{-1}(x^1, \ldots, x^{2q}, y^1, \ldots, y^{2n-4q}, z^1, \ldots, z^{2q}) \\
&\quad = \sum_{k=1}^{q} [\check{\Theta}^{2k-1}(z) e_{2k-1} - \check{\Theta}^{2k}(z) e_{2k}] + \check{\Theta}^{2n+1}(z) e_{2n+1} - \check{\Theta}^{2n+2}(z) e_{2n+2}.
\end{aligned}$$

At any point m' of the domain $\check{U}$ of $\check{\phi}_4$, $\ker S$ is spanned by

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2q}}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{2n-4q}}.$$

For any $W \in T_{m'}M$ and any $j = 1, \ldots, 2n-4q$,

$$S\left(\nabla_W \frac{\partial}{\partial y^j}\right) = \nabla_W \left(S \frac{\partial}{\partial y^j}\right) = 0,$$

therefore

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial y^s}} \frac{\partial}{\partial y^j} &= \sum_{k=1}^{2q} \alpha_{sj}^k \frac{\partial}{\partial x^k} + \sum_{l=1}^{2n-4q} \beta_{sj}^l \frac{\partial}{\partial y^l}, \\
\nabla_{\frac{\partial}{\partial z^s}} \frac{\partial}{\partial y^j} &= \sum_{k=1}^{2q} \gamma_{sj}^k \frac{\partial}{\partial x^k} + \sum_{l=1}^{2n-4q} \delta_{sj}^l \frac{\partial}{\partial y^l}.
\end{aligned}$$

We have

$$\begin{aligned}
&(\check{A}_5 \circ f)_* \left(\frac{\partial}{\partial x^k}\right) = e_k, \\
&(\check{A}_5 \circ f)_* \left(\frac{\partial}{\partial y^l}\right) = e_{2q+l} + \sum_{r=1}^{2q} \frac{\partial \check{\psi}^r}{\partial y^l} e_{2n-2q+r} + \frac{\partial \check{\mathcal{F}}^1}{\partial y^l} e_{2n+1} + \frac{\partial \check{\mathcal{F}}^2}{\partial y^l} e_{2n+2};
\end{aligned}$$

hence

$$\begin{aligned}
&(\check{A}_5 \circ f)_* \left(\nabla_{\partial/\partial y^s} \frac{\partial}{\partial y^j}\right) \\
&\quad = \sum_{k=1}^{2q} \alpha_{sj}^k e_k + \sum_{l=1}^{2n-4q} \beta_{sj}^l e_{2q+l} + \sum_{r=1}^{2q} \left(\sum_{l=1}^{2n-4q} \beta_{sj}^l \frac{\partial \check{\psi}^r}{\partial y^l}\right) e_{2n-2q+r} \\
&\qquad + \left(\sum_{l=1}^{2n-4q} \beta_{sj}^l \frac{\partial \check{\mathcal{F}}^1}{\partial y^l}\right) e_{2n+1} + \left(\sum_{l=1}^{2n-4q} \beta_{sj}^l \frac{\partial \check{\mathcal{F}}^2}{\partial y^l}\right) e_{2n+2}
\end{aligned}$$

and

$$(\check{A}_5 \circ f)_* \left(\nabla_{\partial/\partial z^s} \frac{\partial}{\partial y^j} \right)$$

$$= \sum_{k=1}^{2q} \gamma_{sj}^k e_k + \sum_{l=1}^{2n-4q} \delta_{sj}^l e_{2q+l} + \sum_{r=1}^{2q} \left(\sum_{l=1}^{2n-4q} \delta_{sj}^l \frac{\partial \check{\psi}^r}{\partial y^l} \right) e_{2n-2q+r}$$

$$+ \left(\sum_{l=1}^{2n-4q} \delta_{sj}^l \frac{\partial \check{\mathcal{F}}^1}{\partial y^l} \right) e_{2n+1} + \left(\sum_{l=1}^{2n-4q} \delta_{sj}^l \frac{\partial \check{\mathcal{F}}^2}{\partial y^l} \right) e_{2n+2}.$$

On the other hand, using the Gauss formula we may write

$$(\check{A}_5 \circ f)_* \left(\nabla_{\partial/\partial y^s} \frac{\partial}{\partial y^j} \right)$$

$$= D_{\partial/\partial y^s} (\check{A}_5 \circ f)_* \frac{\partial}{\partial y^j} - h\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{A}_5 \xi + h\left(J \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) J \check{A}_5 \xi$$

$$= \sum_{k=1}^{q} \Bigg[\left(-h\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k-1}(z) + h\left(J \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k}(z) \right) e_{2k-1}$$

$$+ \left(h\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k}(z) + h\left(J \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k-1}(z) \right) e_{2k} \Bigg]$$

$$+ \sum_{r=1}^{2q} \frac{\partial^2 \check{\psi}^r}{\partial y^s \partial y^j} e_{2n-2q+r}$$

$$+ \left(\frac{\partial^2 \check{\mathcal{F}}^1}{\partial y^s \partial y^j} - h\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2n+1}(z) + h\left(J \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2n+2}(z) \right) e_{2n+1}$$

$$+ \left(\frac{\partial^2 \check{\mathcal{F}}^2}{\partial y^s \partial y^j} + h\left(\frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2n+2}(z) + h\left(J \frac{\partial}{\partial y^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2n+1}(z) \right) e_{2n+2}$$

and similarly

$$(\check{A}_5 \circ f)_* \left(\nabla_{\partial/\partial z^s} \frac{\partial}{\partial y^j} \right)$$

$$= D_{\partial/\partial z^s} (\check{A}_5 \circ f)_* \frac{\partial}{\partial y^j} - h\left(\frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) \check{A}_5 \xi + h\left(J \frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) J \check{A}_5 \xi$$

$$= \sum_{k=1}^{q} \Bigg[\left(-h\left(\frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k-1}(z) + h\left(J \frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k}(z) \right) e_{2k-1}$$

$$+ \left(h\left(\frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k}(z) + h\left(J \frac{\partial}{\partial z^s}, \frac{\partial}{\partial y^j} \right) \check{\Theta}^{2k-1}(z) \right) e_{2k} \Bigg]$$

$$+\sum_{r=1}^{2q}\frac{\partial^2\check{\psi}^r}{\partial z^s\partial y^j}e_{2n-2q+r}$$
$$+\left(\frac{\partial^2\check{\mathcal{F}}^1}{\partial z^s\partial y^j}-h\left(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j}\right)\check{\Theta}^{2n+1}(z)+h\left(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j}\right)\check{\Theta}^{2n+2}(z)\right)e_{2n+1}$$
$$+\left(\frac{\partial^2\check{\mathcal{F}}^2}{\partial z^s\partial y^j}+h\left(\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j}\right)\check{\Theta}^{2n+2}(z)+h\left(J\frac{\partial}{\partial z^s},\frac{\partial}{\partial y^j}\right)\check{\Theta}^{2n+1}(z)\right)e_{2n+2}.$$

Since in the second pair of expressions there are no terms containing the basis vectors e_t with $t\in\{2q+1,\ldots,2n-2q\}$, we conclude that $\beta^l_{sj}=0$ and $\delta^l_{sj}=0$ for any l,s,j. Comparing now the coefficients of $e_{t'}$ with $t'\in\{2n-2q+1,\ldots,2n\}$ we obtain

$$\frac{\partial^2\check{\psi}^r}{\partial y^s\partial y^j}=0\quad\text{and}\quad\frac{\partial^2\check{\psi}^r}{\partial z^s\partial y^j}=0$$

for any r,s,j.

It follows that

$$\frac{\partial\check{\psi}^r}{\partial y^j}=E^r_j=\text{const}\quad\text{and}\quad\check{\psi}^r(y,z)=\sum_{j=1}^{2n-4q}E^r_jy^j+\check{\Pi}^r(z).$$

The Cauchy–Riemann equations for $\widehat{\psi}^r$ imply that $\widehat{\Pi}^r:=\check{\Pi}^{2r-1}+i\check{\Pi}^{2r}$ is a holomorphic function and $E^{2r-1}_{2j-1}=E^{2r}_{2j}$, $E^{2r-1}_{2j}=-E^{2r}_{2j-1}$ for $r=1,\ldots,q$, $j=1,\ldots,n-2q$. We set $C^k{}_s:=E^{2k-1}_{2j-1}+iE^{2k}_{2j-1}$ and the lemma follows. ∎

We define the isomorphism $\widehat{A}^0_6$ of $\mathbb{C}^{n+1}$ by

$$\widehat{A}^0_6\widetilde{e}_k:=\begin{cases}\widetilde{e}_k & \text{if } k\notin\{q+1,\ldots,n-q\},\\ \widetilde{e}_k-\sum_{j=1}^q C^j{}_k\widetilde{e}_{n-q+j} & \text{if } k\in\{q+1,\ldots,n-q\}.\end{cases}$$

Let $\widehat{A}_6:=\widehat{A}^0_6\circ\widehat{A}_5$. Then

$$\begin{aligned}\widehat{A}_6\circ f\circ\widehat{\phi}_4^{-1}&(\widetilde{x}^1,\ldots,\widetilde{x}^q,\widetilde{y}^1,\ldots,\widetilde{y}^{n-2q},\widetilde{z}^1,\ldots,\widetilde{z}^q)\\&=(\widetilde{x}^1,\ldots,\widetilde{x}^q,\widetilde{y}^1,\ldots,\widetilde{y}^{n-2q},\widehat{\Pi}^1(\widetilde{z}),\ldots,\widehat{\Pi}^q(\widetilde{z}),\widehat{\mathcal{F}}(\widetilde{y},\widetilde{z}))\end{aligned}$$

and $\widehat{A}_6\xi\circ\widehat{\phi}_4^{-1}$ is given by the same formula as $\widehat{A}_5\xi\circ\widehat{\phi}_4^{-1}$.

LEMMA 13. *If $q>1$ then*

$$\widehat{\Pi}^r(\widetilde{z})=\sum_{s=1}^q C_1{}^r{}_s\widetilde{z}^s+C_2{}^r$$

with $C_1{}^r{}_s$, $C_2{}^r\in\mathbb{C}$.

Proof. Recall that for $q > 1$,

$$\widehat{\Theta}^k(\widetilde{z}^1, \dots, \widetilde{z}^q) = \widetilde{z}^k \quad \text{for } k = 1, \dots, q,$$
$$\widehat{\Theta}^{n+1}(\widetilde{z}^1, \dots, \widetilde{z}^q) \equiv 1.$$

Therefore we have

$$A_6\xi \circ \phi_4^{-1}(x^1, \dots, x^{2q}, y^1, \dots, y^{2n-4q}, z^1, \dots, z^{2q}) = \sum_{k=1}^{q} [z^{2k-1} e_{2k-1} - z^{2k} e_{2k}] + e_{2n+1},$$

and

$$D_{\partial/\partial z^s}(A_6\xi) = (-1)^{s-1} e_s = -(A_6 \circ f)_* \left((-1)^s \frac{\partial}{\partial x^s} \right),$$

which implies

$$S \frac{\partial}{\partial z^s} = (-1)^s \frac{\partial}{\partial x^s}.$$

Applying the covariant derivative ∇_W to the right-hand side of the above formula we obtain zero, therefore

$$S\left(\nabla_W \frac{\partial}{\partial z^s} \right) = \nabla_W \left(S \frac{\partial}{\partial z^s} \right) = 0$$

and so $\nabla_W(\partial/\partial z^s) \in \ker S$ for any tangent vector W. We can now proceed analogously to the proof of Lemma 12. ∎

The matrix $(C_1{}^r{}_s)_{r,s=1,\dots,q}$ is invertible, since $\widehat{A}_6 \circ f$ is an immersion and

$$\widehat{A}_6\xi \circ \widehat{\phi}_4^{-1}(\widetilde{x}^1, \dots, \widetilde{x}^q, \widetilde{y}^1, \dots, \widetilde{y}^{n-2q}, \widetilde{z}^1, \dots, \widetilde{z}^q) = \sum_{k=1}^{q} \overline{\widetilde{z}^k}\, \widetilde{e}_k + \widetilde{e}_{n+1}$$

is a transversal field. Let

$$(C_3{}^i{}_j)_{i,j=1,\dots,q} = [(C_1{}^r{}_s)_{r,s=1,\dots,q}]^{-1}.$$

To complete the proof of the theorem in the case $q > 1$ it remains to apply the affine isomorphism $\widetilde{A}_7^0$ of $\mathbb{C}^{n+1}$, where

$$\widetilde{A}_7^0(\theta^1, \dots, \theta^{n+1}) := \Big(\theta^1, \dots, \theta^{n-q}, \sum_{j_1=1}^{q} C_3{}^1{}_{j_1}(\theta^{n-q+j_1} - C_2{}^{j_1}), \dots, \sum_{j_q=1}^{q} C_3{}^q{}_{j_q}(\theta^{n-q+j_q} - C_2{}^{j_q}), \theta^{n+1} \Big).$$

The linear part of $\widetilde{A}_7^0$ does not change the transversal field $\widehat{A}_6\xi$, therefore we have $\widetilde{A} \circ f$ and $\vec{\widetilde{A}}\xi$ as claimed, with $\widetilde{A} := \widetilde{A}_7^0 \circ \widehat{A}_6$, $\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}$, $\widetilde{\phi} = \widehat{\phi}_4$.

For $q=1$,

$$\widehat{A}_6\circ f\circ\widehat{\phi}_4^{-1}(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widetilde{z})=(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widehat{\Pi}(\widetilde{z}),\widehat{\mathcal{F}}(\widetilde{y},\widetilde{z}))$$

with $\widehat{\Pi}'\neq 0$ in the neighbourhood of $\widetilde{\widetilde{\phi}}_4(m)$. We define a local diffeomorphism

$$\widetilde{\widetilde{\phi}}{}_5^0(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widetilde{z}):=(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widehat{\Pi}(\widetilde{z}))$$

and obtain

$$\widehat{A}_6\circ f\circ\widetilde{\widetilde{\phi}}{}_5^{-1}(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widetilde{z})=(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widetilde{z},\widehat{\widehat{\mathcal{F}}}(\widetilde{y},\widetilde{z})),$$

$$\widehat{A}_6\xi\circ\widetilde{\widetilde{\phi}}{}_5^{-1}(\widetilde{x},\widetilde{y}^1,\dots,\widetilde{y}^{n-2},\widetilde{z})=\overline{\widetilde{\widetilde{\mathcal{G}}}(\widetilde{z})}\,\widetilde{e}_1+\overline{e^{\widetilde{\widetilde{\mathcal{M}}}(\widetilde{z})}}\,\widetilde{e}_{n+1}$$

with $\widehat{\widehat{\mathcal{F}}}=\widehat{\mathcal{F}}\circ\widehat{\Pi}^{-1}$, $\widetilde{\widetilde{\mathcal{M}}}=\widetilde{\mathcal{H}}\circ\widehat{\Pi}^{-1}$, $\widetilde{\widetilde{\mathcal{G}}}=\Theta^1\circ\widehat{\Pi}^{-1}$, $\widetilde{\widetilde{\phi}}_5=\widetilde{\widetilde{\phi}}{}_5^0\circ\widehat{\phi}_4$.

Lemma 14. $\widetilde{\widetilde{\mathcal{G}}}'-\widetilde{\widetilde{\mathcal{M}}}'\widetilde{\widetilde{\mathcal{G}}}=\text{const}\neq 0$.

Proof. Let $\widetilde{\widetilde{\mathcal{G}}}=G^1+iG^2$ and $\widetilde{\widetilde{\mathcal{M}}}=M^1+iM^2$. Then

$$\begin{aligned}&\widetilde{\widetilde{\mathcal{G}}}'-\widetilde{\widetilde{\mathcal{M}}}'\widetilde{\widetilde{\mathcal{G}}}\\&=\left(\frac{\partial G^1}{\partial z^1}+i\frac{\partial G^2}{\partial z^1}\right)-\left(\frac{\partial M^1}{\partial z^1}+i\frac{\partial M^2}{\partial z^1}\right)(G^1+iG^2)\\&=\left(\frac{\partial G^1}{\partial z^1}-\frac{\partial M^1}{\partial z^1}G^1+\frac{\partial M^2}{\partial z^1}G^2\right)+i\left(\frac{\partial G^2}{\partial z^1}-\frac{\partial M^2}{\partial z^1}G^1-\frac{\partial M^1}{\partial z^1}G^2\right).\end{aligned}$$

It is easily seen that

$$\begin{aligned}S\frac{\partial}{\partial z^1}=&\left(-\frac{\partial G^1}{\partial z^1}+\frac{\partial M^1}{\partial z^1}G^1-\frac{\partial M^2}{\partial z^1}G^2\right)\frac{\partial}{\partial x^1}\\&+\left(\frac{\partial G^2}{\partial z^1}-\frac{\partial M^2}{\partial z^1}G^1-\frac{\partial M^1}{\partial z^1}G^2\right)\frac{\partial}{\partial x^2},\end{aligned}$$

because

$$\begin{aligned}D_{\partial/\partial z^1}A_6\xi=&\frac{\partial G^1}{\partial z^1}e_1-\frac{\partial G^2}{\partial z^1}e_2\\&+\left(\frac{\partial M^1}{\partial z^1}e^{M^1}\cos M^2-\frac{\partial M^2}{\partial z^1}e^{M^1}\sin M^2\right)e_{2n+1}\\&+\left(-\frac{\partial M^1}{\partial z^1}e^{M^1}\sin M^2-\frac{\partial M^2}{\partial z^1}e^{M^1}\cos M^2\right)e_{2n+2}\\=&\left(\frac{\partial G^1}{\partial z^1}-\frac{\partial M^1}{\partial z^1}G^1+\frac{\partial M^2}{\partial z^1}G^2\right)e_1\\&+\left(-\frac{\partial G^2}{\partial z^1}+\frac{\partial M^2}{\partial z^1}G^1+\frac{\partial M^1}{\partial z^1}G^2\right)e_2+\frac{\partial M^1}{\partial z^1}A_6\xi-\frac{\partial M^2}{\partial z^1}JA_6\xi.\end{aligned}$$

For any vector W,

$$\nabla_W \frac{\partial}{\partial z^1} \in \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \right\} \subset \ker S,$$

since

$$\begin{aligned} D_W (A_6 \circ f)_* \frac{\partial}{\partial z^1} &= W\left(\frac{\partial \check{\check{\mathcal{F}}}^1}{\partial z^1}\right) e_{2n+1} + W\left(\frac{\partial \check{\check{\mathcal{F}}}^2}{\partial z^1}\right) e_{2n+2} \\ &\in \operatorname{span}_{\mathbb{R}}\{e_1, e_2, A_6\xi, JA_6\xi\}. \end{aligned}$$

It follows that $\nabla_W(S\partial/\partial z^1) = 0$ for any W. Therefore

$$\begin{aligned} -\frac{\partial G^1}{\partial z^1} + \frac{\partial M^1}{\partial z^1} G^1 - \frac{\partial M^2}{\partial z^1} G^2 &=: -B_1 = \text{const}, \\ \frac{\partial G^2}{\partial z^1} - \frac{\partial M^2}{\partial z^1} G^1 - \frac{\partial M^1}{\partial z^1} G^2 &=: B_2 = \text{const}. \end{aligned}$$

Moreover, $(B_1)^2 + (B_2)^2 \neq 0$, because $S \neq 0$. ∎

Let

$$\begin{aligned} \widetilde{\widetilde{\widetilde{\phi}}}{}_6^0(\widetilde{x}, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2}, \widetilde{z}) &:= (\widetilde{x}, \widetilde{y}^1, \ldots, \widetilde{y}^{n-2}, (B_1 + iB_2)\widetilde{z}), \\ \widetilde{\widetilde{\widetilde{A}}}{}_7^0(\zeta^1, \ldots, \zeta^{n+1}) &:= (\zeta^1, \ldots, \zeta^{n-1}, (B_1 + iB_2)\zeta^n, \zeta^{n+1}). \end{aligned}$$

Now $\widetilde{\widetilde{\widetilde{A}}}_7 \circ f \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_6^{-1}$ and $\widetilde{\widetilde{\widetilde{A}}}_7 \xi \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_6^{-1}$ have the same shape as $\widehat{A}_6 \circ f \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_5^{-1}$ and $\widehat{A}_6 \xi \circ \widetilde{\widetilde{\widetilde{\phi}}}{}_5^{-1}$ with $\widetilde{\widetilde{\widetilde{\mathcal{F}}}}$, $\widetilde{\widetilde{\widetilde{\mathcal{G}}}}$, $\widetilde{\widetilde{\widetilde{\mathcal{M}}}}$ replaced by $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{G}}$ $\widetilde{\mathcal{M}}$, where

$$\widetilde{\mathcal{F}}(\widetilde{z}) := \widehat{\widehat{\widetilde{\mathcal{F}}}}\left(\frac{\widetilde{z}}{B_1 + iB_2}\right), \qquad \widetilde{\mathcal{G}}(\widetilde{z}) := \widetilde{\widetilde{\widetilde{\mathcal{G}}}}\left(\frac{\widetilde{z}}{B_1 + iB_2}\right),$$

$$\widetilde{\mathcal{M}}(\widetilde{z}) := \widetilde{\widetilde{\widetilde{\mathcal{M}}}}\left(\frac{\widetilde{z}}{B_1 + iB_2}\right).$$

It is easy to check that

$$\widetilde{\mathcal{G}}' - \widetilde{\mathcal{M}}'\widetilde{\mathcal{G}} = 1.$$

This finishes the proof of the theorem. ∎

There are many examples of functions $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{M}}$ satisfying the above equation. In fact, for any holomorphic $\widetilde{\mathcal{M}}$ (and so for any μ, ν, because $\mu(\partial/\partial z^k) = \partial\mathcal{M}^1/\partial z^k$ and $\nu(\partial/\partial z^k) = -\partial\mathcal{M}^1/\partial z^k$) there exists $\widetilde{\mathcal{G}}$ such that $\widetilde{\mathcal{G}}' - \widetilde{\mathcal{M}}'\widetilde{\mathcal{G}} \equiv 1$. For example

$$\widetilde{\mathcal{G}}(\widetilde{z}) = \widetilde{z}, \qquad \widetilde{\mathcal{M}}(\widetilde{z}) = 0$$

(that is what we obtain also for $q > 1$, with $\mu = \nu = 0$);

$$\begin{aligned} \widetilde{\mathcal{G}}(\widetilde{z}) &= 1, & \widetilde{\mathcal{M}}(\widetilde{z}) &= -\widetilde{z}; \\ \widetilde{\mathcal{G}}(\widetilde{z}) &= e^{\widetilde{z}}, & \widetilde{\mathcal{M}}(\widetilde{z}) &= \widetilde{z} + e^{-\widetilde{z}}. \end{aligned}$$

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: robaszew@im.uj.edu.pl

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