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k-convexity in several complex variables

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Abstract. We define and investigate the notion of k-convexity in the sense of Mejia–Minda for domains in $\mathbb{C}^n$ and also that of k-convex mappings on the Euclidean unit ball.

1. Introduction. Mejia [17] investigated the hyperbolic geometry of k-convex regions in $\mathbb{C}$. Mejia–Minda [18] studied the hyperbolic geometry of k-convex regions in $\mathbb{C}$ and investigated k-convex functions on the unit disk U in $\mathbb{C}$. Ma–Mejia–Minda [16] obtained growth and distortion theorems for k-convex functions on U.

In this paper, we define and investigate the notion of k-convexity in the sense of Mejia-Minda for domains in $\mathbb{C}^n$ and also that of k-convex mappings on the Euclidean unit ball in $\mathbb{C}^n$.

2. Preliminaries. Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. The symbol $'$ means the transpose of vectors and matrices. For a domain Ω in $\mathbb{C}^n$, let $\delta_\Omega(a) = \inf\{\|z - a\| : z \in \partial\Omega\}$ denote the Euclidean distance from a to $\partial\Omega$. For open sets $G_1 \subset \mathbb{C}^n$, $G_2 \subset \mathbb{C}^m$, let $H(G_1, G_2)$ denote the set of holomorphic mappings from G_1 into G_2. Let $B(z_0, r) = \{z \in \mathbb{C}^n : \|z - z_0\| < r\}$. $B(0, r)$ is denoted by B_r and $B(0, 1)$ is denoted by $\mathbb{B}$. If $f \in H(B_r, \mathbb{C}^n)$, we say that f is *normalized* if $f(0) = 0$ and $Df(0) = I$.

For a C^2-curve $C : z = z(t)$ in $\mathbb{C}$, let

$$k(z(t), C) = \frac{1}{|z'(t)|} \Im\left\{\frac{z''(t)}{z'(t)}\right\}$$

denote the Euclidean curvature of C at $z(t)$.

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For a bounded domain D in $\mathbb{C}^n$, the *Carathéodory infinitesimal pseudometric* is defined by

$$\gamma_D(z;X) = \sup\{|Df(z)X| : f \in H(D,U),\ f(z)=0\},$$

where U is the unit disc in $\mathbb{C}$.

Now we recall the notion of strong starlikeness due to Chuaqui [1] (cf. [6]). Let $\mathbb{B} \subset \mathbb{C}^n$. A normalized locally biholomorphic mapping $f \in H(\mathbb{B}, \mathbb{C}^n)$ is called *starlike* if f is biholomorphic on $\mathbb{B}$ and $f(\mathbb{B})$ is a starlike domain, that is,

$$e^{-s} f(\mathbb{B}) \subset f(\mathbb{B}), \qquad s \geq 0.$$

Suffridge [20] showed that if f is a normalized locally biholomorphic mapping on $\mathbb{B}$, then f is starlike if and only if

$$\Re\langle [Df(z)]^{-1} f(z), z\rangle > 0, \qquad z \in \mathbb{B}\setminus\{0\}. \tag{2.1}$$

Let $w(z) = [Df(z)]^{-1} f(z)$. For $z \in \partial\mathbb{B}$ and $\zeta \in U$, let

$$\phi_z(\zeta) = \left\langle \frac{w(\zeta z)}{\zeta}, z \right\rangle$$

for $\zeta \neq 0$ and $\phi_z(0) = 1$. Since $w(0) = 0$ and $Dw(0) = I$, $\phi_z(\cdot)$ is a holomorphic function on U and $\Re\phi_z(\zeta) > 0$ for $\zeta \in U$ from (2.1).

If we put

$$\sigma_z(\zeta) = \frac{\phi_z(\zeta) - 1}{\phi_z(\zeta) + 1},$$

then $\sigma_z(\cdot)$ is a holomorphic function on U such that $\sigma_z(0)=0$ and $|\sigma_z(\zeta)|<1$ for $\zeta \in U$.

Definition 2.1. f is said to be *strongly starlike* if $\phi_z(U)$ is contained in a compact subset of the right half-plane independent of $z \in \partial\mathbb{B}$. Or, equivalently, there exists a constant c with $0 < c < 1$ such that $|\sigma_z(\zeta)| \leq c$ uniformly for $z \in \partial\mathbb{B}$ and $\zeta \in U$.

Let Ω, Ω' be domains in $\mathbb{R}^m$. A homeomorphism $f : \Omega \to \Omega'$ is said to be *quasiconformal* if it is differentiable a.e., ACL (absolutely continuous on lines) and

$$\|D(f;x)\|^m \leq K|\det D(f;x)| \qquad \text{a.e. in } \Omega,$$

where $D(f;x)$ denotes the (real) Jacobian matrix of f, K is a constant and

$$\|D(f;x)\| = \sup\{\|D(f;x)(a)\| : \|a\| = 1\}.$$

Let G be a domain in $\mathbb{C}^n$. A holomorphic mapping $f : G \to \mathbb{C}^n$ is said to be *quasiregular* if

$$\|Df(z)\|^n \leq K|\det Df(z)|, \qquad z \in G,$$

where K is a constant and

$$\|Df(z)\| = \sup\{\|Df(z)(a)\| : \|a\| = 1\}.$$

3. k-convex domains in $\mathbb{C}^n$. Suppose that $k > 0$, $a, b \in \mathbb{C}^n$, $a \neq b$ and $\|a - b\| < 2/k$. Let L be the complex line through a and b. Then there are two distinct closed disks $\overline{U}_1$ and $\overline{U}_2$ of radius $1/k$ in L such that $a, b \in \partial\overline{U}_j$ ($j = 1, 2$). Let $E_k[a, b] = \overline{U}_1 \cap \overline{U}_2$. We also let $E_0[a, b] = [a, b]$, and for $\|a - b\| = 2/k$, $E_k[a, b]$ is the closed disk in L with center $(a + b)/2$ and radius $1/k$.

Definition 3.1. Suppose that $0 \leq k < \infty$. A domain $\Omega \subset \mathbb{C}^n$ is called *k-convex* provided $\|a - b\| < 2/k$ for any pair of points $a, b \in \Omega$ and $E_k[a, b] \subset \Omega$.

Example 3.1. The ellipsoid

$$E = \{z \in \mathbb{C}^n : |z_1|^2/r_1^2 + \ldots + |z_n|^2/r_n^2 < 1\}$$

is k-convex, but is not k'-convex for any $k' > k$, where

$$k = 1/\max\{r_1, \ldots, r_n\},$$

since $E \cap L$ is a disk for any complex line L and the radius of the largest disk contained in E is $\max\{r_1, \ldots, r_n\}$. Thus, for $k > 0$, an open Euclidean ball of radius $1/k$ is k-convex, but is not k'-convex for any $k' > k$.

First, we will give elementary properties of k-convex domains. For $n = 1$, these properties were obtained by Mejia–Minda [18]. By definition, 0-convex is the same as convex. If $0 \leq k' \leq k$ and Ω is k-convex, then Ω is k'-convex. In particular, a k-convex domain is always convex and so simply connected. If $\Omega_1, \ldots, \Omega_\nu$ are k-convex, then $\bigcap \Omega_j$ is k-convex. If $\Omega_1 \subset \Omega_2 \subset \ldots$ is an increasing sequence of k-convex domains, then $\bigcup \Omega_j$ is k-convex.

We can prove the following propositions by an argument similar to Mejia–Minda [18]. The exact proof is left to the reader.

First, recall that if Ω is convex, then for any $a \in \Omega$ and $c \in \partial\Omega$, the half segment $[a, c) \subset \Omega$. The next result gives a refinement of this fact for k-convex domains.

Proposition 3.1. *Suppose that Ω is a k-convex domain. Then for any $a \in \Omega$ and $c \in \partial\Omega$, $E_k[a, c] \setminus \{c\} \subset \Omega$.*

Proposition 3.2. *Suppose that Ω is a k-convex domain. If $c, d \in \partial\Omega$, then* $\operatorname{int} E_k[c, d] \subset \Omega$.

Proposition 3.3. *Suppose that D is an open Euclidean ball or half-space such that $c \in \partial D \cap \partial B(z_0, 1/k)$ and D and $B(z_0, 1/k)$ are externally tangent at c. If $\|a - c\| < 2/k$ and $a \notin \overline{B(z_0, 1/k)}$, then $(E_k[a, c] \setminus \{c\}) \cap D \neq \emptyset$.*

Proposition 3.4. *Suppose that Ω is a k-convex domain. Assume that $a \in \Omega$, $c \in \partial\Omega$ and $\|a - c\| = \delta_\Omega(a)$. If B is the open Euclidean ball of radius $1/k$ that is tangent to the sphere $\|z - a\| = \delta_\Omega(a)$ at c and that contains a in its interior, then $\Omega \subset B$.*

Proposition 3.5. *Suppose that Ω is k-convex. Assume that $a \in \mathbb{C}^n \setminus \Omega$, $c \in \partial\Omega$ and $\|a-c\| = \delta_\Omega(a)$. If B is the open Euclidean ball of radius $1/k$ that is tangent to the sphere $\|z-a\| = \delta_\Omega(a)$ at c and that does not meet the open segment (a,c), then $\Omega \subset B$.*

In the following, we give a necessary and sufficient condition of k-convexity for a bounded domain in $\mathbb{C}^n$ whose boundary is a real hypersurface of class C^2 as follows:

$$\partial\Omega = \{z \in V : \varphi(z) = 0\}, \tag{3.1}$$

where V is a neighborhood of $\partial\Omega$ and φ is a real-valued C^2 function such that $\varphi(z) < 0$ on $V \cap \Omega$ and $\partial\varphi/\partial z(z) \neq 0$ on V. Mejia–Minda [18, Proposition 1] showed the following necessary and sufficient condition for k-convexity using the Euclidean curvature of $\partial\Omega$, when Ω is a simply connected region in $\mathbb{C}$ bounded by a closed Jordan C^2 curve.

Proposition 3.6. *Let $k > 0$ and let Ω be a simply connected domain in $\mathbb{C}$ bounded by a closed Jordan C^2 curve $\partial\Omega$. Then Ω is k-convex if and only if $k(c,\partial\Omega) \geq k$ for all $c \in \partial\Omega$.*

We will give a necessary and sufficient condition for a bounded domain in $\mathbb{C}^n$ with C^2 boundary to be a k-convex domain.

Theorem 3.1. *Let $k \geq 0$ and let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Assume that $\partial\Omega$ is as in* (3.1)*. Then Ω is k-convex if and only if*

$$\Re\left[v' \frac{\partial^2\varphi}{\partial z^2}(c)v\right] + \overline{v}' \frac{\partial^2\varphi}{\partial\overline{z}\partial z}(c)v \geq k\left|\left\langle v, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle\right| \|v\| \tag{3.2}$$

for all $c \in \partial\Omega$ and $v \in T_c(\partial\Omega)$.

Proof. By Krantz [14, Propositions 3.1.6 and 3.1.7], Ω is convex if and only if

$$\Re\left[v' \frac{\partial^2\varphi}{\partial z^2}(c)v\right] + \overline{v}' \frac{\partial^2\varphi}{\partial\overline{z}\partial z}(c)v \geq 0$$

for all $c \in \partial\Omega$ and $v \in T_c(\partial\Omega)$. So, we may assume that $k > 0$ and that Ω is convex. Let L be a complex line such that $\Omega \cap L \neq \emptyset$. We can write L as follows:

$$L = \{c + \zeta u : \zeta \in \mathbb{C}\},$$

where $c \in \partial\Omega \cap L$ and $\|u\| = 1$. Then

$$\partial(\Omega \cap L) = \{\varphi(c + \zeta u) = 0 : c + \zeta u \in V\}.$$

Since Ω is convex and $\Omega \cap L \neq \emptyset$, $\left\langle u, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle \neq 0$. This implies that $\partial(\Omega \cap L)$ is a C^2 curve near c. Let $z(t)$ be a curve in $\mathbb{C}$ such that

$$\varphi(c + z(t)u) = 0, \tag{3.3}$$

$z(0) = 0$ and $|z'(t)| = 1$ for t near 0. Differentiating (3.3) two times at $t = 0$, we have

$$\Re\left\langle v, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle = 0 \tag{3.4}$$

and

$$\Re\left[v'\frac{\partial^2\varphi}{\partial z^2}(c)v\right] + \overline{v}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(c)v + \Re\left\langle \frac{z''(0)}{z'(0)}v, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle = 0, \tag{3.5}$$

where $v = z'(0)u$. Since $\langle v, \frac{\partial\varphi}{\partial\overline{z}}(c)\rangle$ is non-zero and purely imaginary by (3.4), we may assume that

$$\left\langle v, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle = yi$$

with $y > 0$. Therefore,

$$\Re\left\langle \frac{z''(0)}{z'(0)}v, \frac{\partial\varphi}{\partial\overline{z}}(c)\right\rangle = -\frac{1}{|z'(0)|}\Im\left(\frac{z''(0)}{z'(0)}\right)y = -k(c, \partial(\Omega\cap L))|y|. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\Re\left[v'\frac{\partial^2\varphi}{\partial z^2}(c)v\right] + \overline{v}'\frac{\partial^2\varphi}{\partial\overline{z}\partial z}(c)v = k(c, \partial(\Omega\cap L))|y|.$$

Thus, by Proposition 3.6, Ω is k-convex if and only if (3.2) holds. This completes the proof.

Let

$$\lambda_\Omega(z) = \sup_{\|X\|=1}\gamma_\Omega(z; X),$$

where $\gamma_\Omega(z; X)$ denotes the Carathéodory infinitesimal metric on Ω. The following theorem is a generalization of Mejia–Minda [18, Theorem 1].

THEOREM 3.2. *Suppose that Ω is a k-convex domain. Then for $z \in \Omega$,*

$$\lambda_\Omega(z) \geq \frac{1}{\delta_\Omega(z)[2 - k\delta_\Omega(z)]}. \tag{3.7}$$

Proof. First, assume that $\Omega = B(a, 1/k)$. Then

$$\gamma_\Omega(z; X) = \sqrt{\frac{\|X\|^2}{\delta_\Omega(z)(2/k - \delta_\Omega(z))} + \frac{|\langle z - a, X\rangle|^2}{\delta_\Omega(z)^2(2/k - \delta_\Omega(z))^2}}.$$

Therefore,

$$\lambda_\Omega(z) = \frac{1}{\delta_\Omega(z)[2 - k\delta_\Omega(z)]}. \tag{3.8}$$

Next, consider any k-convex domain Ω. Fix $a \in \Omega$. Choose $c \in \partial\Omega$ with $\|a - c\| = \delta_\Omega(a)$. Let B be the open Euclidean ball of radius $1/k$ that is tangent to the sphere $\|z - a\| = \delta_\Omega(a)$ at c and contains a in its interior. By

Proposition 3.4, we have $\Omega \subset B$. Then

$$\gamma_B(a; X) \le \gamma_\Omega(a; X).$$

Therefore,

$$\lambda_B(a) \le \lambda_\Omega(a). \tag{3.9}$$

Since $\delta_\Omega(a) = \delta_B(a)$, we obtain (3.7) from (3.8) and (3.9). This completes the proof.

4. k-convex mappings in several complex variables

DEFINITION 4.1. A holomorphic mapping $f : \mathbb{B} \to \mathbb{C}^n$ is called *k-convex* if f is biholomorphic and $f(\mathbb{B})$ is a k-convex domain. Moreover, for $\alpha > 0$, let $K(k, \alpha)$ denote the family of all k-convex mappings such that $f(0) = 0$, $Df(0) = \alpha I$.

Note that $K(0, 1)$ is the same as the family K of normalized convex mappings on $\mathbb{B}$.

The following theorem is a generalization of Mejia–Minda [18, Corollary 2 to Theorem 1].

THEOREM 4.1. *Suppose that $f \in K(k, \alpha)$. Then $\alpha k \le 1$ and the Euclidean ball $B(0, \alpha/(1 + \sqrt{1 - \alpha k}))$ is contained in $f(\mathbb{B})$.*

Proof. Let $\Omega = f(\mathbb{B})$. Since holomorphic mappings are contractions of the infinitesimal Carathéodory pseudometric, we have

$$\alpha\gamma_\Omega(0, X) = \gamma_\Omega(f(0), Df(0)X) \le \gamma_{\mathbb{B}}(0, X) = \|X\|.$$

Then we have

$$\alpha\lambda_\Omega(0) \le \lambda_{\mathbb{B}}(0) = 1.$$

Also,

$$\frac{\alpha}{\delta_\Omega(0)[2 - k\delta_\Omega(0)]} \le \alpha\lambda_\Omega(0)$$

by Theorem 3.2. Therefore,

$$\frac{\alpha}{\delta_\Omega(0)[2 - k\delta_\Omega(0)]} \le 1.$$

Thus, $\alpha k \le 1$ and

$$\delta_\Omega(0) \ge \frac{1 - \sqrt{1 - \alpha k}}{k} = \frac{\alpha}{1 + \sqrt{1 - \alpha k}}.$$

This completes the proof.

EXAMPLE 4.1. Let $k > 0$. For $u \in \mathbb{C}^n$ with $\|u\| = 1$, let

$$f_{k,u}(z) = \frac{\alpha z}{1 - \sqrt{1 - \alpha k}\,\langle z, u\rangle}.$$

Then $f_{k,u} \in K(k,\alpha)$. This can be verified as follows. We may assume that $u = (1,0,\dots,0)'$. Clearly, $f_{k,u}(0) = 0$, $Df_{k,u}(0) = \alpha I$ and $f_{k,u}$ is biholomorphic on a neighborhood of $\overline{\mathbb{B}}$. Since

$$f_{k,u}^{-1}(w) = \frac{w}{\alpha + \sqrt{1-\alpha k}\, w_1},$$

we have

$$\begin{aligned} f_{k,u}(\mathbb{B}) &= \left\{ w \in \mathbb{C}^n : \left\| \frac{w}{\alpha + \sqrt{1-\alpha k}\, w_1} \right\| < 1 \right\} \\ &= \left\{ w = (w_1, w')' \in \mathbb{C}^n : \frac{|w_1 - k^{-1}\sqrt{1-\alpha k}|^2}{k^{-2}} + \frac{\|w'\|^2}{\alpha k^{-1}} < 1 \right\}. \end{aligned}$$

Since $\sqrt{\alpha k^{-1}} = k^{-1}\sqrt{\alpha k} \le k^{-1}$, $f_{k,u}(\mathbb{B})$ is k-convex by Example 3.1.

Mejia–Minda [18, Corollary 1 to Theorem 8] gave a necessary and sufficient analytic condition for a locally biholomorphic mapping on the unit disc U in $\mathbb{C}$ to be k-convex. We will give a sufficient analytic condition for a locally biholomorphic mapping on the Euclidean unit ball $\mathbb{B}$ in $\mathbb{C}^n$ to be k-convex.

THEOREM 4.2. *Let $k \ge 0$ and let $f : \mathbb{B} \to \mathbb{C}^n$ be a locally biholomorphic mapping. Suppose that*

$$\|v\|^2 - \Re\langle [Df(z)]^{-1} D^2 f(z)(v,v), z\rangle \ge k|\langle z, v\rangle| \|Df(z)v\|$$

for all $z \in \mathbb{B}$ and $v \in \mathbb{C}^n$ with $\Re\langle z, v\rangle = 0$. Then f is k-convex.

Proof. Since

$$\|v\|^2 - \Re\langle [Df(z)]^{-1} D^2 f(z)(v,v), z\rangle \ge 0,$$

f is biholomorphic and $f(\mathbb{B})$ is a convex domain by Kikuchi [12, Theorem 2.1] or Gong–Wang–Yu [4, Theorem 2].

Let $0 < r < 1$ and let $\varphi(w) = \|f^{-1}(w)\|^2 - r^2$. Then

$$\partial f(B_r) = f(\partial B_r) = \{w \in f(\mathbb{B}) : \varphi(w) = 0\}.$$

Let $w_0 \in \partial f(B_r)$ and let $u \in T_{w_0}(\partial f(B_r))$. Then

$$\frac{\partial \varphi}{\partial \overline{w}}(w_0) = \overline{[Df(z_0)']^{-1}} z_0,$$

where $z_0 = f^{-1}(w_0) \in \partial B_r$,

$$\overline{u}' \frac{\partial^2 \varphi}{\partial \overline{w} \partial w}(w_0) u = \overline{u}' \overline{[Df(z_0)']^{-1}} [Df(z_0)]^{-1} u = \|[Df(z_0)]^{-1} u\|^2,$$

and

$$u' \frac{\partial^2 \varphi}{\partial w^2}(w_0) u = -\overline{z_0}' [Df(z_0)]^{-1} D^2 f(z_0)([Df(z_0)]^{-1} u, [Df(z_0)]^{-1} u).$$

Let $v_0 = [Df(z_0)]^{-1}u$. Since $u \in T_{w_0}(f(\partial B_r))$, we have

$$\Re\langle v_0, z_0\rangle = \Re\{u'([Df(z_0)]^{-1})'\overline{z_0}\} = \Re\left\langle u, \frac{\partial\varphi}{\partial\overline{w}}(w_0)\right\rangle = 0.$$

Therefore, $v_0 \in T_{z_0}(\partial B_r)$. Thus,

$$\begin{aligned}\Re\left[u'\frac{\partial^2\varphi}{\partial w^2}(w_0)u\right] &+ \overline{u}'\frac{\partial^2\varphi}{\partial\overline{w}\partial w}(w_0)u \\ &= \|v_0\|^2 - \Re\langle[Df(z_0)]^{-1}D^2f(z_0)(v_0,v_0), z_0\rangle \\ &\geq k|\langle z_0, v_0\rangle|\|Df(z_0)v_0\| \\ &= k\left|\left\langle\frac{\partial\varphi}{\partial\overline{w}}(w_0), u\right\rangle\right|\|u\|.\end{aligned}$$

By Theorem 3.1, $f(B_r)$ is a k-convex domain. Therefore, $f(\mathbb{B})$ is k-convex. This completes the proof.

Example 4.2. For $z = (z_1, z_2)' \in \mathbb{C}^2$, let

$$f(z) = (z_1 + az_2^2, z_2)',$$

where a is a constant. Suffridge [21, Example 9] showed that $f \in K$ if $|a| \leq 1/2$. We will show that if $|a| < 1/2$, then $f \in K(k, 1)$, where

$$k = \frac{1-2|a|}{1+2|a|}.$$

By a direct computation, we have

$$\|v\|^2 - \Re\langle[Df(z)]^{-1}D^2f(z)(v,v), z\rangle = \|v\|^2 - \Re(2av_2^2\overline{z}_1)$$

and

$$Df(z)v = v + 2az_2(v_2, 0)'.$$

Then we have

$$\|v\|^2 - \Re\langle[Df(z)]^{-1}D^2f(z)(v,v), z\rangle \geq (1-2|a|)\|v\|^2$$

and

$$|\langle z, v\rangle|\|Df(z)v\| \leq (1+2|a|)\|v\|^2.$$

Therefore, the assumption of Theorem 4.2 holds for $k = (1-2|a|)/(1+2|a|)$.

For $w = (w_1, \ldots, w_n)' \in \mathbb{C}^n$ and $u \in \mathbb{C}^n$ with $\|u\| = 1$, let

$$S_u(w) = \frac{w}{\alpha - (1-\sqrt{1-\alpha k})\langle w, u\rangle}.$$

We obtain the following result as in Ma–Mejia–Minda [16, Theorem 1].

Theorem 4.3. *If $f \in K(k, \alpha)$, then $S_u \circ f \in K$ for every $u \in \mathbb{C}^n$ with $\|u\| = 1$.*

Proof. It suffices to show the case when $k > 0$. By Theorem 4.1, we have $B(0, \alpha/(1+\sqrt{1-\alpha k})) \subset f(\mathbb{B})$. Also, by Proposition 3.4, $f(\mathbb{B})$ is contained in an open Euclidean ball of radius $1/k$. Thus, for $z \in \mathbb{B}$, we have

$$\|f(z)\| < \frac{2}{k} - \frac{\alpha}{1+\sqrt{1-\alpha k}} = \frac{\alpha}{1-\sqrt{1-\alpha k}}.$$

Hence, $g = S_u \circ f$ is a biholomorphic mapping on $\mathbb{B}$ with $g(0) = 0$, $Dg(0) = I$.

Now, we will show that $g(\mathbb{B})$ is convex. Let L be an arbitrary complex line such that $g(\mathbb{B}) \cap L \neq \emptyset$. It suffices to show that $\Delta = g(\mathbb{B}) \cap L$ is convex. For any point $a \in \Delta$, there exists a point $c \in \partial\Delta$ such that $\|a - c\| = \delta_\Delta(a)$, where $\delta_\Delta(a)$ denotes the Euclidean distance from a to $\partial\Delta$. Let Γ be the circle $\{\zeta \in L : \|\zeta - a\| = \delta_\Delta(a)\}$, l be the tangent line to Γ in L at c, H be the half-plane bounded by l in L and containing a and $d = (S_u)^{-1}(c)$. Since $L' = (S_u)^{-1}(L)$ is a complex line, $(S_u)^{-1}(\Gamma)$ is a circle or a straight line in L' passing through d. Because the open disk in L bounded by Γ is contained in Δ, its image under $(S_u)^{-1}$ lies in $(S_u)^{-1}(\Delta) \subset f(\mathbb{B})$. Since $f(\mathbb{B})$ is bounded by Proposition 3.4, $(S_u)^{-1}(\Gamma)$ must be a circle. Let l' be the circle of radius $1/k$ in L' that is tangent to $(S_u)^{-1}(\Gamma)$ at d such that its interior meets the interior of $(S_u)^{-1}(\Gamma)$ and H' be the open disk in L' bounded by l'. Then $(S_u)^{-1}(\Delta) \subset H'$ by Mejia–Minda [18, Proposition 3]. On the other hand, $S_u(l')$ is a circle or a straight line in L which is tangent to Γ at c. If $S_u(l')$ is a straight line, then $S_u(l') = l$ and $S_u(H') = H$. If $S_u(l')$ is a circle, then $S_u(H')$ is a disk in L contained in H. In both cases, we have $\Delta \subset S_u(H') \subset H$. Let λ_Δ (resp. λ_H) denote the density of the hyperbolic metric on Δ (resp. H). From the monotonicity of the hyperbolic metric, we have

$$\lambda_\Delta(a) \geq \lambda_H(a) = \frac{1}{2\delta_H(a)} = \frac{1}{2\delta_\Delta(a)}.$$

Since $a \in \Delta$ is arbitrary, it follows that $\lambda_\Delta(z) \geq 1/(2\delta_\Delta(z))$ for all $z \in \Delta$. By Mejia–Minda [18, Theorem 2], Δ is convex. This completes the proof.

Let $f \in K$. Then Liu [15], Suffridge [22], FitzGerald–Thomas [2] and the second author [13] independently obtained the following growth theorem (cf. Hamada [5], Hamada–Kohr [9]):

$$\frac{1}{1+\|z\|} \leq \|f(z)\| \leq \frac{1}{1-\|z\|} \quad \text{for } z \in \mathbb{B}. \tag{4.1}$$

Also, Gong–Liu [3] and Pfaltzgraff–Suffridge [19] independently proved the following distortion theorem (cf. Gong–Wang–Yu [4], Hamada–Kohr [8]):

$$\frac{1}{(1+\|z\|)^2} \leq \|Df(z)\| \leq \frac{1}{(1-\|z\|)^2} \quad \text{for } z \in \mathbb{B}. \tag{4.2}$$

THEOREM 4.4. *Let* $f \in K(k,\alpha)$ *with* $k > 0$. *Then*

$$\frac{\alpha\|z\|}{1+\sqrt{1-\alpha k}\|z\|} \le \|f(z)\| \le \frac{\alpha\|z\|}{1-\sqrt{1-\alpha k}\|z\|}, \qquad z \in \mathbb{B}, \tag{4.3}$$

and

$$\frac{\alpha}{(1+\|z\|)(1+\sqrt{1-\alpha k}\|z\|)} \le \|Df(z)\|, \qquad z \in \mathbb{B}. \tag{4.4}$$

Proof. Fix $z_0 \in \mathbb{B} \setminus \{0\}$. There exists a unitary matrix U such that $Uf(z_0) = (\|f(z_0)\|, 0, \dots, 0)'$. Let $u = (-1, 0, \dots, 0)'$. Then $\langle Uf(z_0), u\rangle = -\|f(z_0)\|$. Let $F(z) = Uf(U^{-1}z)$. Then $F \in K(k,\alpha)$. By Theorem 4.3, $S_u \circ F \in K$. By making use of the growth and distortion theorems (4.1) and (4.2) for the class K at $z_1 = Uz_0$, we have

$$\frac{\|f(z_0)\|}{\alpha+(1-\sqrt{1-\alpha k})\|f(z_0)\|} = \|S_u \circ F(z_1)\| \ge \frac{\|z_0\|}{1+\|z_0\|} \tag{4.5}$$

and

$$\left\|\frac{[(\alpha+(1-\sqrt{1-\alpha k})\|f(z_0)\|)I-(1-\sqrt{1-\alpha k})\|f(z_0)\|E_{11}]UDf(z_0)}{(\alpha+(1-\sqrt{1-\alpha k})\|f(z_0)\|)^2}\right\| \ge \frac{1}{(1+\|z_0\|)^2}, \tag{4.6}$$

where $E_{11} = (1,0,\dots,0)(1,0,\dots,0)'$. From (4.5), we have the lower estimate of (4.3). From the lower estimate in (4.3) and (4.6), we have (4.4). If we take $u = (1,0,\dots,0)'$, then we obtain

$$\frac{\|f(z_0)\|}{\alpha-(1-\sqrt{1-\alpha k})\|f(z_0)\|} \le \frac{\|z_0\|}{1-\|z_0\|}$$

as above. This inequality implies the upper estimate in (4.3).

Since a k-convex mapping is convex, it is a starlike mapping. Chuaqui [1] obtained a quasiconformal extension of a quasiconformal strongly starlike mapping with $\|[Df(z)]^{-1}f(z)\|$ uniformly bounded on the Euclidean unit ball $\mathbb{B}$ in $\mathbb{C}^n$. The first author [6] extended this result to a bounded balanced domain Ω with C^1 plurisubharmonic defining functions in $\mathbb{C}^n$, and the authors [10] generalized this to the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$. The authors also gave a quasiconformal extension of a quasiconformal strongly spirallike mapping of type α with $\|[Df(z)]^{-1}f(z)\|$ uniformly bounded on a bounded balanced domain Ω with C^1 plurisubharmonic defining functions in $\mathbb{C}^n$ [7] and on the unit ball with respect to an arbitrary norm on $\mathbb{C}^n$ [11]. As a corollary of the above theorem, we obtain the following theorem.

THEOREM 4.5. *Let $f \in K(k,\alpha)$, where $k > 0$. Assume that f is a quasiregular strongly starlike mapping. Then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.*

Proof. It suffices to show that $[Df(z)]^{-1}f(z)$ is uniformly bounded in $\mathbb{B}$. By Theorem 4.4, there exists a constant $c > 0$ such that

$$\|Df(z)\| \geq c, \quad \|f(z)\| \leq c, \quad z \in \mathbb{B}. \tag{4.7}$$

Also, since f is quasiregular, there exists a constant $K > 0$ such that

$$\|Df(z)\|^n \leq K|\det Df(z)|, \quad z \in \mathbb{B}. \tag{4.8}$$

Fix $z \in \mathbb{B}$ and let $A = Df(z)$. Since A^*A is a Hermitian matrix with $\langle A^*Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$, where $A^* = \overline{A}'$, the eigenvalues of A^*A are real and non-negative. Let $\lambda_1^2, \ldots, \lambda_n^2$ be the eigenvalues of A^*A, where $\lambda_1, \ldots, \lambda_n \geq 0$. We may assume that $\lambda_1 \leq \ldots \leq \lambda_n$. Since $\lambda_1^2 \ldots \lambda_n^2 = \det(A^*A) = |\det(A)|^2 > 0$, it follows that $\lambda_1 > 0$. Also, from (4.7) and (4.8), we have

$$\lambda_n \geq c, \quad \lambda_n^n \leq K\lambda_1 \ldots \lambda_n.$$

The latter inequality implies that $\lambda_n \leq K\lambda_1$.

Fix $y \in \mathbb{C}^n$ with $\|y\| = 1$. Let $x = A^{-1}y$. Then $\|Ax\|^2 = \langle A^*Ax, x\rangle \geq \lambda_1^2\|x\|^2$. Therefore,

$$\|A^{-1}y\| = \|x\| \leq \frac{\|Ax\|}{\lambda_1} = \frac{1}{\lambda_1}.$$

This implies that

$$\|A^{-1}\| \leq \frac{1}{\lambda_1} \leq \frac{K}{\lambda_n} \leq \frac{K}{c}.$$

Thus, we have

$$\|[Df(z)]^{-1}f(z)\| \leq \|[Df(z)]^{-1}\| \cdot \|f(z)\| \leq K.$$

This completes the proof.

References

[1] M. Chuaqui, *Applications of subordination chains to starlike mappings in* $\mathbb{C}^n$, Pacific J. Math. 168 (1995), 33–48.

[2] C. FitzGerald and C. Thomas, *Some bounds of convex mappings in several complex variables*, ibid. 165 (1994), 295–320.

[3] S. Gong and T. Liu, *Distortion theorems for biholomorphic convex mappings on bounded convex circular domains*, Chinese Ann. Math. Ser. B 20 (1999), 297–304.

[4] S. Gong, S. Wang and Q. Yu, *Biholomorphic convex mappings of ball in* $\mathbb{C}^n$, Pacific J. Math. 161 (1993), 287–306.

[5] H. Hamada, *The growth theorem of convex mappings on the unit ball of* $\mathbb{C}^n$, Proc. Amer. Math. Soc. 127 (1999), 1075–1077.

[6] H. Hamada, *Starlike mappings on bounded balanced domains with C^1-plurisubharmonic defining functions*, Pacific J. Math. 194 (2000), 359–371.
[7] H. Hamada and G. Kohr, *The growth theorem and quasiconformal extension of strongly spirallike mappings of type α*, Complex Variables 44 (2001), 281–297.
[8] —, —, *Φ-like and convex mappings in infinite dimensional spaces*, Rev. Roumaine Math. Pures Appl., to appear.
[9] —, —, *Growth and distortion results for convex mappings in infinite dimensional spaces*, Complex Variables, to appear.
[10] —, —, *Quasiconformal extension of strongly starlike mappings on the unit ball of $\mathbb{C}^n$*, submitted.
[11] —, —, *Loewner chains and quasiconformal extension of holomorphic mappings*, submitted.
[12] K. Kikuchi, *Starlike and convex mappings in several complex variables*, Pacific J. Math. 44 (1973), 569–580.
[13] G. Kohr, *On starlikeness and strongly starlikeness of order alpha in $\mathbb{C}^n$*, Mathematica (Cluj) 40 (63) (1998), 95–109.
[14] S. G. Krantz, *Function Theory of Several Complex Variables*, Wadsworth & Brooks/ Cole Advanced Books & Software, 1992.
[15] T. Liu, *The growth theorems, covering theorems and distortion theorems for biholomorphic mappings on classical domains*, thesis, Univ. of Science and Technology of China, 1989.
[16] W. Ma, D. Mejia and D. Minda, *Distortion theorems for Euclidean k-convex functions*, Complex Variables 19 (1992), 259–269.
[17] D. Mejia, *The hyperbolic metric in k-convex regions*, Ph.D. dissertation, Univ. of Cincinnati, 1986.
[18] D. Mejia and D. Minda, *Hyperbolic geometry in k-convex regions*, Pacific J. Math. 141 (1990), 333–354.
[19] J. A. Pfaltzgraff and T. J. Suffridge, *Norm order and geometric properties of holomorphic mappings in $\mathbb{C}^n$*, J. Anal. Math. 82 (2000), 285–313.
[20] T. J. Suffridge, *The principle of subordination applied to functions of several variables*, Pacific J. Math. 33 (1970), 241–248.
[21] —, *Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions*, in: Lecture Notes in Math. 599, Springer, 1976, 146–159.
[22] —, *Biholomorphic mappings of the ball onto convex domains*, Abstracts Amer. Math. Soc. 11 (1990), 46.

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