ANNALES
POLONICI MATHEMATICI
108.1 (2013)

On classes of uniformly starlike functions

by Agnieszka Wiśniowska-Wajnryb (Rzeszów)

Abstract. We geometrically define subclasses of starlike functions related to the class of uniformly starlike functions introduced by A. W. Goodman in 1991. We give an analytic characterization of these classes, some radius properties, and examples of functions in these classes. Our classes generalize the class of uniformly starlike functions, and many results of Goodman are special cases of our results.

1. Introduction. For $r > 0$ let $U_r = \{z \in \mathbb{C} : |z| < r\}$ and let $\bar{U}_r$ be the closure of U_r. Let S denote the class of all functions f that are analytic and univalent in the open unit disk $U = U_1$ and normalized by $f(0) = f'(0) - 1 = 0$.

An open set $D \subset \mathbb{C}$ is said to be *starlike with respect to* w_0, an interior point of D, if the intersection of each half-line beginning at w_0 with the interior of D is connected. We denote by ST the class of all *starlike functions*, i.e. the subclass of S consisting of functions that map U onto domains starlike with respect to $w_0 = 0$ (briefly *starlike domains*). Recall that a function $f \in S$ is starlike if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \text{for all } z \in U.$$

A starlike function takes every disk $U_r \subset U$ onto a starlike domain. Not every function $f \in ST$ maps each disk $\{z : |z - \zeta| < \rho\} \subset U$ onto a region starlike with respect to $f(\zeta)$.

Let CV denote the class of all functions $f \in S$ that are *convex* in U, i.e. such that $f(U)$ is a convex domain. A function in CV maps every disk contained in U onto a convex region (this is a result of Study [S] and Robertson [Rb]).

Let $\gamma : z = z(t)$, $t \in [a, b]$, be a smooth, directed arc and suppose that a function f is analytic on γ. Then the arc $f(\gamma)$ is said to be

2010 *Mathematics Subject Classification*: Primary 30C45.
Key words and phrases: starlike functions, uniformly starlike functions, convex functions, k-uniformly convex functions.

DOI: 10.4064/ap108-1-2

- *starlike with respect to* $w_0 \notin f(\gamma)$ if $\arg(f(z(t))-w_0)$ is a nondecreasing function of t,
- *convex* if the argument of the tangent to $f(\gamma)$ is a nondecreasing function of t.

In 1991 Goodman [G1] introduced geometrically defined classes UCV and UST of *uniformly convex* and *uniformly starlike functions*, respectively. A function $f \in S$ is in the class UCV (resp. UST) if for every circular arc $\gamma \subset U$ with center at $\zeta \in U$, the arc $f(\gamma)$ is convex (resp. starlike with respect to $f(\zeta)$). Goodman obtained the following analytic conditions for UCV and UST expressed by using two complex variables:

$$f \in UCV \;\Leftrightarrow\; \operatorname{Re}\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right) \geq 0 \text{ for all } z,\zeta \in U$$

and

$$f \in UST \;\Leftrightarrow\; \operatorname{Re}\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} \geq 0 \text{ for all } z,\zeta \in U.$$

In [KW] the authors generalized the concept of uniform convexity due to Goodman in the following way. Let $k \geq 0$. A function $f \in S$ is said to be *k-uniformly convex* in U if the image of every circular arc contained in U with center at ζ, where $|\zeta| \leq k$, is convex. Note that 1-$UCV = UCV$ and 0-$UCV = CV$. It was proven in [KW] that $f \in S$ belongs to k-UCV if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad \text{for all } z \in U.$$

For $k = 0$ we get the well known condition for convexity, and for $k = 1$ we get a more applicable characterization of the class UCV obtained by Ma and Minda [MM] and independently by Rønning [R2].

In this paper we introduce the notion of *k-uniform starlikeness* which is intermediate between being starlike and uniformly starlike. The class k-UST of k-uniformly starlike functions coincides with the class ST for $k = 0$ and with the class UST for $k = 1$. We show some properties of k-uniformly starlike functions. Many results of Goodman on UST are special cases of our results. Moreover a result indicated by Goodman in [G2] without proof is a special case of our Theorem 5.

2. The classes of k-uniformly starlike functions. Let $0 \leq k \leq 1$. A function $f \in S$ is said to be *k-uniformly starlike* in U if the image of every circular arc γ contained in U with center at ζ, where $|\zeta| \leq k$, is starlike with respect to $f(\zeta)$. We denote by k-UST the class of all such functions. Note that 1-$UST = UST$, 0-$UST = ST$ and clearly $UST \subset k$-$UST \subset ST$ for every $k \in [0,1]$.

If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ then their *convolution* (or *Hadamard product*) is defined as $(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n$.

We can easily obtain the following analytic characterization of k-uniformly starlike functions.

THEOREM 1. *Let $f\in S$ and $0\le k\le 1$. Then $f\in k$-UST if and only if*

$$\operatorname{Re}\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\ge 0 \quad \text{for all } z\in U \text{ and } \zeta\in\bar{U}_k,$$

or equivalently

$$\operatorname{Re}\frac{f(z)*\frac{z}{(1-\alpha z)(1-kz)}}{f(z)*\frac{z}{(1-\alpha z)^2}}\ge 0 \quad \text{for all } z\in U \text{ and } \alpha\in\bar{U}.$$

Proof. Since for $\gamma: z=z(t)=\zeta+Re^{it}$, $t\in I\subset[0,2\pi]$, the arc $f(\gamma)$ is starlike with respect to $f(\zeta)$ if

$$\frac{d}{dt}\arg\big(f(z(t))-f(\zeta)\big)\ge 0, \qquad t\in I,$$

the first part of the theorem follows after easy calculations.

The second part is a consequence of the first and of the relations

$$\frac{f(\alpha z)-f(kz)}{\alpha-k}=f(z)*\frac{z}{(1-\alpha z)(1-kz)}, \qquad zf'(\alpha z)=f(z)*\frac{z}{(1-\alpha z)^2},$$

which hold for all $z\in U$ and $\alpha\in\bar{U}$. ■

In the next theorems we exhibit some members of the classes k-UST.

LEMMA 2. *Define $f_A(z,\zeta)=(1-Az)/(1-A\zeta)$, where $z,\zeta\in\bar{U}$ and A is a complex number satisfying $0<|A|<1$. Let $0<k\le 1$. Then $\operatorname{Re}f_A(z,\zeta)>0$ for all $z\in\bar{U}$ and $\zeta\in\bar{U}_k$ if and only if $|A|\le 1/\sqrt{1+k^2}$.*

Proof. Fix k. We shall call z,ζ *admissible* if $z\in\bar{U}$ and $\zeta\in\bar{U}_k$. If $\operatorname{Re}f_A(z,\zeta)>0$ for some A and for all admissible z,ζ then the same is true for every A_1 with $|A|=|A_1|$ (because $f_A(z,\zeta)=f_{A_1}(z',\zeta')$ after the rotation $z'=uz$, $\zeta'=u\zeta'$, $u=\exp(it)$ for a suitable real t). So we may assume that A is real and positive. For a fixed admissible pair z,ζ and $0<A<1$ let $P(z)=1-Az$ and let $Q(\zeta)=1-A\zeta$. Consider the functions

$$\bar{U}\ni z\mapsto \operatorname{Arg}(1-Az)=\operatorname{Arg}P(z),$$
$$\bar{U}_k\ni \zeta\mapsto \operatorname{Arg}(1-A\zeta)=\operatorname{Arg}Q(\zeta)$$

and denote

$$\phi_1(A)=\max_{z\in\bar{U}}\operatorname{Arg}P(z), \qquad \phi_2(A)=\min_{\zeta\in\bar{U}_k}\operatorname{Arg}Q(\zeta).$$

By the maximum principle for harmonic functions we get

$$\phi_1(A)=\max_{|z|=1}\operatorname{Arg}P(z), \qquad \phi_2(A)=\min_{|\zeta|=k}\operatorname{Arg}Q(\zeta).$$

Therefore $\operatorname{Re} f_A(z,\zeta) > 0$ if and only if $\phi_1(A) - \phi_2(A) < \pi/2$. Clearly

$$\phi_1(A) = \arcsin A = \operatorname{Arg} P(z_0), \qquad \phi_2(A) = -\arcsin(Ak) = \operatorname{Arg} Q(\zeta_0),$$

where $P(z_0) = P$, $Q(\zeta_0) = Q$ and the segments $0P$, $0Q$ are tangent to the circles $\{w : |w-1| = A\}$, $\{w : |w-1| = Ak\}$, respectively. The angle $\phi_A = \phi_1(A) - \phi_2(A)$ increases when A increases. There is a unique A for which

$$\phi_A = \arcsin A + \arcsin(Ak) = \pi/2.$$

Then we have a rectangle $0P1Q$ with diagonal of length 1 and sides equal to A and Ak. It follows that in this extremal case $A = 1/\sqrt{1+k^2}$. We get the same result solving the equation $\arcsin A + \arcsin(Ak) = \pi/2$. ∎

THEOREM 3. *Let* $0 < k \le 1$. *The function* $f(z) = z/(1 - Az)$, $z \in U$, *belongs to the class* k-*UST if and only if* $|A| \le 1/\sqrt{1+k^2}$.

Proof. If $f(z) = z/(1-Az)$, $z \in U$, then

$$\frac{f(z) - f(\zeta)}{(z-\zeta)f'(z)} = \frac{1-Az}{1-A\zeta} = f_A(z,\zeta),$$

so the result follows from Theorem 1 and Lemma 2. ∎

Goodman [G2] has mentioned that the function of the form $f(z) = z + Az^2$, $z \in U$, is in *UST* iff $|A| \le \sqrt{3}/4$. He omitted the proof which in his words requires a longer analysis. We shall prove this result and we shall generalize it to the class k-*UST* using the following lemma.

LEMMA 4. *Define* $g_A(z,\zeta) = (1 + Az + A\zeta)/(1 + 2Az)$, *where* $z,\zeta \in \bar{U}$ *and* A *is a complex number with* $0 < |A| < 1/2$. *Let* $0 < k \le 1$. *Then* $\operatorname{Re} g_A(z,\zeta) > 0$ *for all* $z \in \bar{U}$ *and* $\zeta \in \bar{U}_k$ *if and only if* $|A| \le \sqrt{3}/\sqrt{4(k^2+3)}$.

Proof. The proof is similar to the proof of the previous lemma. Fix k. We shall call z,ζ *admissible* if $z \in \bar{U}$ and $\zeta \in \bar{U}_k$. We may assume again that A is real and positive. For a fixed, admissible pair z,ζ let $P = 1 + 2Az$ and $Q = 1 + Az + A\zeta$. Also let $R = 1 + Az$. Then $\operatorname{Re} g_A(z,\zeta) > 0$ if and only if the angle $P0Q$ is smaller than $\pi/2$. We denote this angle by $\measuredangle P0Q = \phi = |\operatorname{Arg}(1+2Az) - \operatorname{Arg}(1 + Az + A\zeta)|$. For a given A we let ϕ_A be the supremum of ϕ over all admissible pairs (z,ζ). The set of all admissible pairs is compact and ϕ is a continuous function of (z,ζ) so the maximum ϕ_A is attained for each A at some point (z_0,ζ_0). Moreover the function $\bar{U} \ni z \mapsto \operatorname{Arg}(1+2Az) - \operatorname{Arg}(1+Az+A\zeta)$ for a fixed ζ and the function $\bar{U}_k \ni \zeta \mapsto \operatorname{Arg}(1+2Az) - \operatorname{Arg}(1+Az+A\zeta)$ for a fixed z are harmonic, so, by the maximum principle for harmonic functions, $|z_0| = 1$ and $|\zeta_0| = k$.

We may assume by symmetry that P lies above the real axis. For $P = 1 + 2Az_0$ fixed, the point Q lies on a circle with center R and radius Ak.

Clearly the angle ϕ is largest when the segment $0Q$ is tangent to the circle and the radius RQ points down so this happens for $\zeta = \zeta_0$.

Points $0, P, Q, R, 1$ are shown in Figure 1. The segment $0Q$ is horizontal, to stress the fact that the angle $0QR$ is straight, and the segment 01 is not horizontal. When we increase A to A_1 we can choose z shorter than z_0 and keep $A_1 z = A z_0$ and $\zeta = \zeta_0$ fixed. Then the radius QR increases to $A_1 k$ and ϕ increases and $\phi_{A_1} > \phi_A$. Therefore ϕ_A is an increasing function of A. There is a unique A for which $\phi_A = \pi/2$. We shall compute this A.

If for some A there exists a configuration as in Figure 1 with $\measuredangle P0Q$ equal to $\pi/2$ then $\phi_A \geq \pi/2$. We look for the minimal A for which such a configuration exists. Consider Figure 1. The letters A, B, C, D, E, h, Ak denote lengths of segments. The segment 01 has length 1. The angles $P0Q$, $0QR$ and QBR are straight.

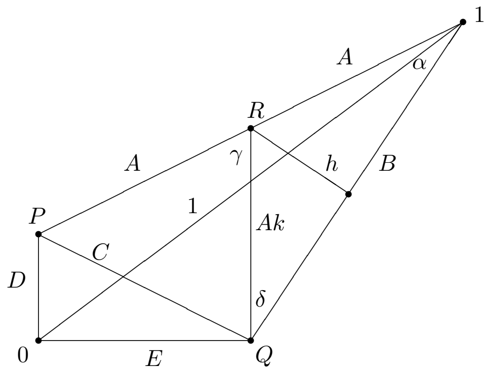

Fig. 1

We have $E = A\sin\gamma = A\sin(\pi-\gamma)$.

From the sine theorem for the triangle $QR1$ we get $B\sin\alpha = Ak\sin(\pi-\gamma)$, therefore $E = (B\sin\alpha)/k$.

Now we compute B from the two pieces of the triangle $QR1$:

$$B = \sqrt{A^2 - A^2\sin^2\alpha} + \sqrt{A^2k^2 - A^2\sin^2\alpha}.$$

From the cosine theorem for the triangle $0Q1$ we get

$$1 = E^2 + B^2 - 2BE\cos(\pi/2+\delta),$$
$$\cos(\pi/2+\delta) = -\sin\delta = -h/Ak = -(A\sin\alpha)/Ak = -(\sin\alpha)/k.$$

Therefore

$$1 = E^2 + B^2 + 2BE(\sin\alpha)/k = B^2(1 + 3(\sin^2\alpha/k^2).$$

Let $\sin^2\alpha = x$. Then $B = A(\sqrt{1-x} + \sqrt{k^2-x})$. Hence

$$1 = A^2(\sqrt{k^2-x} + \sqrt{1-x})^2(1+3x/k^2).$$

We let

$$h(x) = k^2/A^2 = (\sqrt{k^2 - x} + \sqrt{1-x})^2(k^2 + 3x), \qquad x \in [0, k^2].$$

The maximum of $h(x)$ corresponds to the minimal value of A for which the configuration in Figure 1 exists. Since

$$h'(x) = (\sqrt{k^2 - x} + \sqrt{1-x})^2\left(3 - \frac{k^2 + 3x}{\sqrt{k^2 - x}\sqrt{1-x}}\right),$$

$h'(x) = 0$ if and only if $3\sqrt{(k^2 - x)(1-x)} = k^2 + 3x$. This happens when $x = (9k^2 - k^4)/(15k^2 + 9)$. For this value of x,

$$1 - x = (k^2 + 3)^2/(15k^2 + 9), \qquad k^2 - x = 16k^4/(15k^2 + 9),$$

hence $h(x) = 4k^2(k^2 + 3)/3$.

It is indeed the maximal value of $h(x)$, greater than the value at 0 and at k^2. Therefore the value of A for which $\phi_A = \pi/2$ is equal to $A = \sqrt{3}/\sqrt{4(k^2+3)}$. ∎

THEOREM 5. *Let $0 < k \le 1$. The function $f(z) = z + Az^2$, $z \in U$, belongs to the class k-UST if and only if $|A| \le \sqrt{3}/\sqrt{4(k^2+3)}$.*

Proof. For $f(z) = z + Az^2, z \in U$, we get

$$\frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} = \frac{1 + A(z + \zeta)}{1 + 2A\zeta} = g_A(z, \zeta),$$

so the result follows directly from Theorem 1 and Lemma 4. ∎

REMARK 6. When we let $k = 0$ in Theorems 3 and 5 we get simple results for starlike functions. For $k = 1$ we obtain a result proven by Goodman [G2] and the more difficult result mentioned before Lemma 4, respectively.

THEOREM 7. *Let $0 < k \le 1$ and $f(z) = z + Az^n$, $z \in U$, for some integer $n \ge 2$. Then*

$$|A| \le \frac{1}{n}\sqrt{\frac{n+1}{n+1+(n-1)k^2}} \;\Rightarrow\; f \in UST.$$

Proof. We may assume $A > 0$. If $f(z) = z + Az^n$, $z \in U$, then

$$\begin{aligned}\frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} &= \frac{1 + A(z^{n-1} + \zeta z^{n-2} + \cdots + \zeta^{n-1})}{1 + nAz^{n-1}} \\ &= \frac{1 + A(1 + k\alpha + k^2\alpha^2 + \cdots + k^{n-1}\alpha^{n-1})w}{1 + nAw} = \frac{1 + A(1+\gamma)w}{1 + nAw},\end{aligned}$$

where

$$w = z^{n-1}, \quad \zeta = k\alpha z, \quad \gamma = k\alpha + (k\alpha)^2 + \cdots + (k\alpha)^{n-1}, \quad |\alpha| \le 1.$$

The function $G(w) = \frac{1+A(1+\gamma)w}{1+nAw}$, $w \in U$, maps $|w| = 1$ onto the circle with center

$$c = \frac{1 - nA^2(1+\gamma)}{1 - n^2A^2}$$

and radius

$$R = \left| \frac{1 + A(1+\gamma)}{1 + nA} - \frac{1 - nA^2(1+\gamma)}{1 - n^2A^2} \right| = \frac{A|(n-1) - \gamma|}{|1 - n^2A^2|}.$$

Hence $G(U)$ is in the right half-plane if $R \le \operatorname{Re} c$, that is,

$$\left| \frac{(n-1) - \gamma}{1 - n^2A^2} \right| A \le \frac{1 - nA^2(1 + \operatorname{Re}\gamma)}{1 - n^2A^2}.$$

Since k-$UST \subset ST$ it is clear that $A < 1/n$. Note that $|\gamma| \le k(n-1)$ and $1 + \operatorname{Re}\gamma < 1 + |\gamma| < 1 + k(n-1) < n$, which shows that $\operatorname{Re} c > 0$. Moreover $|(n-1)-\gamma|^2 = (n-1)^2 - 2(n-1)\operatorname{Re}\gamma + |\gamma|^2 \le (n-1)^2(1+k^2) - 2(n-1)\operatorname{Re}\gamma$. Thus the required inequality holds when

$$A^2[(1+k^2)(n-1)^2 - 2(n-1)x] \le (1 - nA^2)^2 - 2nA^2(1 - nA^2)x + n^2A^4x^2,$$

where $x = \operatorname{Re}\gamma$, or equivalently when

$$n^2A^4x^2 + 2(n^2A^2 - 1)A^2x + (1 - nA^2)^2 - (1 + k^2)(n-1)^2A^2 \ge 0.$$

For $x = (1 - n^2A^2)/(n^2A^2)$ the left side of the last inequality takes its minimum value. After substitution and simplification we get

$$n + 1 - 2n^2A^2 - (1 + k^2)(n-1)n^2A^2 \ge 0,$$

hence

$$A^2 \le \frac{n+1}{n^2[n + 1 + (n-1)k^2]}. \ \blacksquare$$

REMARK 8. For $k = 0$ we get a sufficient condition for $f(z) = z + Az^n$, $z \in U$, to be in the class ST. For $k = 1$ we get Theorem 5 from [MS], which improves the bound $|A| \le 1/(\sqrt{2}\,n)$ of Goodman [G2]. Our bound does not seem to be the best possible except when $n = 2$ (compare Theorem 5) or when $k = 0$.

It is known that every convex function is starlike, so the ST radius in CV is equal to 1. It was proven in [MS] and [R1] that the number $1/\sqrt{2}$ is the radius of uniform starlikeness in CV. To prove a radius result for k-UST we need the following result of Ruscheweyh and Sheil–Small [RS]:

LEMMA 9. *If ϕ is a normalized convex univalent function in U and $g \in ST$, then*

$$\operatorname{Re} \frac{\phi * (Fg)}{\phi * g} \ge 0, \qquad z \in U,$$

whenever F is an analytic function with positive real part in U.

THEOREM 10. *Let $0 < k \le 1$. If $f \in CV$, then the function $U \ni z \mapsto (1/r_k)\, f(r_k z)$, where $r_k = 1/\sqrt{1+k^2}$, belongs to the class k-UST. The radius r_k is the best possible.*

Proof. Let k be fixed. The function $F(z) = (1-\alpha rz)/(1-krz)$ is analytic in U and from Lemma 1, $\operatorname{Re} F(z) > 0$ for all $z \in U$ and $\alpha \in \bar{U}$ iff $r \le 1/\sqrt{1+k^2}$. Note that the function $g(z) = z/(1-\alpha rz)^2$ is starlike in U for every $r \in (0,1)$ and $\alpha \in \bar{U}$. If $f \in CV$, then from Lemma 9 we conclude that

$$\operatorname{Re} \frac{f(z) * \left(\frac{1-\alpha rz}{1-krz} \frac{z}{(1-\alpha rz)^2}\right)}{f(z) * \frac{z}{(1-\alpha rz)^2}} \ge 0 \quad \text{for } z \in U$$

and all $\alpha \in \bar{U}$ and $0 < r \le 1/\sqrt{1+k^2}$. By the properties of convolution this condition is equivalent to

$$\operatorname{Re} \frac{\frac{1}{r} f(rz) * \frac{z}{(1-kz)(1-\alpha z)}}{\frac{1}{r} f(rz) * \frac{z}{(1-\alpha z)^2}} \ge 0 \quad \text{for } z \in U,\ \alpha \in \bar{U},\ 0 < r \le 1/\sqrt{1+k^2}.$$

Thus by Theorem 1 we have $(1/r) f(rz) \in k$-UST if $0 < r \le 1/\sqrt{1+k^2}$. The result is sharp. Choosing $f(z) = z/(1-z) \in CV$ we see that $(1/r) f(rz) = z/(1-rz)$ is in k-UST iff $r \le 1/\sqrt{1+k^2}$, so the number $1/\sqrt{1+k^2}$ is the best possible. ∎

REMARK 11. Note that for $k = 0$ and $k = 1$ we obtain the above mentioned ST radius and UST radius in the class CV, respectively.

Acknowledgements. The author would like to thank the referee for helpful remarks and suggestions that improved the paper.

References

[G1] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. 56 (1991), 87–92.

[G2] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. 155 (1991), 364–370.

[KW] S. Kanas and A. Wiśniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. 104 (1999), 327–336.

[MM] W. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. 52 (1992), 165–175.

[MS] E. Merkes and M. Salmassi, *Subclasses of uniformly starlike functions*, Int. J. Math. Math. Sci. 15 (1992), 449–454.

[Rb] M. S. Robertson, *On the theory of univalent functions*, Ann. of Math. 37 (1936), 374–408.

[R1] F. Rønning, *On uniform starlikeness and related properties of univalent functions*, Complex Variables Theory Appl. 24 (1994), 233–239.

[R2] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. 118 (1993), 189–196.

[RS] St. Ruscheweyh and T. Sheil-Small, *Hadamard products of schlicht functions and the Pólya–Schoenberg conjecture*, Comment. Math. Helv. 48 (1973), 119–135.

[S] E. Study, *Konforme Abbildung einfachzusammenhängender Bereiche*, Vorlesungen über ausgewählte Gegenstände der Geometrie, Heft 2, Teubner, Leipzig, 1913.

Agnieszka Wiśniowska-Wajnryb
Department of Mathematics
Rzeszów University of Technology
Al. Powstańców Warszawy 12
35-959 Rzeszów, Poland
E-mail: agawis@prz.edu.pl

Received 10.1.2012
and in final form 22.4.2012 (2636)