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# Inertial manifolds for retarded second order in time evolution equations in admissible spaces

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**Abstract.** Using the Lyapunov–Perron method, we prove the existence of an inertial manifold for the process associated to a class of non-autonomous semilinear hyperbolic equations with finite delay, where the linear principal part is positive definite with a discrete spectrum having a sufficiently large distance between some two successive spectral points, and the Lipschitz coefficient of the nonlinear term may depend on time and belongs to some admissible function spaces.

**1. Introduction and statement of the main result.** One of the main problems in the theory of nonlinear differential equations is to study the behavior of their solutions as time goes to infinity. It is now well-known that for many dissipative equations, this behavior can be described by a global attractor with finite Hausdorff and fractal dimensions. Such an attractor is the largest bounded invariant set and attracts all bounded sets (see e.g. [20]). The concept of inertial manifolds for evolutionary equations introduced by Foiaş, Sell & Temam [9] allows us to go further in the study of the long-time behavior of the solutions. These manifolds, which are finite-dimensional Lipschitz manifolds, contain the global attractor and attract exponentially all the solutions of the system under consideration. Moreover, the dynamices in some absorbing set, when restricted to the inertial manifold, reduces to a system of ordinary differential equations called the inertial form of the given evolutionary equation.

The notion of inertial manifold has been translated and extended to more general classes of differential equations like non-autonomous differential equations (see e.g. [10, 12, 13]), retarded partial differential equations (see e.g. [3, 17, 19]), or differential equations with random or stochastic perturbations (see e.g. [2, 5, 8]). However, to the best of our knowledge,

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the most popular conditions for existence of inertial manifolds are the spectral gap condition on the linear principal part $A$ and the uniform Lipschitz condition on the nonlinear term, i.e., the Lipschitz coefficient of the nonlinearity does not depend on time. In some recent works [1, 11], the authors constructed inertial manifolds for a class of nonautonomous semilinear parabolic equations with or without delay under two conditions. First, the linear operator $A$ is positive definite with a discrete spectrum having a sufficiently large distance between some two successive spectral points, which can be considered in some sense as a slight generalization of the restrictive spectral gap conditions in [7, 18]. Second, the nonlinear term $B(t, u_t)$ is non-uniformly Lipschitz continuous in some interpolation space, i.e., $\|B(t, u_t) - B(t, v_t)\| \leq \varphi(t)|u_t - v_t|_{C_\alpha}$ for $\varphi$ being a real and positive function which belongs to an addmissible function space defined in Definition 2.2 below and satisfies certain conditions.

The aim of this paper is to study the existence of inertial manifolds for second order in time, retarded PDEs, where the Lipschitz coefficient of the nonlinear term may depend on time and belongs to some admissible function space. In what follows, we will formulate the problem and the result obtained in detail.

Hypothesis A. *Let $A$ be a positive definite operator with discrete spectrum in a separable Hilbert space $H$ (with a norm $\|\cdot\|$) and there exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $H$ such that*

$$Ae_k = \mu_k e_k \quad \textit{with} \quad 0 < \mu_1 \leq \mu_2 \leq \cdots, \qquad \lim_{k\to\infty} \mu_k = \infty.$$

*In the usual way, we can associate with $A$ its powers $A^\alpha$ defined on the domain $D(A^\alpha)$ endowed with the norm $\|\cdot\|_\alpha = \|A^\alpha \cdot\|$, in particular $D(A^0)$ $= H$.*

For $r > 0$ and $0 \leq \alpha \leq 1/2$ we denote by $C_\alpha = C([-r, 0]; D(A^\alpha))$ the space of strongly continuous functions on the interval $[-r, 0]$ with values in $D(A^\alpha)$. It is a Banach space with the norm

$$|v|_{C_\alpha} \equiv \sup_{\theta\in[-r,0]} \|v(\theta)\|_\alpha.$$

In this paper, we will study the existence of an inertial manifold for the following retarded second order in time nonautonomous evolution equation arising in the theory of nonlinear oscillations:

$$\begin{cases} \dfrac{d^2u}{dt^2} + 2\epsilon\dfrac{du}{dt} + Au = B(t, u_t) \quad \text{for } t > \tau,\ \epsilon > 0, \\ u|_{t=\tau+\theta} = u^{\tau,0}(\theta) \quad \text{for } \theta \in [-r, 0], \quad \left.\dfrac{du}{dt}\right|_{t=\tau} = u^{\tau,1}, \end{cases} \tag{1.1}$$

where $u^{\tau,0} \in C_\alpha$, $u^{\tau,1} \in H$, $\tau \in \mathbb{R}$ are given, and $u_t = u_t(\theta)$ denotes the element of $C_\alpha$ such that for all $\theta \in [-r, 0]$, we have $u_t(\theta) = u(t+\theta)$. Besides the above assumption on the operator $A$, we assume that the nonlinearity $B$ satisfies the following hypothesis.

HYPOTHESIS B. *Let $E$ be an admissible Banach function space on $\mathbb{R}$ (see Definition 2.2 below) and $\varphi$ be a positive function belonging to $E$.*

*Assume that the function $B : \mathbb{R} \times C_\alpha \to H$ is $\varphi$-Lipschitz, that is, for a.e. $t \in \mathbb{R}$, and all $u_t, v_t \in C_\alpha$:*

(i) $\|B(t, u_t)\| \le \varphi(t)(1 + |u_t|_{C_\alpha})$;
(ii) $\|B(t, u_t) - B(t, v_t)\| \le \varphi(t)|u_t - v_t|_{C_\alpha}$.

Let $\mathcal{H} = D(A^{1/2}) \times H$. It is clear that $\mathcal{H}$ is a separable Hilbert space with the inner product

$$(U, V) = (Au^0, v^0) + (u^1, v^1),$$

where $U = (u^0; u^1)$ and $V = (v^0; v^1)$ are elements of $\mathcal{H}$. In $\mathcal{H}$ problem (1.1) can be rewritten as a system of first order:

$$\begin{cases} \dfrac{dU}{dt} + \mathcal{A}U(t) = \mathcal{B}(t, U_t), \quad t > \tau, \\ U|_{t=\tau,\, \theta\in[-r,0]} = U_\tau, \end{cases} \tag{1.2}$$

where $U(t) = (u(t); \dot{u}(t))$, $U_\tau = (u^{\tau,0}; u^{\tau,1})$. Here the linear operator $\mathcal{A}$ and the mapping $\mathcal{B}$ are defined by

$$\mathcal{A}U = (-u^1; Au^0 + 2\epsilon u^1), \qquad D(\mathcal{A}) = D(A) \times D(A^{1/2}),$$
$$\mathcal{B}(t, U_t) = (0; B(t, u_t^0)) \quad \text{for } U = (u^0; u^1).$$

It is easy to verify that the eigenvalues and eigenvectors of the operator $\mathcal{A}$ have the form

$$\lambda_n^\pm = \epsilon \pm \sqrt{\epsilon^2 - \mu_n}, \qquad f_n^\pm = (e_n; -\lambda_n^\pm e_n), \qquad n = 1, 2, \ldots,$$

where $\mu_n$ and $e_n$ are eigenvalues and eigenvectors of the operator $A$.

Let the condition $\epsilon^2 > \mu_{N+1}$ hold for some integer $N$. We consider the decomposition of the space $\mathcal{H}$ into the orthogonal sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2,$$

where

$$\mathcal{H}_1 = \operatorname{span}\{(e_k; 0), (0; e_k) : k = 1, \ldots, N\},$$

and $\mathcal{H}_2$ is defined as the closure of the set

$$\operatorname{span}\{(e_k; 0), (0; e_k) : k \ge N + 1\}.$$

As in [16], we will use the following inner products in $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\begin{aligned} \langle U, V\rangle_1 &= \epsilon^2(u^0, v^0) - (Au^0, v^0) + (\epsilon u^0 + u^1, \epsilon v^0 + v^1), \\ \langle U, V\rangle_2 &= (Au^0, v^0) - (\epsilon^2 - 2\mu_{N+1})(u^0, v^0) + (\epsilon u^0 + u^1, \epsilon v^0 + v^1). \end{aligned} \tag{1.3}$$

Here $U = (u^0; u^1)$ and $V = (v^0; v^1)$ are elements from the corresponding subspace $\mathcal{H}_i$. Using (1.3) we define a new inner product and norm in $\mathcal{H}$ by

$$\langle U, V\rangle = \langle U_1, V_1\rangle_1 + \langle U_2, V_2\rangle_2, \qquad |U| = \langle U, U\rangle^{1/2},$$

where $U = U_1 + U_2$ and $V = V_1 + V_2$ are decompositions of the elements $U$ and $V$ into the orthogonal terms $V_i, U_i \in \mathcal{H}_i$, $i = 1, 2$.

LEMMA 1.1 ([7, Lemma 7.1]). *The estimates*

$$\begin{aligned} |U|_1 &\geq \frac{1}{\mu_N^\alpha}\sqrt{\epsilon^2 - \mu_N}\|u^0\|_\alpha, \quad && U = (u^0; u^1) \in \mathcal{H}_1,\\ |U|_2 &\geq \frac{1}{\mu_{N+1}^\alpha}\delta_{N,\epsilon}\|u^0\|_\alpha, && U = (u^0; u^1) \in \mathcal{H}_2, \end{aligned}$$

*hold for* $0 \leq \alpha \leq 1/2$. *Here*

$$\delta_{N,\epsilon} = \sqrt{\mu_{N+1}}\min\Bigg(1, \sqrt{\frac{\epsilon^2 - \mu_{N+1}}{\mu_{N+1}}}\Bigg). \tag{1.4}$$

In particular, this lemma implies the estimate

$$\|u^0\|_\alpha \leq \mu_{N+1}^\alpha \delta_{N,\epsilon}^{-1}|U| \tag{1.5}$$

for any $U = (u^0; u^1) \in \mathcal{H}$, where $0 \leq \alpha \leq 1/2$ and $\delta_{N,\epsilon}$ has the form (1.4).

LEMMA 1.2. *Let* $\mathcal{B}(t, U_t) = (0; B(t, u_t^0))$, *where* $U = (u^0; u^1) \in \mathcal{H}$ *and* $B(t, u_t^0)$ *satisfies Hypothesis B. Then*

$$|\mathcal{B}(t, U_t)| \leq \varphi(t) + \psi(t)\sup_{\theta\in[-r,0]}|U(t+\theta)|, \tag{1.6}$$

$$|\mathcal{B}(t, U_t) - \mathcal{B}(t, V_t)| \leq \psi(t)\sup_{\theta\in[-r,0]}|U(t+\theta) - V(t+\theta)|, \tag{1.7}$$

*where*

$$\psi(t) = \varphi(t)\mu_{N+1}^\alpha\delta_{N,\epsilon}^{-1} = \varphi(t)\mu_{N+1}^{\alpha-1/2}\max\Bigg\{1, \sqrt{\frac{\mu_{N+1}}{\epsilon^2 - \mu_{N+1}}}\Bigg\}.$$

*Proof.* By Hypothesis B, using (1.5) we have

$$\begin{aligned} |\mathcal{B}(t, U_t)| &= \|B(t, u_t^0)\| \leq \varphi(t)(1 + |u_t^0|_{C_\alpha})\\ &= \varphi(t)\Big(1 + \sup_{\theta\in[-r,0]}\|u^0(t+\theta)\|_\alpha\Big)\\ &\leq \varphi(t)\Big(1 + \mu_{N+1}^\alpha\delta_{N,\epsilon}^{-1}\sup_{\theta\in[-r,0]}|U(t+\theta)|\Big)\\ &= \varphi(t) + \psi(t)\sup_{\theta\in[-r,0]}|U(t+\theta)|,\\ |\mathcal{B}(t, U_t) - \mathcal{B}(t, V_t)| &= \|B(t, u_t^0) - B(t, v_t^0)\| \leq \varphi(t)|u_t^0 - v_t^0|_{C_\alpha}\\ &= \varphi(t)\sup_{\theta\in[-r,0]}\|u^0(t+\theta) - v^0(t+\theta)\|_\alpha \end{aligned}$$

$$\leq \varphi(t)\mu_{N+1}^{\alpha}\delta_{N,\epsilon}^{-1}\sup_{\theta\in[-r,0]}|U(t+\theta)-V(t+\theta)|$$
$$=\psi(t)\sup_{\theta\in[-r,0]}|U(t+\theta)-V(t+\theta)|. \blacksquare$$

DEFINITION 1.1. A function $U(t)\equiv(u(t);\dot{u}(t))\in C([\tau-r,T];\mathcal{H})$ is said to be a *mild solution* of problem (1.2) on the interval $[\tau,T]$ if $u(\tau+\theta)=u^{\tau,0}(\theta)$ for $\theta\in[-r,0]$, $\dot{u}(\tau)=u^{\tau,1}$ and $U$ satisfies the integral equation

$$U(t)=e^{-(t-\tau)\mathcal{A}}U(\tau)+\int_{\tau}^{t}e^{-(t-s)\mathcal{A}}\mathcal{B}(s,U_s)\,ds \tag{1.8}$$

for all $t\in[\tau,T]$.

The proof of the existence and uniqueness of a mild solution is standard. First, an auxiliary nonretarded linear problem is considered, namely (1.1), with a given $h(t)\in L^{\infty}(\mathbb{R};H)$ instead of $B$. Using Galerkin approximate solutions and the compactness method we obtain the existence and uniqueness of solution for the auxiliary problem. This allows one to define a linear semigroup $e^{-t\mathcal{A}}$ in $\mathcal{H}$ which is a contraction for $\epsilon>0$ (see e.g. [7, 20]). Then, using the standard fixed point method, one easily proves the existence and uniqueness of a mild solution of (1.2).

Now, we can define an evolution semigroup $S(t,\tau)$ in the space $C_{\mathcal{H}}\equiv C([-r,0];\mathcal{H})$ by

$$S(t,\tau)U_\tau\equiv[S(t,\tau)U_\tau](\theta)=\begin{cases}U(t+\theta) & \text{if } t+\theta>\tau,\\ U(\tau+\theta) & \text{if } t+\theta\leq\tau,\end{cases}$$

for any $\theta\in[-r,0]$, $U_\tau\in C_{\mathcal{H}}$, and any $\tau\leq t$, where $U(t)$ is the mild solution of problem (1.2) with the initial datum $U_\tau$ at time $\tau$.

We now fix an integer $N$ and consider the subspaces

$$\mathcal{H}_1^{\pm}=\operatorname{span}\{f_k^{\pm}:k\leq N\},$$

which are orthogonal for the hermitian product $\langle\cdot,\cdot\rangle$, so $\mathcal{H}_1=\mathcal{H}_1^{+}\oplus\mathcal{H}_1^{-}$. We denote by $P_{\mathcal{H}_i}$ the orthoprojectors onto the subspace $\mathcal{H}_i$ in $\mathcal{H}$, $i=1,2$.

LEMMA 1.3 ([7, pp. 195–196]). *We have*

$$|e^{-\mathcal{A}t}P_{\mathcal{H}_2}|=e^{-\lambda_{N+1}^{-}t},\quad |e^{\mathcal{A}t}P_{\mathcal{H}_1^{-}}|=e^{\lambda_N^{-}t},\quad |e^{-\mathcal{A}t}P_{\mathcal{H}_1^{+}}|=e^{-\lambda_N^{+}t},\quad t\geq 0.$$

*Here $|\cdot|$ is the operator norm induced by the corresponding vector norm.*

We set $P\equiv P_{\mathcal{H}_1^{-}}$ and $Q=I-P=P_{\mathcal{H}_1^{+}}+P_{\mathcal{H}_2}$. Lemma 1.3 implies the dichotomy conditions

$$|e^{\mathcal{A}t}P|\leq e^{\lambda_N^{-}|t|}\quad\text{for } t\in\mathbb{R}\quad\text{and}\quad|e^{-\mathcal{A}t}Q|\leq e^{-\lambda_{N+1}^{-}t}\quad\text{for } t>0. \tag{1.9}$$

We can define the Green function as follows:

$$G(t,s) := \begin{cases} e^{-(t-s)\mathcal{A}}Q & \text{for } t > s, \\ -e^{-(t-s)\mathcal{A}}P & \text{for } t \le s. \end{cases}$$

Then $G(t,s)$ maps $\mathcal{H}$ into $\mathcal{H}$, and for $\sigma = (\lambda_N^- + \lambda_{N+1}^-)/2$ we have

$$e^{\sigma(t-s)}|G(t,s)| \le e^{-\mu|t-s|} \quad \text{for all } t \ne s, \tag{1.10}$$

where $\mu = (\lambda_{N+1}^- - \lambda_N^-)/2$.

We also define the $N$-dimensional projector $\hat{P}$ in $C_{\mathcal{H}}$ by

$$\hat{P}U = (\hat{P}U)(\theta) = \sum_{k=1}^{N} e^{-\lambda_k^- \theta}\langle U(0), f_k^-\rangle f_k^- \equiv e^{-\mathcal{A}\theta}PU(0),$$

where $-r \le \theta \le 0$ and $U = U(\theta)$ is an element of $C_{\mathcal{H}}$.

DEFINITION 1.2. The *inertial manifold* of problem (1.2) is a collection $\mathcal{M} = \{\mathcal{M}_t\}_{t\in\mathbb{R}}$ of surfaces in $C_{\mathcal{H}}$ of the form

$$\mathcal{M}_t = \{\hat{p}(\theta) + \Phi_t(\hat{p}(0))(\theta) : \hat{p}(\theta) \in \hat{P}C_{\mathcal{H}}\} \subset C_{\mathcal{H}} \quad \text{for } t \in \mathbb{R}, \tag{1.11}$$

where $\Phi_t(\cdot)$ is a mapping from $P\mathcal{H}$ into $\hat{Q}C_{\mathcal{H}}$ with the following properties:

(i) For every $t \in \mathbb{R}$, $\mathcal{M}_t$ is a finite-dimensional Lipschitz manifold, i.e.,

$$|\Phi_t(p_1) - \Phi_t(p_2)|_{C_{\mathcal{H}}} \le \ell_\Phi |p_1 - p_2|$$

for all $p_1, p_2 \in P\mathcal{H}$ with $\ell_\Phi$ independent of $p_1$, $p_2$ and $t$.

(ii) $\mathcal{M}$ is *invariant* with respect to $S(t,\tau)$, i.e., $S(t,\tau)\mathcal{M}_\tau = \mathcal{M}_t$ for all $t \ge \tau$.

(iii) $\mathcal{M}$ is *exponentially attracting*, i.e., there exists a positive constant $\sigma$ such that for every $\tau \in \mathbb{R}$ and $U_\tau \in C_{\mathcal{H}}$ there exists a $V_\tau \in \mathcal{M}_\tau$ with

$$|S(t,\tau)U_\tau - S(t,\tau)V_\tau|_{C_{\mathcal{H}}} \le K e^{-\sigma(t-\tau)}$$

for $t \ge \tau$ and $K = K(\tau, U_\tau) > 0$.

From now on, we frequently use the following notations:

$$\sigma := \frac{\lambda_{N+1}^- + \lambda_N^-}{2}, \qquad \mu := \frac{\lambda_{N+1}^- - \lambda_N^-}{2},$$

$$\ell_1 := \frac{N_1 e^{\sigma r}}{1 - e^{-\mu}}\|\Lambda_1\psi\|_\infty, \quad \ell_2 := \frac{N_2 e^{\sigma r}}{1 - e^{-\mu}}\|\Lambda_1\psi\|_\infty, \quad \ell := \ell_1 + \ell_2,$$

with

$$\psi(t) = \varphi(t)\mu_{N+1}^{\alpha-1/2}\max\left\{1, \sqrt{\frac{\mu_{N+1}}{\epsilon^2 - \mu_{N+1}}}\right\},$$

and $\Lambda_1\psi, N_1, N_2$ are given in Definition 2.2 below.

The main result of this paper is the following

THEOREM 1.1. *Let Hypotheses A and B hold. If for some integer $N$ we have $\epsilon^2 > \mu_{N+1}$, $\ell < 1$, and*

$$\delta = \frac{\ell_2(1+\ell)}{1-\ell} + \ell < 1, \tag{1.12}$$

*then the process $S(t,\tau)$ associated to problem* (1.2) *possesses an inertial manifold $\mathcal{M} = \{\mathcal{M}_t\}_{t\in\mathbb{R}}$.*

REMARK 1.1. It is easy to check that when $0 \le \alpha \le 1/2$, condition (1.12) is fulfilled if the following two conditions hold:

(i) the spectral gap $\lambda^-_{N+1} - \lambda^-_N$ is sufficiently large,
(ii) the norm

$$\|\varLambda_1\psi\|_\infty = \mu_{N+1}^{\alpha-1/2} \max\left\{1, \sqrt{\frac{\mu_{N+1}}{\epsilon^2 - \mu_{N+1}}}\right\} \cdot \sup_{t\in\mathbb{R}} \int_{t-1}^{t} \varphi(s)\,ds$$

is sufficiently small.

The plan of the paper is as follows. In Section 2, for the convenience of the reader, we recall some background material on admissible function spaces. In Section 3, using a slightly modified version of the Lyapunov–Perron method, we give the construction of the inertial manifold and establish some of its properties. In the last section, we give an example to illustrate the result obtained.

**2. Admissible function spaces.** Denote by $\mathcal{B}$ the Borel algebra and by $\lambda$ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,\mathrm{loc}}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo $\lambda$-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\,dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$ (see [15, Chapter 2, §20]). We can now define Banach function spaces as follows.

DEFINITION 2.1. A normed vector space $E$ of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo $\lambda$-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(i) $E$ is a Banach lattice with respect to the norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and $\psi$ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \le |\varphi(\cdot)|$ $\lambda$-a.e, then $\psi \in E$ and $\|\psi\|_E \le \|\varphi\|_E$;
(ii) the characteristic functions $\chi_A$ belong to $E$ for all $A \in \mathcal{B}$ having finite measure, and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$;
(iii) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$, i.e., for each seminorm $p_n$ of $L_{1,\mathrm{loc}}(\mathbb{R})$, there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \le \beta_{p_n}\|f\|_E$ for all $f \in E$.

We remark that condition (iii) in the above definition means that for each compact interval $J \subset \mathbb{R}$, there exists a number $\beta_J \geq 0$ such that $\int_J |f(t)|\,dt \leq \beta_J \|f\|_E$ for all $f \in E$.

We now introduce the notion of admissibility.

DEFINITION 2.2. The Banach function space $E$ is called *admissible* if

(i) there is a constant $M \geq 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have
$$\int_a^b |\varphi(t)|\,dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \quad \text{for all } \varphi \in E;$$

(ii) for any $\varphi \in E$, the function $\Lambda_1 \varphi$ defined by $\Lambda_1\varphi(t) := \int_{t-1}^t \varphi(s)\,ds$ belongs to $E$;

(iii) $E$ is $T_\tau^+$-invariant and $T_\tau^-$-invariant, where $T_\tau^+$ and $T_\tau^-$ are defined, for $\tau \in \mathbb{R}$, by
$$T_\tau^+\varphi(t) := \varphi(t-\tau) \quad \text{for } t \in \mathbb{R},$$
$$T_\tau^-\varphi(t) := \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}.$$
Moreover, there are constants $N_1, N_2$ such that $\|T_\tau^+\| \leq N_1$ and $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{R}^+$.

EXAMPLE 1. Besides the space $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, and the space
$$M(\mathbb{R}) := \Big\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_{t-1}^t |f(s)|\,ds < \infty \Big\}$$
endowed with the norm $\|f\|_M := \sup_{t\in\mathbb{R}} \int_{t-1}^t |f(s)|\,ds$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty$ (see [4, Theorem 3, p. 284]), and, more generally, rearrangement invariant function spaces over $(\mathbb{R}, \mathcal{B}, \lambda)$ (see [14, 2.a]) are admissible.

REMARK 2.1. If $E$ is an admissible Banach function space, then $E \hookrightarrow M(\mathbb{R})$. Indeed, put $\beta := \sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$ (see Definition 2.1). Then, from Definition 2.2 we derive
$$\int_{t-1}^t |\varphi(s)|\,ds \leq \frac{M}{\beta}\|\varphi\|_E \quad \text{for all } t \in \mathbb{R} \text{ and } \varphi \in E.$$
Therefore, if $\varphi \in E$, then $\varphi \in M(\mathbb{R})$ and $\|\varphi\|_M \leq (M/\beta)\|\varphi\|_E$, so $E \hookrightarrow M(\mathbb{R})$.

We now collect some properties of admissible Banach function spaces (see [15, 23.V.1]).

Proposition 2.1. *Let $E$ be an admissible Banach function space. Then the following assertions hold:*

(i) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R})$ be such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where $\Lambda_1$ is as in Definition 2.2. For $\sigma > 0$ define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)\,ds,$$

$$\Lambda''_\sigma(t) := \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)\,ds.$$

*Then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to $E$. In particular, if $\sup_{t\in\mathbb{R}}\int_{t-1}^{t}\varphi(s)\,ds < \infty$ (this will be satisfied if $\varphi \in E$, see Remark 2.1), then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover,*

$$\|\Lambda'_\sigma\varphi\|_\infty \leq \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma\varphi\|_\infty \leq \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty,$$

*where the constants $N_1, N_2$ are defined in Definition 2.2.*

(ii) *$E$ contains all exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constant $\alpha > 0$.*

(iii) *$E$ contains no exponentially growing function $f(t) := e^{b|t|}$ for $t \in \mathbb{R}$ and any fixed constant $b > 0$.*

## 3. Proof of the main result

**3.1. Integral equation for determination of an inertial manifold.** We rely on a version of the Lyapunov–Perron method presented in [6] for the nonretarded case. For $\tau \in \mathbb{R}$ we introduce the space

$$\mathcal{C}^-_{\sigma,\tau} = \Big\{V \in C((-\infty,\tau];\mathcal{H}) : |V|^-_\sigma = \sup_{t\in(-\infty,\tau]} e^{\sigma(t-\tau)}|V(t)| < \infty\Big\},$$

which is a Banach space endowed with the norm $|\cdot|^-_\sigma$. For $V \in \mathcal{C}^-_{\sigma,\tau}$ and $p \in P\mathcal{H}$, we consider the integral equation

$$V(p)(t) = \mathcal{T}(V,p)(t), \tag{3.1}$$

where $\mathcal{T}(V,p)(t)$ is the map defined by

$$\mathcal{T}(V,p)(t) = e^{-(t-\tau)\mathcal{A}}p + \int_{-\infty}^{\tau} G(t,s)\mathcal{B}(s,V_s)\,ds$$

for all $t \leq \tau$, and $V_s$ is an element from $C_\mathcal{H}$ defined by $V_s(\theta) = V(s+\theta)$ for all $\theta \in [-r,0]$.

Equation (3.1) is called the *Lyapunov–Perron equation*; it will be used to determine an inertial manifold for (1.2). Our construction of the inertial manifold is based on the fact that, for suitable $\sigma$, a function $V \in \mathcal{C}^-_{\sigma,\tau}$ is a solution of (3.1) if and only if $V$ is a fixed point of $\mathcal{T}$. The idea then is to prove that for suitable $\sigma$, the map $\mathcal{T}$ is well-defined from $\mathcal{C}^-_{\sigma,\tau} \times P\mathcal{H}$ into $\mathcal{C}^-_{\sigma,\tau}$, and is a strict contraction in $\mathcal{C}^-_{\sigma,\tau}$, uniformly in $P\mathcal{H}$. Hence, for each $p \in P\mathcal{H}$, there exists a unique $V \in \mathcal{C}^-_{\sigma,\tau}$ such that $\mathcal{T}(V,p) = V(p)$; in other words, there will be a map $V : P\mathcal{H} \to \mathcal{C}^-_{\sigma,\tau}$ such that $\mathcal{T}(V,p) = V(p)$. We can then define a map $\Phi_\tau : P\mathcal{H} \to \hat{Q}C_{\mathcal{H}}$ which gives the inertial manifold by

$$\Phi_\tau(p)(\theta) = \int_{-\infty}^{\tau} G(\tau+\theta,s)\mathcal{B}(s,V_s)\,ds \equiv V(p)(\tau+\theta) - e^{-\theta\mathcal{A}}p$$

for all $\theta \in [-r,0]$.

### 3.2. Construction of an invariant manifold

LEMMA 3.1. *We have $\mathcal{T} : \mathcal{C}^-_{\sigma,\tau} \times P\mathcal{H} \to \mathcal{C}^-_{\sigma,\tau}$ provided $\sigma = (\lambda^-_N + \lambda^-_{N+1})/2$.*

*Proof.* Take $V \in \mathcal{C}^-_{\sigma,\tau}$ and $p \in P\mathcal{H}$. From Lemma 1.2, we have

$$\begin{aligned}|\mathcal{B}(t,V_t)| &\le \varphi(t) + \psi(t) \sup_{\theta\in[-r,0]} |V(t+\theta)| \\ &\le \varphi(t) + \psi(t)e^{\sigma r}e^{-\sigma(t-\tau)} \sup_{\theta\in[-r,0]} e^{\sigma(t+\theta-\tau)}|V(t+\theta)| \\ &\le \varphi(t) + \psi(t)e^{\sigma r}e^{-\sigma(t-\tau)}|V|^-_\sigma \\ &\le \psi^*(t)e^{\sigma r}e^{-\sigma(t-\tau)}(1+|V|^-_\sigma),\end{aligned}$$

where $\psi^*(t) = \max\{\varphi(t),\psi(t)\}$, for all $t \le \tau$. From (3.1), we have

$$\begin{aligned}e^{\sigma(t-\tau)}|\mathcal{T}(V,p)(t)| &\le e^{\sigma(t-\tau)}|G(t,\tau)|\cdot|p| + e^{\sigma(t-\tau)}\int_{-\infty}^{\tau} |G(t,s)|\cdot|\mathcal{B}(s,V_s)|\,ds \\ &\le e^{-\mu|t-\tau|}|p| + \int_{-\infty}^{\tau} e^{\sigma(t-s)}|G(t,s)|\psi^*(s)\,ds \cdot e^{\sigma r}(1+|V|^-_\sigma) \\ &\le |p| + \frac{N_1+N_2}{1-e^{-\mu}}\|\Lambda_1\psi^*\|_\infty e^{\sigma r}(1+|V|^-_\sigma).\end{aligned}$$

This implies that $|\mathcal{T}(V,p)|^-_\sigma < \infty$. Here, we have used the estimates

$$\begin{aligned}(3.2)\quad \int_{-\infty}^{\tau} e^{\sigma(t-s)}|G(t,s)|\psi^*(s)\,ds &\le \int_{-\infty}^{\tau} e^{-\mu|t-s|}\psi^*(s)\,ds \\ &\le \int_{-\infty}^{t} e^{-\mu(t-s)}\psi^*(s)\,ds + \int_{t}^{\tau} e^{-\mu(s-t)}\psi^*(s)\,ds \\ &\le \frac{N_1}{1-e^{-\mu}}\|\Lambda_1\psi^*\|_\infty + \frac{N_2}{1-e^{-\mu}}\|\Lambda_1\psi^*\|_\infty.\end{aligned}$$

The continuity of $t \mapsto \mathcal{T}(V,p)(t)$ from $(-\infty,\tau]$ into $\mathcal{H}$ can be proved in the same way. Indeed, assume that $t_1, t_2 \in (-\infty,\tau]$ and $t_1 < t_2$. It is evident that

$$\begin{aligned}(3.3)\qquad \mathcal{T}(V,p)(t_2) &= e^{-t_2\mathcal{A}}p - \int_{t_2}^{\tau} e^{-(t_2-s)\mathcal{A}}P\mathcal{B}(s,V_s)\,ds \\ &\quad + \int_{-\infty}^{t_2} e^{-(t_2-s)\mathcal{A}}Q\mathcal{B}(s,V_s)\,ds \\ &= e^{-(t_2-t_1)\mathcal{A}}e^{-t_1\mathcal{A}}p - e^{-(t_2-t_1)\mathcal{A}}\Big(\int_{t_2}^{t_1} + \int_{t_1}^{\tau}\Big)e^{-(t_1-s)\mathcal{A}}P\mathcal{B}(s,V_s)\,ds \\ &\quad + e^{-(t_2-t_1)\mathcal{A}}\Big(\int_{-\infty}^{t_1} + \int_{t_1}^{t_2}\Big)e^{-(t_1-s)\mathcal{A}}Q\mathcal{B}(s,V_s)\,ds \\ &= e^{-(t_2-t_1)\mathcal{A}}\mathcal{T}(V,p)(t_1) + \int_{t_1}^{t_2} e^{-(t_2-s)\mathcal{A}}\mathcal{B}(s,V_s)\,ds.\end{aligned}$$

We see that if $t_2 \to t_1$, then

$$|\mathcal{T}(V,p)(t_1) - e^{-(t_2-t_1)\mathcal{A}}\mathcal{T}(V,p)(t_1)| \to 0.$$

Therefore, it is sufficient to estimate the second term on the right-hand side of (3.3). Equation (1.9) implies that

$$\begin{aligned}\Big|\int_{t_1}^{t_2} e^{-(t_2-s)\mathcal{A}}\mathcal{B}(s,V_s)\,ds\Big| &\le \int_{t_1}^{t_2} e^{-\lambda_1^-(t_2-s)}|\mathcal{B}(s,V_s)|\,ds \le \int_{t_1}^{t_2} ds \max_{s\in[t_1,t_2]}|\mathcal{B}(s,V_s)| \\ &\le (t_2-t_1)\max_{s\in[t_1,t_2]}|\mathcal{B}(s,V_s)|,\end{aligned}$$

which converges to 0 as $t_2 \to t_1$. Thus, $\mathcal{T}(V,p) \in \mathcal{C}^-_{\sigma,\tau}$, which shows that $\mathcal{T}(V,p)$ is well-defined as a map from $\mathcal{C}^-_{\sigma,\tau} \times P\mathcal{H}$ into $\mathcal{C}^-_{\sigma,\tau}$. ∎

LEMMA 3.2. *Assume that the conditions in Theorem* 1.1 *hold. Then for any fixed $\tau \in \mathbb{R}$ and any $p \in P\mathcal{H}$, there exists a unique function $V(p) \in \mathcal{C}^-_{\sigma,\tau}$ satisfying the integral equation* (3.1) *for all $t \in (-\infty,\tau]$ with $PV(p)(\tau) = p$. Moreover,*

$$(3.4)\qquad |V(p)|^-_\sigma < \infty, \qquad |V(p)-V(q)|^-_\sigma \le (1-\ell)^{-1}|p-q|,$$

*where $\ell < 1$.*

*Proof.* Take $U, V \in \mathcal{C}^-_{\sigma,\tau}$ and $p,q \in P\mathcal{H}$. By Lemma 1.2, for all $t \le \tau$,

$$|\mathcal{B}(t,U_t) - \mathcal{B}(t,V_t)| \le \psi(t)e^{\sigma r}e^{-\sigma(t-\tau)}|U-V|^-_\sigma.$$

From (3.1) we have

$$\begin{aligned}&e^{\sigma(t-\tau)}|\mathcal{T}(U,p)(t) - \mathcal{T}(V,q)(t)| \\ &\qquad \le e^{\sigma(t-\tau)}|G(t,s)|\,|p-q| + e^{\sigma(t-\tau)}\int_{-\infty}^{\tau}|G(t,s)|\,|\mathcal{B}(s,U_s) - B(s,V_s)|\,ds\end{aligned}$$

$$\leq e^{-\mu(\tau-t)}|p-q| + \int_{-\infty}^{\tau} e^{\sigma(t-s)}|G(t,s)|\psi(s)\,ds \cdot e^{\sigma r}|U-V|_\sigma^-$$
$$\leq |p-q| + \ell|U-V|_\sigma^-.$$

Therefore,

$$|\mathcal{T}(U,p) - \mathcal{T}(V,q)|_\sigma^- \leq |p-q| + \ell|U-V|_\sigma^-, \tag{3.5}$$

which implies that $\mathcal{T}$ is a strict contraction in $\mathcal{C}_{\sigma,\tau}^-$, uniformly in $P\mathcal{H}$ (if $p=q$). Therefore, there exists a unique fixed point $V(p) = \mathcal{T}(V,p)$, which is the unique solution of the integral equation (3.1).

Using (3.5) and the fact that $V(p) = \mathcal{T}(V,p)$, $V(q) = \mathcal{T}(V,q)$, we have

$$|V(p) - V(q)|_\sigma^- \leq |p-q| + \ell|V(p)-V(q)|_\sigma^-.$$

Hence,

$$|V(p) - V(q)|_\sigma^- \leq (1-\ell)^{-1}|p-q|. \ \blacksquare$$

Lemma 3.2 enables us to define a collection $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ of manifolds by

$$\mathcal{M}_t = \{\hat{p}(\theta) + \Phi_t(p)(\theta) : \hat{p}(\theta) \in \hat{P}C_{\mathcal{H}}\} \subset C_{\mathcal{H}},$$

where $p = \hat{p}(0)$, and

$$\Phi_t(p)(\theta) = \int_{-\infty}^{t} G(t+\theta,s)\mathcal{B}(s,V_s)\,ds \tag{3.6}$$

for all $\theta \in [-r,0]$. Here $V_s(\theta) = V(s+\theta)$, $\theta \in [-r,0]$ and $V(s) = V(p)(s)$ is the solution to (3.1) for all $s \leq t$. Some properties of the manifolds $\mathcal{M}_t$ and the function $\Phi_t(p)(\theta)$ are given in the following assertion.

Proposition 3.1. *Assume that the conditions in Theorem* 1.1 *are satisfied. Then the collection* $\mathcal{M} \equiv \{\mathcal{M}_t\}_{t\in\mathbb{R}}$ *has the following properties:*

(i) $\mathcal{M}_t$ *is a Lipschitzian surface and*

$$|\Phi_t(p) - \Phi_t(q)|_{C_{\mathcal{H}}} \leq \ell_\Phi|p-q|$$

*for all* $p,q \in P\mathcal{H}$ *and* $t \in \mathbb{R}$, *where* $\ell_\Phi = \ell e^{\sigma r}/(1-\ell)$.

(ii) $\mathcal{M}$ *is invariant with respect to* $S(t,\tau)$, *i.e.,* $S(t,\tau)\mathcal{M}_\tau = \mathcal{M}_t$.

*Proof.* Take $p,q \in P\mathcal{H}$. For $\sigma = (\lambda_{N+1}^- + \lambda_N^-)/2$ and all $\theta \in [-r,0]$, from (3.6) and (3.4), we have

$$|\Phi_t(p)(\theta) - \Phi_t(q)(\theta)| \leq \int_{-\infty}^{t} |G(t+\theta,s)|\cdot|\mathcal{B}(s,V_s(p)) - \mathcal{B}(s,V_s(q))|\,ds \tag{3.7}$$
$$\leq \int_{-\infty}^{t} e^{-\sigma(s-t)}|G(t+\theta,s)|\psi(s)\,ds \cdot e^{\sigma r}|V(p)-V(q)|_\sigma^-$$

$$\leq e^{-\sigma\theta} \int_{-\infty}^{t} e^{\sigma(t+\theta-s)} |G(t+\theta,s)| \psi(s)\, ds \cdot \frac{e^{\sigma r}}{1-\ell} |p-q|$$

$$\leq \frac{e^{-\sigma\theta}}{1-\ell} \cdot \frac{N_1+N_2}{1-e^{-\mu}} \|\Lambda_1 \psi\|_\infty \cdot e^{\sigma r} |p-q| = \frac{\ell e^{-\sigma\theta}}{1-\ell} |p-q|.$$

Hence,

$$|\Phi_t(p) - \Phi_t(q)|_{C_{\mathcal{H}}} = \sup_{\theta \in [-r,0]} |\Phi_t(p)(\theta) - \Phi_t(q)(\theta)| \leq \frac{\ell e^{\sigma r}}{1-\ell} |p-q|.$$

We now prove (ii). To do this, let $U$ be a solution of problem (1.2) with initial datum $U_\tau \in \mathcal{M}_\tau$, i.e., $U_\tau(\theta) = \hat{p}(\theta) + \Phi_\tau(\hat{p}(0))(\theta)$, where $\hat{p} \in \hat{P}C_{\mathcal{H}}$. We have to prove that $U_t = S(t,\tau)U_\tau \in \mathcal{M}_t$.

Fix $t \in [\tau, \infty)$ and define a function $W(t)$ on $(-\infty, t]$ by

$$W(s) = \begin{cases} U(s) & \text{for } s \in [\tau, t], \\ V(s) & \text{for } s \in (-\infty, \tau], \end{cases}$$

where $V(s) = V(p)(s)$ is the unique solution of (3.1) with $p = \hat{p}(0)$. From (3.5), since $\hat{p}(0) = e^{-0\mathcal{A}} p = p$, we have

$$\Phi_\tau(p(0))(\theta) = \int_{-\infty}^{\tau+\theta} e^{-(\tau+\theta-s)\mathcal{A}} Q\mathcal{B}(s, V_s)\, ds - \int_{\tau+\theta}^{\tau} e^{-(\tau+\theta-s)\mathcal{A}} P\mathcal{B}(s, V_s)\, ds.$$

Hence, for all $t \in [\tau - r, \tau]$, we have $\theta = t - \tau \in [-r, 0]$ and

$$\begin{aligned} U(t) &= U(\tau+\theta) = U_\tau(\theta) = \hat{p}(\theta) + \Phi_\tau(\hat{p}(0))(\theta) \\ &= e^{-\theta\mathcal{A}} \hat{p}(0) + \int_{-\infty}^{\tau+\theta} e^{-(\tau+\theta-s)\mathcal{A}} Q\mathcal{B}(s, V_s)\, ds - \int_{\tau+\theta}^{\tau} e^{-(\tau+\theta-s)A} P\mathcal{B}(s, V_s)\, ds \\ &= e^{-(t-\tau)\mathcal{A}} \hat{p}(0) + \int_{\tau}^{t} e^{-(t-s)\mathcal{B}} P\mathcal{B}(s, V_s)\, ds + \int_{-\infty}^{t} e^{-(t-s)\mathcal{B}} Q\mathcal{B}(s, V_s)\, ds. \end{aligned}$$

For all $t \geq \tau$, (1.8) implies that

$$\begin{aligned} U(t) &= e^{-(t-\tau)\mathcal{A}} U_\tau(0) + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} \mathcal{B}(s, U_s)\, ds \\ &= e^{-(t-\tau)\mathcal{A}} \hat{p}(0) + e^{-(t-\tau)\mathcal{A}} \Phi_\tau(\hat{p}(0))(0) \\ &\quad + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} P\mathcal{B}(s, U_s)\, ds + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} Q\mathcal{B}(s, U_s)\, ds \\ &= e^{-(t-\tau)\mathcal{A}} \hat{p}(0) + e^{-(t-\tau)\mathcal{A}} \int_{-\infty}^{\tau} e^{-(\tau-s)\mathcal{B}} Q\mathcal{B}(s, V_s)\, ds \\ &\quad + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} P\mathcal{B}(s, U_s)\, ds + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} Q\mathcal{B}(s, U_s)\, ds \end{aligned} \tag{3.8}$$

$$= e^{-(t-\tau)\mathcal{A}}\hat{p}(0) + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} Q\mathcal{B}(s, W_s)\, ds.$$

So, we obtain

$$U(t) = e^{-(t-\tau)\mathcal{A}}\hat{p}(0) + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds + \int_{-\infty}^{t} e^{-(t-s)\mathcal{A}} Q\mathcal{B}(s, W_s)\, ds \tag{3.9}$$

for all $t \geq \tau - r$.

Now, for all $t \geq \tau$, it follows from (3.9) that

$$(\hat{P}U_t)(\theta) = e^{-\theta\mathcal{A}} PU(t) = e^{-(t+\theta-\tau)\mathcal{A}}\hat{p}(0) + e^{-\theta\mathcal{A}} \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds \tag{3.10}$$

for all $\theta \in [-r, 0]$, and

$$\begin{aligned}
\hat{Q}U_t(\theta) &\equiv U_t(\theta) - (\hat{P}U_t)(\theta) \\
&= e^{-(t+\theta-\tau)\mathcal{A}}\hat{p}(0) + \int_{\tau}^{t+\theta} e^{-(t+\theta-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds \\
&\quad + \int_{-\infty}^{t+\theta} e^{-(t+\theta-s)\mathcal{A}} Q\mathcal{B}(s, W_s)\, ds \\
&\quad - e^{-(t+\theta-\tau)\mathcal{A}}\hat{p}(0) - \int_{\tau}^{t} e^{-(t+\theta-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds \\
&= \int_{t}^{t+\theta} e^{-(t+\theta-s)\mathcal{A}} P\mathcal{B}(s, W_s)\, ds + \int_{-\infty}^{t+\theta} e^{-(t+\theta-s)\mathcal{A}} Q\mathcal{B}(s, W_s)\, ds \\
&= \Phi_t(PU(t))(\theta).
\end{aligned}$$

Therefore, in order to prove $U_t \in \mathcal{M}_t$, it is sufficient to check that the function $W(\cdot)$ is a solution of (3.1) with $p = PU(t)$ and $s \leq t$. Let us do this. From (3.1) and (3.9) we have

$$W(s) = e^{-(s-\tau)\mathcal{A}}\hat{p}(0) + \int_{\tau}^{s} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, W_r)\, dr + \int_{-\infty}^{s} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, W_r)\, dr \tag{3.11}$$

for all $s \le t$. Equation (3.10) implies

$$\hat{p}(0) = e^{(t-\tau)\mathcal{A}} PU(t) - e^{(t-\tau)\mathcal{A}} \int_{\tau}^{t} e^{-(t-r)\mathcal{A}} P\mathcal{B}(r, W_r)\, dr,$$

hence

$$e^{-(s-\tau)\mathcal{A}} \hat{p}(0) = e^{-(s-t)\mathcal{A}} PU(t) - \int_{\tau}^{t} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, W_r)\, dr.$$

From (3.11) we get

$$W(s) = e^{-(s-t)\mathcal{A}} PU(t) + \int_{t}^{s} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, W_r)\, dr + \int_{-\infty}^{s} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, W_r)\, dr.$$

This implies that $W(\cdot)$ is a solution of (3.1) with $p = PU(t)$ and $s \le t$. So we have proved that $U(t,\tau)\mathcal{M}_\tau \subset \mathcal{M}_t$.

Conversely, if $U_t \in \mathcal{M}_t$, there exists $\hat{p} \in \hat{P}C_{\mathcal{H}}$ such that

$$U_t(\theta) = \hat{p}(\theta) + \Phi_t(\hat{p}(0))(\theta) = e^{-\theta\mathcal{A}} p(t) + \Phi_t(p(t))(\theta) = V(p(t))(t+\theta)$$

for all $\theta \in [-r, 0]$. For a given $p(t)$ there exists a function $V(s) = V(p(t))(s)$ which is a solution of (3.1) with $p = p(t)$ and $s \le t$. So, we obtain

$$\begin{aligned}
&V(s) - e^{-(s-\tau)\mathcal{A}} V(\tau) \\
&\quad = e^{-(s-t)\mathcal{A}} p(t) + \int_{t}^{s} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, V_r)\, dr + \int_{-\infty}^{s} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, V_r)\, dr \\
&\qquad - e^{-(s-\tau)\mathcal{A}} \left[ e^{-(\tau-t)\mathcal{A}} p(t) + \int_{t}^{\tau} e^{-(\tau-r)\mathcal{A}} P\mathcal{B}(r, V_r)\, dr \right] \\
&\qquad - e^{-(s-\tau)\mathcal{A}} \int_{-\infty}^{\tau} e^{-(\tau-r)\mathcal{A}} Q\mathcal{B}(r, V_r)\, dr \\
&\quad = \int_{t}^{s} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, V_r)\, dr + \int_{-\infty}^{s} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, V_r)\, dr \\
&\qquad + \int_{\tau}^{t} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, V_r)\, dr - \int_{-\infty}^{\tau} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, V_r)\, dr \\
&\quad = \int_{\tau}^{s} e^{-(s-r)\mathcal{A}} P\mathcal{B}(r, V_r)\, dr + \int_{\tau}^{s} e^{-(s-r)\mathcal{A}} Q\mathcal{B}(r, V_r)\, dr \\
&\quad = \int_{\tau}^{s} e^{-(s-r)\mathcal{A}} \mathcal{B}(r, V_r)\, dr \qquad \text{for all } \tau \le s \le t.
\end{aligned}$$

Hence,

$$V(s) = e^{-(s-\tau)\mathcal{A}} V_\tau(0) + \int_\tau^s e^{-(s-r)\mathcal{A}} \mathcal{B}(r, V_r)\, dr.$$

This implies that $V(\cdot)$ is a solution of problem (1.2) with the initial datum $V_\tau = V(p(t))(\tau)$. Therefore, $V_s = S(s,\tau)\mathcal{M}_\tau$, i.e., $\mathcal{M}_t \subset S(t,\tau)\mathcal{M}_\tau$. Thus, $S(t,\tau)\mathcal{M}_\tau = \mathcal{M}_t$. ∎

**3.3. Asymptotic completeness.** In this subsection, we show that the collection $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ determined as in the previous subsection has the property of exponential uniform attraction and hence is an inertial manifold for problem (1.2). More precisely, Proposition 3.2 below states that $\{\mathcal{M}_t\}$ is an *exponentially asymptotically complete inertial manifold*, i.e., for any solution $U_t = S(t,\tau)U_\tau$, there exists a solution $U_t^* = S(t,\tau)U_\tau^*$ lying in the manifold (i.e. $U_t^* \in \mathcal{M}_t$ for all $t \ge \tau$) such that

$$|U_t - U_t^*|_{C_\mathcal{H}} \le Ce^{-\sigma(t-\tau)}, \qquad \sigma > 0,\ t \ge \tau.$$

In this case, the solution $U^*(t)$ is called an *induced trajectory* for $U(t)$ on the manifold $\{\mathcal{M}_t\}$. In particular, the existence of such induced trajectories means that the solution to the original infinite-dimensional problem (1.2) can be naturally associated to the solution of the system (3.1)–(3.3) of ordinary differential equations.

Proposition 3.2. *Assume that the conditions in Theorem* 1.1 *hold. Then the collection* $\{\mathcal{M}_t\}_{t\in\mathbb{R}}$ *of manifolds given by formula* (1.11) *is the inertial manifold for problem* (1.2). *Moreover, for any solution* $U_t = S(t,\tau)U_\tau$, *there exists an induced trajectory* $U_t^* = S(t,\tau)U_\tau^*$ *such that* $U_t^* \in \mathcal{M}_t$ *for* $t \ge \tau$ *and*

$$|U_t^* - U_t|_{C_\mathcal{H}} \le (1-\delta)^{-1} |\Phi_\tau(PU_\tau(0)) - (I - \hat{P})U_\tau|_{C_\mathcal{H}} e^{-\sigma(t-\tau)}.$$

*Proof.* We will find the induced trajectory in the form $U^*(t) = U(t) + W(t)$ with

$$|W|_\sigma^+ = \sup_{t \ge \tau - r} e^{\sigma(t-\tau)} |W(t)| < \infty. \tag{3.12}$$

For simplicity of presentation we put

$$\mathcal{F}(t, W_t) = \mathcal{B}(t, U_t + W_t) - B(t, U_t),$$

and set

$$\mathcal{C}_{\sigma,\tau}^+ = \Big\{ V \in C([\tau - r, \infty); \mathcal{H}) : \sup_{t \ge \tau - r} e^{\sigma(t-\tau)} |V(t)| < \infty \Big\}$$

endowed with the norm $|\cdot|_\sigma^+$ defined as in (3.12).

Since $U$ and $U^*$ are solutions of (1.2), for $t \geq \tau$ one has

$$W(t) = e^{-(t-\tau)\mathcal{A}} QW(\tau) + \int_{\tau}^{t} e^{-(t-s)\mathcal{A}} Q\mathcal{F}(s, W_s)\, ds - \int_{t}^{\infty} e^{-(t-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds. \tag{3.13}$$

This gives for all $\theta \in [-r, 0]$,

$$\begin{aligned}(\hat{P}W_\tau)(\theta) &= e^{-\theta\mathcal{A}} PW_\tau(0) = e^{-\theta\mathcal{A}} PW(\tau) \\ &= -e^{-\theta\mathcal{A}} \int_{\tau}^{\infty} e^{-(\tau-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds \\ &= -\int_{\tau}^{\infty} e^{-(\tau+\theta-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds.\end{aligned} \tag{3.14}$$

Since by definition of an induced trajectory, $U^*_\tau = U_\tau + W_\tau \in \mathcal{M}_\tau$, we have

$$\begin{aligned}(I - \hat{P})(U_\tau + W_\tau)(\theta) &= \Phi_\tau(\hat{P}(U_\tau + W_\tau)(0))(\theta) \\ &= \Phi_\tau\Big(PU_\tau(0) - \int_{\tau}^{\infty} e^{-(\tau-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds\Big)(\theta).\end{aligned}$$

Hence

$$\begin{aligned}&(I - \hat{P})W_\tau(\theta) \\ &\quad = -(I - \hat{P})U_\tau(\theta) + \Phi_\tau\Big(PU_\tau(0) - \int_{\tau}^{\infty} e^{-(\tau-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds\Big)(\theta).\end{aligned} \tag{3.15}$$

So, (3.14) and (3.15) give the formula for $W_\tau(\theta)$, $\theta \in [-r, 0]$:

$$\begin{aligned}W_\tau(\theta) &= (I - \hat{P})W_\tau(\theta) + (\hat{P}W_\tau)(\theta) \\ &= -(I - \hat{P})U_\tau(\theta) + \Phi_\tau\Big(PU_\tau(0) - \int_{\tau}^{\infty} e^{-(\tau-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds\Big)(\theta) \\ &\quad - \int_{\tau}^{\infty} e^{-(\tau+\theta-s)\mathcal{A}} P\mathcal{F}(s, W_s)\, ds.\end{aligned} \tag{3.16}$$

Now, we define the map $\mathcal{T} : \mathcal{C}^+_{\sigma,\tau} \to \mathcal{C}^+_{\sigma,\tau}$ given by the right-hand side of (3.13) and (3.16). Our goal is to prove that $\mathcal{T}$ is a contraction in the space $\mathcal{C}^+_{\sigma,\tau}$.

Indeed, for $W(\cdot) \in \mathcal{C}^+_{\sigma,\tau}$, we have

$$|\mathcal{F}(t, W_t)| \leq \psi(t) e^{\sigma r} e^{-\sigma(t-\tau)} |W|^+_\sigma,$$

therefore we can estimate for all $t \geq \tau - r$. For $t \in [\tau - r, \tau]$, noting that

$\theta \in [-r, 0]$ and $t = \tau + \theta$, we have

$$
\begin{aligned}
e^{\sigma(t-\tau)}|\mathcal{T}(W)(t)| &= e^{\sigma\theta}|W_\tau(\theta)| \\
&\le e^{\sigma\theta}|\Phi_\tau(PU_\tau(0))(\theta) - (I-\hat{P})U_\tau(\theta)| \\
&\quad + e^{\sigma\theta}\Big|\Phi_\tau\Big(PU_\tau(0) - \int_\tau^\infty e^{-(\tau-s)\mathcal{A}}P\mathcal{F}(s,W_s)\,ds\Big)(\theta) - \Phi_\tau(PU_\tau(0))(\theta)\Big| \\
&\quad + e^{\sigma\theta}\Big|\int_\tau^\infty e^{-(\tau+\theta-s)\mathcal{A}}P\mathcal{F}(s,W_s)\,ds\Big| \\
&\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} \\
&\quad + e^{\sigma\theta}\frac{\ell e^{-\sigma\theta}}{1-\ell}\Big|\int_\tau^\infty e^{-(\tau-s)\mathcal{A}}P\mathcal{F}(s,W_s)\,ds\Big| \\
&\quad + e^{\sigma\theta}\Big|\int_\tau^\infty e^{-(\tau+\theta-s)\mathcal{A}}P\mathcal{F}(s,W_s)\,ds\Big| \\
&\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} \\
&\quad + e^{\sigma\theta}\frac{\ell e^{-\sigma\theta}}{1-\ell}\Big|\int_\tau^\infty e^{\sigma r}e^{-\lambda_N^-(\tau-s)}e^{\sigma(\tau-s)}\psi(s)\,ds\Big||W|_\sigma^+ \\
&\quad + e^{\sigma\theta}\Big|\int_\tau^\infty e^{\sigma r}e^{-\lambda_N^-(\tau+\theta-s)}e^{\sigma(\tau-s)}\psi(s)\,ds\Big||W|_\sigma^+ \\
&\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} \\
&\quad + \Big(\frac{\ell}{1-\ell} + e^{\mu\theta}\Big)\Big|\int_\tau^\infty e^{\sigma r}e^{\mu(\tau-s)}\psi(s)\,ds\Big||W|_\sigma^+ \\
&\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \Big(\frac{\ell}{1-\ell}+1\Big)\ell_2|W|_\sigma^+ \\
&\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \frac{\ell_2}{1-\ell}|W|_\sigma^+.
\end{aligned}
$$

From (3.16), we deduce that

$$
|QW_\tau(\theta)| \le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \frac{\ell e^{-\sigma\theta}}{1-\ell}\ell_2|W|_\sigma^+.
$$

So, (3.13) and the last inequality show that for all $t \ge \tau$,

$$
\begin{aligned}
&e^{\sigma(t-\tau)}|(\mathcal{T}W)(t)| \\
&\quad\le e^{-(\lambda_{N+1}^- - \sigma)(t-\tau)}|QW(\tau)| + e^{\sigma r}|W|_\sigma^+\int_\tau^\infty e^{\sigma(t-s)}|G(t,s)|\cdot\psi(s)\,ds \\
&\quad\le e^{-\mu(t-\tau)}\Big(|\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \frac{\ell}{1-\ell}\ell_2|W|_\sigma^+\Big) + \ell|W|_\sigma^+ \\
&\quad\le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \Big(\frac{\ell_2}{1-\ell}+1\Big)\ell|W|_\sigma^+.
\end{aligned}
$$

It follows that $(\mathcal{T}W)(\cdot) \in \mathcal{C}^+_{\sigma,\tau}$, and

$$(3.17) \qquad |\mathcal{T}W|^+_\sigma \le |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} + \delta |W|^+_\sigma,$$

where $\delta = \ell_2(1+\ell)/(1-\ell) + \ell$. Therefore, the transformation $\mathcal{T}$ acts from $\mathcal{C}^+_{\sigma,\tau}$ into itself.

We take $W^1, W^2 \in \mathcal{C}^+_{\sigma,\tau}$, and use the fact that

$$|\mathcal{F}(t, W^1_t) - \mathcal{F}(t, W^2_t)| \le \psi(t) e^{\sigma r} e^{-\sigma(t-\tau)} |W^1 - W^2|^+_\sigma$$

to obtain for $\theta \in [-r, 0]$, $t = \tau + \theta$,

$$(3.18) \qquad e^{\sigma(t-\tau)} |(\mathcal{T}W^1)(t) - (\mathcal{T}W^2)(t)| \le \frac{\ell_2}{1-\ell} |W^1 - W^2|^+_\sigma,$$

and for $t \ge \tau$,

$$(3.19) \quad e^{\sigma(t-\tau)} |(\mathcal{T}W^1)(t) - (\mathcal{T}W^2)(t)| \le |Q(W^1(\tau) - W^2(\tau))| + \ell |W^1 - W^2|^+_\sigma.$$

We deduce from (3.16) and (3.7) with $\theta = 0$ that

$$|Q(W^1(\tau) - W^2(\tau)| \le \frac{\ell}{1-\ell} \ell_2 |W^1 - W^2|^+_\sigma.$$

Therefore,

$$|\mathcal{T}W^1 - \mathcal{T}W^2|^+_\sigma \le \delta |W^1 - W^2|^+_\sigma.$$

Hence, if

$$\delta = \frac{\ell_2(1+\ell)}{1-\ell} + \ell < 1,$$

then $\mathcal{T} : \mathcal{C}^+_{\sigma,\tau} \to \mathcal{C}^+_{\sigma,\tau}$ is a contraction. Thus, there exists a unique $W(\cdot) \in \mathcal{C}^+_{\sigma,\tau}$ such that $\mathcal{T}W = W$. By the definition of $\mathcal{T}$, $W(\cdot)$ is the unique solution in $\mathcal{C}^+_{\sigma,\tau}$ of equations (3.13) and (3.16) for $t \ge \tau - r$. Also using (3.17) we obtain

$$|W|^+_\sigma \le (1-\delta)^{-1} |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}}.$$

Furthermore, by determination of $W$, we obtain the existence of the solution $U^* = U + W$ to (1.2) such that $U^*_t \in \mathcal{M}_t$ for $t \ge \tau$, and $U^*$ satisfies

$$\begin{aligned} |U^*_t(\theta) - U_t(\theta)| &= |W(t+\theta)| \le e^{-\sigma(t-\tau)} |W|^+_\sigma \\ &\le (1-\delta)^{-1} |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} e^{-\sigma(t-\tau)} \end{aligned}$$

for all $t \ge \tau$ and $\theta \in [-r, 0]$. Hence,

$$|U^*_t - U_t|_{C_{\mathcal{H}}} \le (1-\delta)^{-1} |\Phi_\tau(PU_\tau(0)) - (I-\hat{P})U_\tau|_{C_{\mathcal{H}}} e^{-\sigma(t-\tau)}. \ \blacksquare$$

**4. An example.** Consider the following Cauchy–Dirichlet problem for the semilinear damped wave equation with delay:

(4.1)
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) + 2\epsilon \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + f\left(x,t,u(x,t-r), \dfrac{\partial u}{\partial x}(x,t-r)\right), \\ \hfill 0 < x < \pi,\ t > \tau, \\ u(0,t) = u(\pi,t) = 0, \quad t > \tau, \\ u(x,\tau+\theta) = \phi_1(x,\theta),\ \dfrac{\partial u}{\partial t}(x,\tau) = \phi_2(x), \quad 0 < x < \pi,\ -r \le \theta \le 0, \end{cases}$$

where $r$ is a positive real number, $\phi_1$ and $\phi_2$ are given initial functions, and $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(x,t,u_1,\xi_1) - f(x,t,u_2,\xi_2)| \le \varphi_1(t)(L_1|u_1-u_2| + L_2|\xi_1-\xi_2|) \tag{4.2}$$

for all $x \in [0,\pi]$, $t \ge \tau$, and

$$\int_0^\pi [f(x,t,0,0)]^2\,dx \le [L_3\varphi_2(t)]^2,$$

where $L_j$, $j = 1,2,3$, are nonnegative numbers, and $\varphi_1, \varphi_2$ belong to an admissible function space $E$.

We choose the Hilbert space $H = L^2(0,\pi)$ and consider the operator $A : H \to H$ defined by

$$Au = -\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad D(A) = H_0^1(0,\pi) \cap H^2(0,\pi).$$

Then $A$ is a positive operator with discrete point spectrum

$$1^2, 2^2, \ldots, n^2, \ldots.$$

Note that $D(A^{1/2}) = H_0^1(0,\pi)$.

Let $B : \mathbb{R} \times C_{1/2} \to H$ be defined by

$$B(t,u_t)(x) = f\left(x,t,u_t(-r)(x), \frac{\partial u_t(-r)}{\partial x}(x)\right), \qquad u_t \in C_{1/2},\ x \in [0,\pi].$$

Then $B$ is well-defined because $f$ is continuous. Since for all $u_t, v_t \in C_{1/2}$,

$$\left\| \frac{\partial u_t(-r)}{\partial x} - \frac{\partial v_t(-r)}{\partial x} \right\|^2 = \|A^{1/2}(u_t(-r) - v_t(-r))\|^2 \le |u_t - v_t|^2_{C_{1/2}},$$
$$\|u_t(-r) - v_t(-r)\|^2 \le \|A^{1/2}(u_t(-r) - v_t(-r))\|^2 \le |u_t - v_t|^2_{C_{1/2}},$$

the mapping $B$ satisfies

$$\|B(t,u_t) - B(t,v_t)\| \le \varphi_1(t)(L_1+L_2)|u_t - v_t|_{C_{1/2}},$$
$$\|B(t,u_t)\| \le \varphi_1(t)(L_1+L_2)|u_t|_{C_{1/2}} + L_3\varphi_2(t),$$

for all $t \in \mathbb{R}$ and $u_t, v_t \in C_{1/2}$. Therefore, $B$ is $\varphi$-Lipschitz with

$$\varphi(t) = \max\{(L_1 + L_2)\varphi_1(t), L_3\varphi_2(t)\}.$$

Hence, for each initial data, problem (4.1) has a unique mild solution $u(t)$. Thus, we can define a process associated to problem (4.1), which has an inertial manifold if the condition (1.12) is fulfilled.

In particular, if $N_1 = N_2$ (this assumption holds, for example, in the case where $\varphi(t) \equiv L$ does not depend on time $t$), we have $\ell_1 = \ell_2 = \ell/2$. Then (1.12) becomes

$$\frac{\ell(1+\ell)}{2(1-\ell)} + \ell < 1,$$

that is,

$$\ell < \frac{5-\sqrt{17}}{2} \text{ (since } \ell < 1).$$

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