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Exponential limit shadowing

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Abstract. We introduce the notion of exponential limit shadowing and show that it is a persistent property near a hyperbolic set of a dynamical system. We show that Ω-stability implies the exponential limit shadowing property.

1. Introduction. The theory of shadowing in dynamical systems has been extended by many researchers [LS, P1, JTT, Y]. Let us explain this theory by considering a set M and a map $\phi : M \to M$. In numerical computation of the orbit of ϕ with initial value $x_0 \in M$ we can approximate $\phi(x_0)$ by x_1. To continue the process we can compute the value x_2 close to $\phi(x_1)$ and so on. Sometimes this sequence can play the role of a shadow for the orbit $\mathcal{O}(x,\phi) = \{\phi^n(x)\}_{n\in\mathbb{Z}}$ for some $x \in M$. A natural question is: when, for a given shadow, can we find a real orbit close to it? This leads us to consider shadowing properties.

Furthermore, we may consider shadowing as a weak form of stability of dynamical systems with respect to C^0 perturbations. More precisely, let M be a compact smooth manifold with a metric r and let $\phi : M \to M$ be a C^1 diffeomorphism. Then the dynamical system ϕ has the *pseudo orbit tracing property* (POTP) on M if for each $\epsilon > 0$ there is $d > 0$ such that for any given sequence $\xi = \{x_k\}_{k\in\mathbb{Z}}$ with

$$r(\phi(x_k), x_{k+1}) < d \quad \text{for } k \in \mathbb{Z}$$

(called a *d-pseudo orbit*) there exists a point $p \in M$ such that

$$r(\phi^k(p), x_k) < \epsilon \quad \text{for } k \in \mathbb{Z}$$

(see [B]). The dynamical system ϕ has the *Lipschitz shadowing property* (LpSP) on M if there exist constants $L > 0$ and $d_0 > 0$ such that for any sequence $\xi = \{x_k\}_{k\in\mathbb{Z}}$ with

$$r(\phi(x_k), x_{k+1}) < d < d_0 \quad \text{for } k \in \mathbb{Z}$$

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there is a point $p \in M$ such that

$$r(\phi^k(p), x_k) < Ld \quad \text{for } k \in \mathbb{Z}.$$

Another type of shadowing is the *limit shadowing* [ENP]. The dynamical system ϕ has the *limit shadowing property* (LmSP) on M if for any sequence $\xi = \{x_k\}_{k\geq 0}$ with

$$r(\phi(x_k), x_{k+1}) \to 0 \quad \text{as } k \to \infty$$

there exists a point $p \in M$ so that

$$r(\phi^k(p), x_k) \to 0 \quad \text{as } k \to \infty$$

From the numerical point of view, this property means that if we apply a numerical method that approximates ϕ with "improving accuracy", so that one-step errors tend to zero as time goes to infinity, then the numerically obtained trajectories tend to real ones.

We now introduce a kind of shadowing where one-step errors tend to zero with exponential rate, and each one-step error is determined in terms of two constants and the step number.

Definition 1.1. We say that a dynamical system ϕ has the *exponential limit shadowing property* (ELmSP) on M if there exist constants $L > 0$ and $\lambda \in (0, 1)$ such that for any sequence $\xi = \{x_k\}_{k\geq 0}$ with

$$r(\phi(x_k), x_{k+1}) < \lambda^k \quad \text{for all } k \text{ greater than a } k_1 \in \mathbb{N} \tag{1.1}$$

there exists a point $p \in M$ and $k_2 \in \mathbb{N}$ so that

$$r(\phi^k(p), x_k) < L\lambda^{k/2} \quad \text{for } k \geq k_2. \tag{1.2}$$

In this paper, $\Omega(\phi)$ denotes the set of non-wandering points of ϕ. The system ϕ is called *Ω-stable* if given $\epsilon > 0$ there is $\delta > 0$ such that for any diffeomorphism ψ such that $\rho_1(\phi, \psi) = \sup_{x\in M}\{\|\phi(x) - \psi(x)\|, \|D\phi(x) - D\psi(x)\|\} < \delta$ there is a homeomorphism $h : \Omega(\phi) \to \Omega(\psi)$ with

- $h \circ \phi = \psi \circ h$;
- $r(x, h(x)) < \epsilon$ for each $x \in \Omega(\phi)$.

We recall that a compact, ϕ-invariant subset $\Lambda \subset M$ is called a *hyperbolic set* for ϕ if there are constants $\lambda \in (0, 1)$, $C > 0$ and a family of subspaces $E^s(x) \subset T_xM$ and $E^u(x) \subset T_xM$, $x \in \Lambda$, so that for every $x \in \Lambda$:

(1) $T_xM = E^s(x) \oplus E^u(x)$;
(2) $\|(D_x\phi^n)v\| \leq C\lambda^n\|v\|$ for each $v \in E^s(x)$ and $n \geq 0$;
(3) $\|(D_x\phi^{-n})v\| \leq C\lambda^n\|v\|$ for every $v \in E^u(x)$ and $n \geq 0$;
(4) $(D_x\phi)E^s(x) = E^s(x)$ and $(D_x\phi)E^u(x) = E^u(x)$.

The subspace $E^s(x)$ (respectively $E^u(x)$) is called the *stable* (resp. *unstable*) *subspace* at x, and the family $\{E^s(x)\}_{x\in\Lambda}$ (res. $\{E^u(x)\}_{x\in\Lambda}$) is called the *stable* (resp. *unstable*) *distribution* of $\phi|_\Lambda$.

A system ϕ satisfies *Axiom A* if

- $\Omega(\phi)$ is a hyperbolic set;
- the set $\mathrm{Per}(\phi)$ of periodic points of ϕ is dense in $\Omega(\phi)$.

2. Theorems. In this paper we prove the following main theorems:

THEOREM 2.1. *Let Λ be a hyperbolic set for a diffeomorphism ϕ of M. Then there exists a neighborhood W of Λ such that ϕ has the ELmSP on W.*

THEOREM 2.2. *If a diffeomorphism ϕ is Ω-stable, then it has the ELmSP.*

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism. Then f is called *expansive* if there is a constant $e > 0$ such that if $d(f^i(x), f^i(y)) \leq e$ for all $i \in \mathbb{Z}$, then $x = y$.

For $\Delta > 0$ and $x \in X$, let $W^s_\Delta(x) = \{y \in X : d(f^n(x), f^n(y)) \leq \Delta$ for all $n \geq 0\}$ and $W^u_\Delta = \{y \in X : d(f^n(x), f^n(y)) \leq \Delta$ for all $n \leq 0\}$. The mapping f is called *$\mathcal{L}$-hyperbolic* [Sa] if

- f is a Lipschitz homeomorphism;
- there is $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, there exists $\delta > 0$ so that for any $x, y \in X$ with $d(x, y) < \delta$, the set $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ consists of a single point $\alpha(x, y)$;
- there is a constant $\mathcal{K} > 0$ such that
$$d(\alpha(x,y), x) \leq \mathcal{K} d(x,y), \qquad d(\alpha(x,y), y) \leq \mathcal{K} d(x,y);$$
- there are $0 < \Delta, \nu < 1$ such that for all $x \in X$,
$$\begin{aligned} y \in W^s_\Delta(x) \ &\Rightarrow\ d(f^n(x), f^n(y)) \leq \nu^n d(x,y),\ n \geq 0, \\ y \in W^u_\Delta(x) \ &\Rightarrow\ d(f^{-n}(x), f^{-n}(y)) \leq \nu^n d(x,y),\ n \geq 0. \end{aligned}$$

We will also prove the following theorem.

THEOREM 2.3. *Let $f : X \to X$ be an $\mathcal{L}$-hyperbolic homeomorphism on a compact metric space (X, d). Then f has the ELmSP.*

3. Proofs of theorems. Let us recall the following theorem which is a stronger version of the shadowing lemma:

THEOREM 3.1 ([P1]). *If Λ is a hyperbolic set for a diffeomorphism ϕ then there exists a neighborhood W of Λ on which ϕ has the LpSP.*

To prove Theorem 2.1 we will use the following lemma.

LEMMA 3.2 ([ENP]). *If Λ is a hyperbolic set for a diffeomorphism ϕ then there exist a neighborhood U of Λ and constants $\delta > 0$ and $\nu \in (0,1)$ such that if two points x and y in M have the properties $\phi^k(x), \phi^k(y) \in U$ and $r(\phi^k(x), \phi^k(y)) \leq \delta$ for all $k \geq 0$ then*
$$r(\phi^k(x), \phi^k(y)) \leq 2\nu^k r(x,y) \quad \textit{for all } k \geq 0.$$

Proof of Theorem 2.1. By Theorem 3.1 there exists a neighborhood U_0 of Λ on which ϕ has the LpSP with constants L_0 and d_0. Take a neighborhood U_1 of Λ and numbers $\nu \in (0,1)$ and $\delta > 0$ as given by Lemma 3.2. If

$$U = U_0 \cap U_1$$

then we can find a neighborhood W of Λ such that $N(W,\delta) \subset U$ (by decreasing δ), where $N(W,\delta)$ is the δ-neighborhood of W. We claim that for $\lambda = \nu \in (0,1)$, ϕ has the ELmSP on W. To prove this let $\xi = \{x_k\}_{k\geq 0}$ be a sequence on W so that

$$r(x_{k+1}, \phi(x_k)) \leq \lambda^k \quad \text{for } k \geq k_1.$$

We choose $k_2 \geq k_1$ large enough so that $\lambda^k < \min\{d_0, \delta/2L_0\}$ for $k \geq k_2$. Since ϕ has the LpSP on $W \subset U_0$, for each $k \geq k_2$ there exists $y_k \in M$ such that

$$r(\phi^i(y_k), x_i) \leq L_0\lambda^k < \delta/2 \quad \text{for } i \geq k \geq k_2.$$

If $p = y_{k_2}$, then

$$r(\phi^i(y_k), \phi^i(p)) \leq r(\phi^i(y_k), x_i) + r(x_i, \phi^i(p)) < \delta/2 + \delta/2 = \delta$$

for $i \geq k \geq k_2$. Fix $k \geq k_2$; then for $i \geq 0$ we deduce

$$r(\phi^{i+k}(y_k), \phi^{i+k}(p)) < \delta \quad \text{and} \quad \phi^{i+k}(y_k), \phi^{i+k}(p) \in U \subset U_1.$$

Hence Lemma 3.2 implies

$$r(\phi^{i+k}(y_k), \phi^{i+k}(p)) \leq 2r(\phi^k(y_k), \phi^k(p))\lambda^i < 2\delta\lambda^i \quad \text{for } i \geq 0.$$

So

$$d(\phi^{2k}(p), x_{2k}) \leq d(\phi^{2k}(p), \phi^{2k}(y_k)) + d(\phi^{2k}(y_k), x_{2k}) < (L_0 + 2\delta)\lambda^k \quad \text{for } k \geq k_2.$$

Thus

$$d(\phi^k(p), x_k) < (L_0 + 2\delta)\lambda^{k/2} \quad \text{for } k \geq 2k_2.$$

If $L = L_0 + 2\delta$ then $r(\phi^k(p), x_k) \leq L\lambda^{k/2}$ for $k \geq 2k_2$. ∎

Let S^1 be the unit circle with coordinate $x \in [0,1)$. Then we have the following proposition:

PROPOSITION 3.3. *Let $\phi : S^1 \to S^1$ be an orientation preserving diffeomorphism. Moreover suppose that the set* $\mathrm{Fix}(\phi)$ *of fixed points of ϕ is a nonempty hyperbolic, nowhere dense set. Then ϕ has the ELmSP.*

Proof. Since $\mathrm{Fix}(\phi)$ is a hyperbolic set, by Theorem 2.1 there is a neighborhood W of $\mathrm{Fix}(\phi)$ such that ϕ has the ELmSP on W with constants $\lambda \in (0,1)$ and $L > 0$. Suppose that the sequence $\xi = \{x_k\}_{k\geq k_1}$ for some $k_1 \in \mathbb{N}$ satisfies

$$|\phi(x_k) - x_{k+1}| < \lambda^k \quad \text{for } k \geq k_1.$$

Then Theorem 3.1.2 of [P1] implies that there is a fixed point x such that $x_k \to x$ and so $\{x_k\}_{k\geq k_2} \subset W$ for some $k_2 \geq k_1$. Hence there exists $p \in S^1$

such that

$$|\phi^k(p) - x_k| < \lambda^{k/2} \quad \text{for } k \ge k_3$$

for some $k_3 \in \mathbb{N}$, i.e. ϕ has the ELmSP. ∎

Let us recall Smale's spectral decomposition theorem [Sm]. If ϕ satisfies Axiom A, then there is a unique representation

$$\Omega(\phi) = \Omega_1 \cup \cdots \cup \Omega_k$$

of $\Omega(\phi)$ as a disjoint union of closed ϕ-invariant sets (called *basic sets*) such that

- each Ω_i is a locally maximal hyperbolic set of ϕ;
- ϕ is topologically transitive on each Ω_i;
- each Ω_i is a disjoint union of closed sets Ω_i^j, $1 \le j \le m_i$, cyclically permuted by ϕ, and ϕ^{m_i} is topologically mixing on each Ω_i^j.

For the proof of the following lemma, see the proof of Lemma 1 of [P2].

LEMMA 3.4 ([P2]). *If a diffeomorphism ϕ is Ω-stable and the sequence $\{x_k\}$ satisfies $\lim_{k\to\infty} d(\phi(x_k), x_{k+1}) = 0$ then there exists a basic set Ω_i such that*

$$r(x_k, \Omega_i) \to 0 \quad \textit{as } k \to \infty.$$

Proof of Theorem 2.2. Let

$$\Omega(\phi) = \Omega_1 \cup \cdots \cup \Omega_n$$

be the spectral decomposition of ϕ. Since ϕ is Ω-stable, it satisfies Axiom A, so $\Omega(\phi)$ is a hyperbolic set. Therefore by Theorem 2.1 there exists a neighborhood W of $\Omega(\phi)$ on which ϕ has the ELmSP with constants $\lambda \in (0,1)$ and L. Now if a sequence $\{x_k\}$ satisfies (1.1), then by Lemma 3.4 there exists a basic set Ω_i such that

$$r(x_k, \Omega_i) \to 0 \quad \text{as } k \to \infty.$$

So there exists $k_0 \ge k_1$ such that $\{x_k\}_{k \ge k_0} \subset W$. Hence there exist $p \in M$ and $k_2 \in \mathbb{N}$ such that

$$r(\phi^{2k}(p), x_{2k}) \le L\lambda^k \quad \text{for } k \ge k_2.$$

Thus $d(\phi^k(p), x_k) < L\lambda^{k/2}$ for $k \ge 2k_2$. ∎

THEOREM 3.5 ([Sa]). *Let f be a homeomorphism on a compact metric space X. Then the following conditions are equivalent:*

- *f is expansive and has the POTP.*
- *There is a compatible metric D for X such that f is $\mathcal{L}$-hyperbolic.*
- *(X, f) is a Smale space.*

THEOREM 3.6 ([Sa]). *Let $f : X \to X$ be an $\mathcal{L}$-hyperbolic homeomorphism on a compact metric space (X, d). Then f has the LpSP.*

Proof of Theorem 2.3. Suppose that the sequence $\xi = \{x_k\}$ satisfies

$$d(f(x_k), x_{k+1}) \le \lambda^k \quad \text{for } k \ge k_1,$$

for some $k_1 \in \mathbb{N}$, where $\lambda = \nu \in (0,1)$ is the constant in the definition of $\mathcal{L}$-hyperbolicity. By Theorem 3.6 there exist constants $L_0, \delta > 0$ such that f has the LpSP with these constants. Choose $k_2 \ge k_1$ so that $k > k_2$ implies that $\lambda^k < \min\{\delta, \Delta/2L_0\}$. Hence for any $k > k_2$ there exists $y_k \in X$ such that

$$d(f^i(y_k), x_i) \le L_0 \lambda^k < \Delta/2 \quad \text{for } i \ge k \ge k_2.$$

If $y = y_{k_2}$ then

$$d(f^i(y_k), f^i(y)) \le d(f^i(y_k), x_i) + d(x_i, f^i(y)) < \Delta/2 + \Delta/2 = \Delta$$

for $i \ge k \ge k_2$. If $k \ge k_2$ is a fixed integer then

$$d(f^i(y_k), f^i(y)) < \Delta \quad \text{for } i \ge k.$$

Hence

$$d(f^{i+k}(y_k), f^{i+k}(y)) < \Delta \quad \text{for } i \ge 0.$$

Therefore $f^k(y_k) \in W^s_\Delta(f^k(y))$. The property (4) in the definition of $\mathcal{L}$-hyperbolicity implies

$$d(f^{i+k}(y_k), f^{i+k}(y)) < \nu^i \Delta \quad \text{for } i \ge 0.$$

Thus

$$d(f^{2k}(y), x_{2k}) \le d(f^{2k}(y), f^{2k}(y_k)) + d(f^{2k}(y_k), x_{2k}) < (L_0 + \Delta)\lambda^k \quad \text{for } k \ge k_2.$$

So

$$d(f^k(y), x_k) < (L_0 + \Delta)\lambda^{k/2} \quad \text{for } k \ge 2k_2.$$

If $L = L_0 + \Delta$ then $d(f^k(y), x_k) \le L\lambda^{k/2}$ for $k \ge 2k_2$. ∎

Theorems 2.3 and 3.5 imply the following corollaries:

COROLLARY 3.7. *Let $f : X \to X$ be an expansive homeomorphism on a compact metric space having the POTP. Then there is a compatible metric D on X such that f has the ELmSP with respect to D.*

COROLLARY 3.8. *Let (X, f) be a Smale space. Then there is a compatible metric D on X such that f has the ELmSP with respect to D.*

4. Examples. We begin by an example of a diffeomorphism which has the LmSP but does not have the ELmSP.

EXAMPLE 4.1. Consider the unit circle with coordinate $x \in [0, 1)$. Let ϕ be a dynamical system on S^1 generated by the mapping $f : [0.1) \to [0, 1)$ defined by $f(x) = x - x^2(x - 1/2)(x - 1)^2$. Then ϕ has two fixed points $\{0, 1/2\}$, and 0 is not a hyperbolic point because $f'(0) = 1$. Theorem 3.1.2

in [P1] ensures that ϕ has the LmSP on S^1. We now show that it does not have the ELmSP.

Suppose ϕ has the ELmSP with constants $L > 0$ and $\lambda \in (0,1)$. Take a natural number n_0 such that $\sum_{i=n_0}^{\infty} \lambda^i < 1/4$. We define a sequence $\{x_k\}$ by

$$x_{n_0} = 1/4, \quad x_{k+1} = \phi(x_k) + \lambda^k \quad \text{for } k \geq n_0.$$

Then it is easy to see that $\xi = \{x_k\}_{k \geq n_0}$ is a subset of $(0, 1/2)$,

$$|\phi(x_k) - x_{k+1}| \leq \lambda^k \quad \text{for } k \geq n_0 \tag{4.1}$$

and $x_k \to 0$ as $k \to \infty$. Hence there exist a point $p \in S^1$ and $n_1 \in \mathbb{N}$ so that

$$|\phi^k(p) - x_k| < L\lambda^{k/2} \quad \text{for } k \geq n_1. \tag{4.2}$$

Thus

$$|\phi^{2k+1}(p) - x_{2k+1}| < L\lambda^k \quad \text{for } k \geq n_1.$$

The following three cases can happen:

Case 1. If $p = 1/2$, then p does not satisfy (4.2).

Case 2. If $p \in [0, 1/2)$, then we show that there exists an index k_1 such that $\phi^{k_1}(p) < x_{k_1}$. If $p = 0$ then there is nothing to prove.

If $p \in (0, 1/2)$ then there is a natural number i_0 such that

$$\phi^{n_0+i_0}(p) < x_{n_0}.$$

If $\operatorname{Card}\{i : x_i \leq x_{i+1}\} > i_0$, then obviously

$$\phi^{n_0+i_0+k}(p) < x_{n_0+i_0+k} \quad \text{for some } k \in \mathbb{N},$$

and this proves our claim with $k_1 = n_0 + i_0 + k$.

Now suppose $\operatorname{Card}\{i : x_i \leq x_{i+1}\} \leq i_0$. Then for all large enough $k \in \mathbb{N}$ we have $x_{k+1} < x_k$. Assume that $0 < \cdots < a_2 < a_1 = 1/4$ is a decreasing sequence of real numbers such that $\operatorname{diam}([a_{n-1}, a_n]) \to 0$ and consider the sequence $\{I_n = (a_{n+1}, a_n] : n \geq 1\}$ of intervals. For $j \geq 1$ we put

$$i_j = \operatorname{Card}\{\{x_{n_0+i}\}_{i=0}^{\infty} \cap I_j\}, \quad i'_j = \operatorname{Card}\{\mathcal{O}(\phi, p) \cap I_j\}.$$

Obviously for any n, there is at most one k such that $x_{n+1} \leq \phi^k(p) \leq x_n$. So we can choose a_n's so that $i_n > i'_n$.

We show that $\sum_{j=1}^{m}(i_j - i'_j) > i_0$ for some m. If $\sum_{j=1}^{m}(i_j - i'_j) \leq i_0$ for each $m \in \mathbb{N}$ then $\sum_{j=1}^{\infty}(i_j - i'_j) \leq i_0$. The inequality $i_j \geq i'_j$ implies $i_j = i'_j$ for $j \geq j_0$, for some j_0. Let l be the smallest index such that $x_l \in I_{j_0}$ and m be the smallest natural number such that $\phi^m(p) \in I_{j_0}$. If $\phi^m(p) \leq x_l$ then $\epsilon = x_{l+1} - \phi^{m+1}(p) > 0$ and

$$x_{l+i_{j_0}+\cdots+i_j} - \phi^{m+i_{j_0}+\cdots+i_j}(p) > \epsilon \quad \text{for } j > j_0.$$

But $\operatorname{diam}(I_n) \to 0$ so there exists $N \in \mathbb{N}$ such that $\operatorname{diam}(I_N) < \epsilon$ and for some j we have $x_{l+i_{j_0}+\cdots+i_j} \in I_N$. So $\phi^{m+i_{j_0}+\cdots+i_j}(p) \notin I_N$, which contradicts our hypothesis.

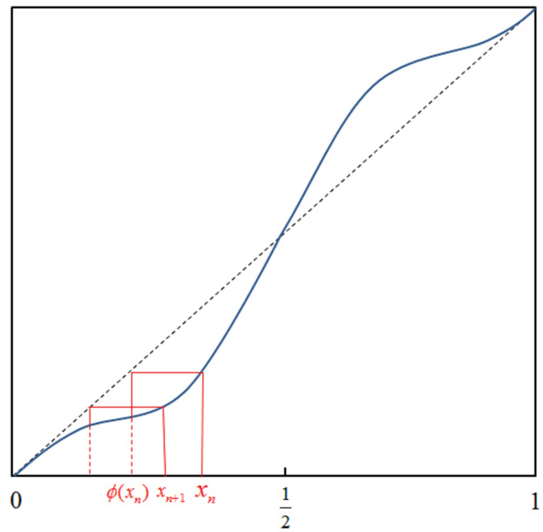

Fig. 1. The sequence $\{x_k\}$ of Example 1

If $\phi^m(p) > x_l$ then $\phi^{m+1}(p) \le x_l$ and

$$x_{l+i_{j_0}+\cdots+i_j} - \phi^{m+1+i_{j_0}+\cdots+i_j}(p) > 2\epsilon$$

for some $\epsilon > 0$ and $j \ge j_0$. Since $\phi^{m+i_{j_0}+\cdots+i_j}(p) - \phi^{m+1+i_{j_0}+\cdots+i_j}(p) \to 0$ as $j \to \infty$, there is $j_1 \ge j_0$ such that

$$x_{l+i_{j_0}+\cdots+i_j} - \phi^{m+1+i_{j_0}+\cdots+i_j}(p) < \epsilon \quad \text{for } j \ge j_1$$

and we find a contradiction.

So $\sum_{j=1}^{m}(i_j - i'_j) > i_0$ for some m. Thus

$$\phi^{i_0+(\sum_{j=1}^{m} i'_j)+1}(p) \in I_{m+1} \quad \text{and} \quad x_{n_0+\sum_{j=1}^{m} i_j} \in I_m.$$

So

$$\phi^{i_0+(\sum_{j=1}^{m} i'_j)+1}(p) < x_{n_0+\sum_{j=1}^{m} i_j}.$$

Therefore

$$\phi^{n_0+\sum_{j=1}^{m} i_j}(p) \le \phi^{n_0+(\sum_{j=1}^{m} i'_j)+i_0}(p) \le \phi^{(\sum_{j=1}^{m} i'_j)+i_0+1}(p) < x_{n_0+\sum_{j=1}^{m} i_j},$$

and this proves our claim with $k_1 = n_0 + \sum_{j=1}^{m} i_j$.

Now since $\lim_{x\to 0} \phi'(x) = 1$, there exists a positive real ϵ such that $\phi'(x) > \lambda^{1/2}$ for any $x \in [0,\epsilon)$. Hence, the mean value theorem implies that $\phi(x+h) > \phi(x) + \lambda^{1/2}h$ for any $h > 0$ with $x+h \in [0,\epsilon)$.

Since $x_k \to 0$, there is k_2 such that $x_k \in (0,\epsilon)$ for $k \ge k_2$. If $k_0 = \max\{k_1+1, k_2\}$, then $\phi^{2k+1}(p) < \phi^{2k-k_0+1}(x_{k_0})$ for $k \ge k_0$. So

$$
\begin{aligned}
x_{2k+1} - \phi^{2k+1}(p) &= \phi(x_{2k}) + \lambda^{2k} - \phi^{2k+1}(p) \\
&= \phi(x_{2k-1} + \lambda^{2k-1}) + \lambda^{2k} - \phi^{2k+1}(p) \\
&> \phi^2(x_{2k-1}) + \lambda^{2k-1+1/2} + \lambda^{2k} - \phi^{2k+1}(p) > \cdots \\
&> \phi^{i+1}(x_{2k-i}) + \lambda^{2k-i+1/2} + \lambda^{2k-(i-1)+1/2} + \cdots + \lambda^{2k-1+1/2} \\
&\quad + \lambda^{2k} - \phi^{2k+1}(p)
\end{aligned}
$$

and with the new variable $i = 2k - k_0$ we deduce

$$
\begin{aligned}
x_{2k+1} - \phi^{2k+1}(p) &> \phi^{2k-k_0}(x_{k_0}) + \lambda^{k+1/2}\Big(\sum_{j=0}^{2k-k_0} \lambda^{k-j}\Big) - \phi^{2k+1}(p) \\
&= \phi^{2k-k_0}(x_{k_0}) + \lambda^{k+1/2}\Big(\sum_{j=0}^{k} \lambda^{j} + \sum_{j=0}^{k-k_0} \lambda^{-j}\Big) - \phi^{2k+1}(p) \\
&> \lambda^{k+1/2}\Big(\sum_{j=0}^{k} \lambda^{j} + \sum_{j=0}^{k-k_0} \lambda^{-j}\Big).
\end{aligned}
$$

Therefore

$$
L > \lambda^{1/2}\Big(\sum_{j=0}^{k} \lambda^{j} + \sum_{j=0}^{k-k_0} \lambda^{-j}\Big) \qquad \text{for } k > \max\{k_0, n_1\},
$$

which is a contradiction.

CASE 3. If $p \in (1/2, 1)$ then

$$
r(\phi^k(p), x_k) > r(\phi^k(1-p), x_k).
$$

Since $1 - p \in (0, 1/2)$, p does not satisfy (4.2) by Case 2. So no point $p \in S^1$ satisfies (4.2), i.e. ϕ does not have the ELmSP.

REMARK 4.2. If $\phi : S^1 \to S^1$ is a diffeomorphism generated by a function $f : [0,1) \to [0,1)$ with the properties:

- f is differentiable and increasing;
- f has a non-hyperbolic attracting fixed point p so that $(p, p+\epsilon) \cap \mathrm{Fix}(f) = \emptyset$ or $(p-\epsilon, p) \cap \mathrm{Fix}(f) = \emptyset$ for some $\epsilon > 0$,

then as in Example 4.1 we can find a sequence $\xi = \{x_k\}$ which satisfies (1.1), but there is no p such that (1.2) holds, i.e. ϕ does not have the ELmSP.

5. Conclusion. We know from [ENP] that near a hyperbolic set we have the limit shadowing property. In this paper we show that near a hyperbolic set we have a new kind of shadowing which is not equivalent to the limit shadowing property. Moreover, we show that an Ω-stable dynamical system has this property and in a compact metric space any $\mathcal{L}$-hyperbolic map has the ELmSP.

We say that the dynamical system ϕ has the *strong exponential limit shadowing property* (SELmSP) on M if there exist constants $L > 0$ and $\lambda \in (0, 1)$ such that for any sequence $\xi = \{x_k\}$ with (1.1) there exist a point $p \in M$ and $k_2 \in \mathbb{N}$ such that

$$r(\phi^k(p), x_k) < L\lambda^k \quad \text{for } k \geq k_2. \tag{5.1}$$

Obviously the SELmSP implies the ELmSP. Example 4.1 shows that the SELmSP is a different concept from the LmSP. The consideration of the SELmSP is a topic for further research.

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