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Non-trivial solutions for a two-point boundary value problem

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Abstract. We prove the existence of at least one non-trivial solution for Dirichlet quasilinear elliptic problems. The approach is based on variational methods.

1. Introduction. We investigate the existence of at least one non-trivial weak solution to the quasilinear elliptic problem

$$\begin{cases} -u'' = \big[\lambda f(x,u) + g(u)\big] h(x,u') & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for all $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

Motivated by the fact that such problems are used to describe a large class of physical phenomena, many authors looked for existence of solutions for second order ordinary differential non-linear equations.

In this paper, we generalize the results obtained in [4] with $g \equiv 0$ and $h \equiv 1$ (see Remark 3.9). Our analysis is mainly based on a recent critical point theorem of Bonanno [1], contained in Theorem 2.1 below. This theorem has been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [2, 3, 4, 6, 7, 8, 11]).

As an example, we state here the following special case of our results.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$16\int_0^5 f(x)\,dx < 25\int_0^1 f(x)\,dx.$$

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Then, for each

$$\lambda \in \left]\frac{10}{\int_0^1 f(x)\,dx}, \frac{15}{\int_0^5 f(x)\,dx}\right[,$$

the problem

$$\begin{cases} -u'' + u = \lambda f(u) & \text{in } (0,1),\\ u(0) = u(1) = 0 \end{cases}$$

admits at least one positive classical solution $\bar{u}$ *such that* $|\bar{u}(x)| < 5$ *for all* $x \in [0,1]$.

2. Preliminaries. Our main tool is the Ricceri variational principle [13, Theorem 2.5] as given in [1, Theorem 5.1] which is recalled below (see also [1, Proposition 2.1]).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$.

THEOREM 2.1 ([1, Theorem 5.1]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]1/\rho(r_1, r_2), 1/\beta(r_1, r_2)[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for all $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0, +\infty)$ be a bounded and continuous function with $m := \inf_{(x,t) \in [0,1] \times \mathbb{R}} h(x,t) > 0$.

We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-*Carathéodory function* if

(a) $x \mapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;
(b) $\xi \mapsto f(x, \xi)$ is continuous for almost every $x \in [0,1]$;

(c) for every $\rho > 0$ there is a function $l_\rho \in L^1([0,1])$ such that
$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$
for almost every $x \in [0,1]$.

Corresponding to f, g and h we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $H : [0,1] \times \mathbb{R} \to [0,+\infty)$ as follows:
$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \qquad G(t) := -\int_0^t g(\xi)\,d\xi,$$
$$H(x,t) := \int_0^t \left(\int_0^\tau \frac{1}{h(x,\delta)}\,d\delta \right) d\tau,$$
for all $x \in [0,1]$ and $t \in \mathbb{R}$.

Throughout, we let $M := \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t)$ and suppose that the Lipschitz constant $L > 0$ of g satisfies the condition $LM < 4$.

Let X be the Sobolev space $W_0^{1,2}([0,1])$ equipped with the norm
$$\|u\| := \left(\int_0^1 |u'(x)|^2\,dx \right)^{1/2}.$$
We say that a function $u \in X$ is a *weak solution* of problem (1.1) if
$$\int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\,d\tau \right) v'(x)\,dx - \lambda \int_0^1 f(x,u(x))v(x)\,dx - \int_0^1 g(u(x))v(x)\,dx = 0$$
for all $v \in X$. By standard regularity results, if f is continuous in $[0,1] \times \mathbb{R}$, then weak solutions of problem (1.1) belong to $C^2([0,1])$, thus they are classical solutions.

For other basic notations and definitions, we refer the reader to [5, 10, 14, 16].

3. Main results. Put
$$A := \frac{4 - LM}{8M}, \qquad B := \frac{4 + Lm}{8m},$$
and suppose that $B \le 4A$.

Given a non-negative constant c_1 and two positive constants c_2 and d with $c_1^2 < 8d^2 < c_2^2$, put
$$a(c_2,d) := \frac{\int_0^1 \sup_{|t|\le c_2} F(x,t)\,dx - \int_{1/4}^{3/4} F(x,d)\,dx}{Bc_2^2 - 8Bd^2},$$
$$b(c_1,d) := \frac{\int_{1/4}^{3/4} F(x,d)\,dx - \int_0^1 \sup_{|t|\le c_1} F(x,t)\,dx}{8Bd^2 - Ac_1^2}.$$

We formulate our main result as follows.

Theorem 3.1. *Assume that there exist a non-negative constant* c_1 *and two positive constants* c_2 *and* d *with* $c_1^2 < 8d^2 < c_2^2$ *such that*

(A_1) $F(x,t) \geq 0$ *for all* $(x,t) \in ([0,1/4] \cup [3/4,1]) \times [0,d]$;
(A_2) $a(c_2,d) < b(c_1,d)$.

Then, for each $\lambda \in]1/b(c_1,d), 1/a(c_2,d)[$, *problem* (1.1) *admits at least one non-trivial weak solution* $\bar{u} \in X$ *such that*

$$\frac{A}{B}c_1^2 < \|\bar{u}\|^2 < \frac{B}{A}c_2^2.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, for each $u \in X$, we define $\Phi, \Psi : X \to \mathbb{R}$ by

$$\Phi(u) := \int_0^1 H(x, u'(x))\,dx + \int_0^1 G(u(x))\,dx,$$

$$\Psi(u) := \int_0^1 F(x, u(x))\,dx,$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u), \qquad u \in X.$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$ given by

$$\Phi'(u)(v) = \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)}\,d\tau \right) v'(x)\,dx - \int_0^1 g(u(x))v(x)\,dx,$$

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)\,dx$$

for every $v \in X$. Also, the functionals Φ and Ψ satisfy all regularity assumptions imposed in Theorem 2.1 (for more details, see the proof of [9, Theorem 2.1]). Note that the weak solutions of (1.1) are exactly the critical points of I_λ.

Since g is Lipschitz continuous and satisfies $g(0) = 0$, while h is bounded away from zero, the inequality

$$\max_{x\in[0,1]} |u(x)| \leq \frac{1}{2}\|u\| \qquad \text{for all } u \in X \tag{3.1}$$

(see, e.g., [15]) yields for any $u \in X$ the estimate

$$A\|u\|^2 \leq \Phi(u) \leq B\|u\|^2. \tag{3.2}$$

Now, put

$$r_1 := Ac_1^2, \qquad r_2 := Bc_2^2, \qquad w(x) := \begin{cases} 4dx & \text{if } x \in [0, 1/4[, \\ d & \text{if } x \in [1/4, 3/4], \\ 4d(1-x) & \text{if } x \in]3/4, 1]. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular,

$$\|w\|^2 = 8d^2.$$

So, from (3.2), we have

$$8Ad^2 \le \Phi(w) \le 8Bd^2.$$

From the condition $c_1^2 < 8d^2 < c_2^2$, we obtain $r_1 < \Phi(w) < r_2$. Since $B \le 4A$, for all $u \in X$ such that $\Phi(u) < r_2$, taking (3.1) into account, one has $|u(x)| < c_2$ for all $x \in [0, 1]$, which implies

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) = \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \int_0^1 F(x, u(x))\, dx \le \int_0^1 \sup_{|t| \le c_2} F(x, t)\, dx.$$

Arguing as before, we obtain

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le \int_0^1 \sup_{|t| \le c_1} F(x, t)\, dx.$$

Since $0 \le w(x) \le d$ for each $x \in [0, 1]$, assumption (A_1) ensures that

$$\int_0^{1/4} F(x, w(x))\, dx + \int_{3/4}^1 F(x, w(x))\, dx \ge 0,$$

and so

$$\Psi(w) \ge \int_{1/4}^{3/4} F(x, d)\, dx.$$

Therefore,

$$\begin{aligned} \beta(r_1, r_2) &\le \frac{\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \\ &\le \frac{\int_0^1 \sup_{|t| \le c_2} F(x, t)\, dx - \int_{1/4}^{3/4} F(x, d)\, dx}{Bc_2^2 - 8Bd^2} = a(c_2, d). \end{aligned}$$

On the other hand,

$$\begin{aligned} \rho(r_1, r_2) &\ge \frac{\Psi(w) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(w) - r_1} \\ &\ge \frac{\int_{1/4}^{3/4} F(x, d)\, dx - \int_0^1 \sup_{|t| \le c_1} F(x, t)\, dx}{8Bd^2 - Ac_1^2} = b(c_1, d). \end{aligned}$$

Hence, from assumption (A_2), one has $\beta(r_1, r_2) < \rho(r_1, r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in]1/b(c_1, d), 1/a(c_2, d)[$, the functional I_λ admits at least one critical point $\bar{u}$ such that

$$r_1 < \Phi(\bar{u}) < r_2,$$

that is,

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2,$$

and the conclusion is achieved. ■

Now, we point out an immediate consequence of Theorem 3.1.

THEOREM 3.2. *Assume that there exist two positive constants c and d with $2\sqrt{2}d < c$ such that assumption* (A_1) *in Theorem* 3.1 *holds. Furthermore, suppose that*

$$(\mathrm{A}_3) \quad \frac{\int_0^1 \sup_{|t| \le c} F(x,t)\, dx}{c^2} < \frac{1}{8} \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{d^2}.$$

Then, for each

$$\lambda \in \left] \frac{8Bd^2}{\int_{1/4}^{3/4} F(x,d)\, dx}, \frac{Bc^2}{\int_0^1 \sup_{|t| \le c} F(x,t)\, dx} \right[,$$

problem (1.1) *admits at least one non-trivial weak solution $\bar{u} \in X$ such that $|\bar{u}(x)| < c$ for all $x \in [0,1]$.*

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to assumption (A_3), one has

$$\begin{aligned} a(c,d) &= \frac{\int_0^1 \sup_{|t| \le c} F(x,t)\, dx - \int_{1/4}^{3/4} F(x,d)\, dx}{Bc^2 - 8Bd^2} \\ &< \frac{(1 - 8d^2/c^2) \int_0^1 \sup_{|t| \le c} F(x,t)\, dx}{B(c^2 - 8d^2)} = \frac{1}{Bc^2} \int_0^1 \sup_{|t| \le c} F(x,t)\, dx. \end{aligned}$$

On the other hand,

$$b(0,d) = \frac{\int_{1/4}^{3/4} F(x,d)\, dx}{8Bd^2}.$$

Hence, taking assumption (A_3) and (3.1) into account, Theorem 3.1 yields the conclusion. ■

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) := \int_0^t f(\xi)\, d\xi$ for all $t \in \mathbb{R}$. We have the following result as a direct consequence of Theorem 3.1 in the autonomous case.

COROLLARY 3.3. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with $c_1^2 < 8d^2 < c_2^2$ such that*

(A_4) *$f(t) \geq 0$ for all $t \in [-c_2, \max\{c_2, d\}]$;*

(A_5) $\dfrac{F(c_2) - \frac{1}{2}F(d)}{Bc_2^2 - 8Bd^2} < \dfrac{F(c_1) - \frac{1}{2}F(d)}{Ac_1^2 - 8Bd^2}$.

Then, for each

$$\lambda \in \left] \frac{Ac_1^2 - 8Bd^2}{F(c_1) - \frac{1}{2}F(d)}, \frac{Bc_2^2 - 8Bd^2}{F(c_2) - \frac{1}{2}F(d)} \right[,$$

the problem

$$\begin{cases} -u'' = \big[\lambda f(u) + g(u)\big] h(x, u') & \text{in } (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

admits at least one non-trivial classical solution $\bar{u}$ such that

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Proof. From the condition $c_1^2 < 8d^2 < c_2^2$, we obtain $c_1 < c_2$. Therefore, assumption (A_4) means $f(t) \geq 0$ for each $t \in [-c_1, c_1]$ and $f(t) \geq 0$ for each $t \in [-c_2, c_2]$, which implies

$$\max_{t \in [-c_1, c_1]} F(t) = F(c_1), \qquad \max_{t \in [-c_2, c_2]} F(t) = F(c_2).$$

So, the conclusion follows from Theorem 3.1. ∎

Now, we point out a special situation of our main result when the nonlinear term has separated variables. To be precise, let $\alpha \in L^1([0,1])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0,1]$, $\alpha \not\equiv 0$, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Consider the following Dirichlet boundary value problem,

$$\begin{cases} -u'' = \big[\lambda \alpha(x) \gamma(u) + g(u)\big] h(x, u') & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.3}$$

Put $\Gamma(t) := \int_0^t \gamma(\xi)\, d\xi$ for all $t \in \mathbb{R}$, and set $\|\alpha\|_1 := \int_0^1 \alpha(x)\, dx$.

Corollary 3.4. *Assume that there exist a non-negative constant c_1 and two positive constants c_2 and d with $c_1^2 < 8d^2 < c_2^2$ such that*

(A_6) $\dfrac{\Gamma(c_2)\|\alpha\|_1 - \Gamma(d) \int_{1/4}^{3/4} \alpha(x)\, dx}{Bc_2^2 - 8Bd^2} < \dfrac{\Gamma(d) \int_{1/4}^{3/4} \alpha(x)\, dx - \Gamma(c_1)\|\alpha\|_1}{8Bd^2 - Ac_1^2}$.

Then, for each

$$\lambda \in \left] \frac{8Bd^2 - Ac_1^2}{\Gamma(d) \int_{1/4}^{3/4} \alpha(x)\, dx - \Gamma(c_1)\|\alpha\|_1}, \frac{Bc_2^2 - 8Bd^2}{\Gamma(c_2)\|\alpha\|_1 - \Gamma(d) \int_{1/4}^{3/4} \alpha(x)\, dx} \right[,$$

problem (3.3) *admits at least one positive weak solution $\bar{u} \in X$ such that*

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Proof. Put $f(x,\xi) := \alpha(x)\gamma(\xi)$ for all $(x,\xi) \in [0,1]\times\mathbb{R}$. Clearly, $F(x,t) = \alpha(x)\Gamma(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. Therefore, taking into account that Γ is a non-decreasing function, Theorem 3.1 and the strong maximum principle (see, e.g., [12, Theorem 11.1]) yield the conclusion. ∎

An immediate consequence of Corollary 3.4 is the following.

COROLLARY 3.5. *Assume that there exist positive constants c and d with $2\sqrt{2}d < c$ such that*

$$(\mathrm{A}_7)\quad \frac{\Gamma(c)\|\alpha\|_1}{c^2} < \frac{1}{8}\,\frac{\Gamma(d)\int_{1/4}^{3/4}\alpha(x)\,dx}{d^2}.$$

Then, for each

$$\lambda \in \left]\frac{8Bd^2}{\Gamma(d)\int_{1/4}^{3/4}\alpha(x)\,dx}, \frac{Bc^2}{\Gamma(c)\|\alpha\|_1}\right[,$$

problem (3.3) *admits at least one positive weak solution $\bar{u} \in X$ such that $|\bar{u}(x)| < c$ for all $x \in [0,1]$.*

Proof. This follows directly from Theorem 3.2. ∎

REMARK 3.6. Theorem 1.1 in the introduction is an immediate consequence of Corollary 3.5, on choosing $g(u) = -u$, $h \equiv 1$, $c = 5$ and $d = 1$.

Here, we point out another relevant consequence of Corollary 3.5.

THEOREM 3.7. *Assume that*

$$(\mathrm{A}_8)\quad \lim_{t\to 0^+}\frac{\gamma(t)}{t} = +\infty.$$

Then, for each

$$\lambda \in \left]0, \frac{B}{\|\alpha\|_1}\sup_{c>0}\frac{c^2}{\Gamma(c)}\right[,$$

problem (3.3) *admits at least one positive weak solution.*

Proof. For fixed λ as in the conclusion, there is $c > 0$ such that $\lambda < Bc^2/\|\alpha\|_1\Gamma(c)$. Moreover, assumption (A_8) implies that $\lim_{t\to 0^+}\Gamma(t)/t^2 = +\infty$. Therefore, there is $d < \frac{\sqrt{2}}{4}c$ such that

$$\frac{\Gamma(d)\int_{1/4}^{3/4}\alpha(x)\,dx}{8Bd^2} > \frac{1}{\lambda}.$$

Hence, Corollary 3.5 implies the conclusion. ∎

REMARK 3.8. Taking (A_8) into account, fix $\upsilon > 0$ such that $\gamma(t) > 0$ for all $t \in \,]0,\upsilon[$. Then, put

$$\lambda_\upsilon := \frac{B}{\|\alpha\|_1}\sup_{c\in]0,\upsilon[}\frac{c^2}{\Gamma(c)}.$$

Now, fix $\lambda \in]0, \lambda_\upsilon[$ and argue as in the proof of Theorem 3.7 to find $c \in]0, \upsilon[$ and $d < \frac{\sqrt{2}}{4}c$ such that

$$\frac{8Bd^2}{\Gamma(d)\int_{1/4}^{3/4}\alpha(x)\,dx} < \lambda < \frac{Bc^2}{\|\alpha\|_1\Gamma(c)}.$$

Hence, Corollary 3.5 ensures that, for each $\lambda \in]0, \lambda_\upsilon[$, problem (3.3) admits at least one positive weak solution $\bar{u} \in X$ such that $|\bar{u}(x)| < \upsilon$ for all $x \in [0,1]$.

Remark 3.9. We would like to stress that our results generalize those of [4]. In fact, we can consider problem (1.1) as a generalization of problem (D_λ) of [4]. Specifically, Theorem 3.1 improves Theorem 3.1 of [4]. Corollaries 3.4 and 3.5 provide extensions of Corollary 3.2 and Theorem 3.3 in [4], respectively. Theorem 3.7 and Remark 3.8 also extend Remark 3.9 and Theorem 3.8 in [4], respectively.

Finally, we present the following example to illustrate the result.

Example 3.10. Consider the problem

(3.4)
$$\begin{cases} -u'' = [\lambda e^x(1 + e^{-u^+}u^+(2-u^+)) + u^+](2 + x + \cos u')^{-1} & \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $u^+ := \max\{u, 0\}$. Let $\alpha(x) = e^x$, $\gamma(t) = 1 + e^{-t^+}t^+(2-t^+)$, $g(t) = t^+$ and $h(x,t) = (2 + x + \cos t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$, where $t^+ := \max\{t, 0\}$. It is clear that $\lim_{t\to 0^+}\gamma(t)/t = +\infty$. Pick $\upsilon = 1$. Hence, taking Remark 3.8 into account, by applying Theorem 3.7, since $B = 17/8$, for every $\lambda \in \left]0, \frac{17}{8(e-1)}\frac{e}{e+1}\right[$, problem (3.4) has at least one positive classical solution $\bar{u} \in X$ such that $\|\bar{u}\|_\infty < 1$.

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