## Fully starlike and fully convex harmonic mappings of order $\alpha$

by SUMIT NAGPAL (Delhi) and V. RAVICHANDRAN (Delhi and Penang)

Abstract. The hereditary properties of convexity and starlikeness for conformal mappings do not generalize to univalent harmonic mappings. This failure leads to the notions of fully starlike and fully convex mappings. In this paper, properties of fully starlike mappings of order  $\alpha$  and fully convex mappings of order  $\alpha$  ( $0 \leq \alpha < 1$ ) are studied; in particular, the bounds for the radius of full starlikeness of order  $\alpha$  as well as the radius of full convexity of order  $\alpha$  are determined for certain families of univalent harmonic mappings. Unlike the analytic case, convexity is not preserved under the convolution of univalent harmonic convex mappings. Given two univalent harmonic convex mappings fand g, the radius  $r_0$  such that their harmonic convolution f \* g is a univalent harmonic convex mapping in  $|z| < r_0$  is also investigated.

**1. Introduction.** Let  $\mathcal{H}$  denote the class of all complex-valued harmonic functions f in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  normalized by f(0) = $0 = f_z(0) - 1$ . Let  $\mathcal{S}_H$  be the subclass of  $\mathcal{H}$  consisting of univalent and sense-preserving functions. A function  $f \in S_H$  can be represented in the form  $f = h + \bar{q}$ , where

(1.1) 
$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic and |g'(z)| < |h'(z)| in  $\mathbb{D}$ . The class  $\mathcal{S}_H$  includes the class  $\mathcal{S}$ of normalized univalent analytic functions. In 1984, Clunie and Sheil-Small [2] investigated  $\mathcal{S}_H$  as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on  $\mathcal{S}_H$  and its subclasses  $\mathcal{S}_H^*$ ,  $\mathcal{K}_H$  and  $\mathcal{C}_H$  of harmonic mappings that map  $\mathbb{D}$ onto starlike, convex and close-to-convex domains, respectively. Let  $S_H^0 = \{f \in S_H : b_1 = f_{\bar{z}}(0) = 0\}$ . Define  $S_H^{*0} = S_H^* \cap S_H^0$ ,  $\mathcal{K}_H^0 = \mathcal{K}_H \cap S_H^0$  and  $\mathcal{C}_H^0 = \mathcal{C}_H \cap \mathcal{S}_H^0.$ 

Let  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ . Convexity and starlikeness are hereditary properties for conformal mappings: if f is univalent analytic in  $\mathbb{D}$  and  $f(\mathbb{D})$  is

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a convex (resp. starlike) domain, then, for each 0 < r < 1,  $f(\mathbb{D}_r)$  is also a convex (resp. starlike) domain. The harmonic half-plane mapping

(1.2) 
$$L(z) = M(z) + \overline{N(z)},$$
$$M(z) := \frac{z - \frac{1}{2}z^2}{(1-z)^2}, \quad N(z) := -\frac{-\frac{1}{2}z^2}{(1-z)^2} \quad (z \in \mathbb{D}),$$

constructed by shearing the conformal mapping l(z) = z/(1-z) vertically with dilatation w(z) = -z, shows that convexity is not a hereditary property for harmonic mappings. Although L maps the unit disc onto the half-plane Re w > -1/2, the image of the disks  $\mathbb{D}_r$  under the mapping L is not convex for every r in the interval  $\sqrt{2} - 1 < r < 1$ . Similarly, starlikeness is also not a hereditary property for harmonic mappings. Chuaqui, Duren and Osgood [1] introduced the notions of fully starlike and fully convex functions that do inherit the properties of starlikeness and convexity respectively. This concept is generalized to fully starlike functions of order  $\alpha$  and fully convex functions of order  $\alpha$  for  $0 \leq \alpha < 1$  in Section 2, analogous to the subclasses  $\mathcal{S}^*(\alpha)$  and  $\mathcal{K}(\alpha)$  of  $\mathcal{S}$ , in the analytic case, consisting of respectively starlike functions of order  $\alpha$  and convex functions of order  $\alpha$  (see [16]). Recall that these classes are defined analytically by the equivalences

(1.3) 
$$f \in \mathcal{S}^*(\alpha) \iff \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \ (z \in \mathbb{D}),$$
$$f \in \mathcal{K}(\alpha) \iff \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \ (z \in \mathbb{D}),$$

with  $\mathcal{S}^*(0) = \mathcal{S}^*$  and  $\mathcal{K}(0) = \mathcal{K}$ .

For  $f = h + \bar{g} \in S_H^0$ , Clunie and Sheil-Small [2] conjectured that the Taylor coefficients of the series of h and g satisfy the inequalities

(1.4) 
$$|a_n| \le \frac{1}{6}(2n+1)(n+1), \quad |b_n| \le \frac{1}{6}(2n-1)(n-1) \text{ for all } n \ge 2,$$

and verified it for typically real functions. Later, Sheil-Small [19] proved it for all functions  $f \in S_H^0$  for which  $f(\mathbb{D})$  is starlike with respect to the origin or  $f(\mathbb{D})$  is convex in one direction. Wang, Liang and Zhang [20] verified the conjecture for close-to-convex functions in  $S_H^0$ . However, this coefficient conjecture remains an open problem for the full class  $S_H^0$ . Equality occurs in (1.4) for the harmonic Koebe function

(1.5) 
$$K(z) = H(z) + \overline{G(z)},$$
$$H(z) := \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, \quad G(z) := \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \quad (z \in \mathbb{D}),$$

constructed by shearing the Koebe function  $k(z) = z/(1-z)^2$  horizontally with dilatation w(z) = z. Note that K maps the unit disk  $\mathbb{D}$  onto the slitplane  $\mathbb{C} \setminus (-\infty, -1/6]$ . For convex harmonic mappings  $f \in \mathcal{K}_H^0$ , Clunie and Sheil-Small [2] proved that the Taylor coefficients of h and g satisfy the inequalities

(1.6) 
$$|a_n| \le \frac{n+1}{2}$$
 and  $|b_n| \le \frac{n-1}{2}$  for all  $n \ge 2$ 

Equality occurs in (1.6) for the harmonic half-plane mapping (1.2).

Let  $\mathcal{F}$  be the family of all functions of the form  $f = h + \overline{g}$  where h and g are given by (1.1). In [10], the radii of univalence and starlikeness of the family  $\mathcal{F}$  are determined if the coefficients of the series satisfy the conditions (1.4) and (1.6). These results are generalized to the context of fully starlike and fully convex functions of order  $\alpha$  ( $0 \le \alpha < 1$ ) in Section 3. This, in turn, provides a bound for the radius of full starlikeness (resp. full convexity) of order  $\alpha$  for convex, starlike and close-to-convex mappings in  $\mathcal{S}_H$ . Recently, a similar analysis [15] has been carried out for certain classes of analytic functions with fixed second coefficient.

For analytic functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  and  $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ , their convolution (or Hadamard product) is defined as

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n \quad (z \in \mathbb{D}).$$

In the harmonic case, with

$$f = h + \bar{g} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \quad (z \in \mathbb{D})$$

and

$$F = H + \bar{G} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n} \quad (z \in \mathbb{D}),$$

their harmonic convolution is defined as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n z^n \quad (z \in \mathbb{D}).$$

Harmonic convolutions were investigated in [2, 3, 4, 6, 14, 18]. The convolution of convex analytic functions is again convex: if  $f_1, f_2 \in \mathcal{K}$ , then  $f_1 * f_2 \in \mathcal{K}$ . However, it is easy to see that the Hadamard product of two functions in  $\mathcal{K}_H$  is not necessarily convex, or even univalent. In Section 4, the radius of univalence of the family

$$\mathcal{G} = \left\{ f = h + \bar{g} \in \mathcal{H} : |a_n| \le \left(\frac{n+1}{2}\right)^2 \text{ and } |b_n| \le \left(\frac{n-1}{2}\right)^2 \text{ for } n \ge 1 \right\}$$

is determined, which turns out to be  $r_0 \approx 0.129831$ . This number is also the radius of starlikeness of  $\mathcal{G}$ . The radius of convexity of  $\mathcal{G}$  is  $s_0 \approx 0.0712543$ .

This, in particular, shows that if  $f, g \in \mathcal{K}_H^0$ , then f \* g is univalent and convex in at least  $|z| < s_0 \approx 0.0712543$ .

2. Full starlikeness and convexity of order  $\alpha$ . In this section, some basic properties of fully starlike functions of order  $\alpha$  ( $0 \le \alpha < 1$ ) and fully convex functions of order  $\alpha$  ( $0 \le \alpha < 1$ ) are investigated. The following definitions introduce fully starlike/convex functions of order  $\alpha$ .

DEFINITION 2.1. A harmonic mapping f of the unit disk  $\mathbb{D}$  with f(0) = 0 is said to be *fully starlike of order*  $\alpha$  ( $0 \leq \alpha < 1$ ) if it maps every circle |z| = r < 1 in a one-to-one manner onto a curve that bounds a domain starlike with respect to the origin satisfying

(2.1) 
$$\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) > \alpha, \quad 0 \le \theta < 2\pi, \ 0 < r < 1.$$

If  $\alpha = 0$ , then f is fully starlike.

DEFINITION 2.2. A harmonic mapping f of the unit disk  $\mathbb{D}$  is said to be fully convex of order  $\alpha$  ( $0 \le \alpha < 1$ ) if it maps every circle |z| = r < 1 in a one-to-one manner onto a convex curve satisfying

(2.2) 
$$\frac{\partial}{\partial \theta} \left( \arg \left( \frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) > \alpha, \quad 0 \le \theta < 2\pi, \ 0 < r < 1.$$

If  $\alpha = 0$ , then f is fully convex.

Let  $\mathcal{FS}_{H}^{*}(\alpha)$  denote the subclass of  $\mathcal{S}_{H}^{*}$  consisting of fully starlike functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ), with  $\mathcal{FS}_{H}^{*} := \mathcal{FS}_{H}^{*}(0)$ , and let  $\mathcal{FS}_{H}^{*0}(\alpha) = \mathcal{FS}_{H}^{*}(\alpha) \cap \mathcal{S}_{H}^{*0}$ . Let  $\mathcal{FK}_{H}(\alpha)$  ( $0 \leq \alpha < 1$ ) denote the subclass of  $\mathcal{K}_{H}$  consisting of fully convex functions of order  $\alpha$ , with  $\mathcal{FK}_{H} := \mathcal{FK}_{H}(0)$ , and let  $\mathcal{FK}_{H}^{0}(\alpha) = \mathcal{FK}_{H}(\alpha) \cap \mathcal{K}_{H}^{0}$ . The hereditary property does not generalize for harmonic mappings, and therefore  $\mathcal{FS}_{H}^{*} \subsetneq \mathcal{S}_{H}^{*}$  and  $\mathcal{FK}_{H} \subsetneq \mathcal{K}_{H}$ . These subclasses were discussed earlier in [8, 9, 13].

Since fully convex harmonic mappings are univalent in  $\mathbb{D}$  (see [1]), fully convex harmonic mappings of order  $\alpha$  ( $0 \leq \alpha < 1$ ) are also univalent in  $\mathbb{D}$ . The affine mappings  $f(z) = \alpha z + \gamma + \beta \overline{z}$  with  $|\alpha| > |\beta|$  are fully convex of order  $(|\alpha| - |\beta|)/(|\alpha| + |\beta|)$ . If  $f \in \mathcal{K}_H$ , then f is fully convex in  $|z| < \sqrt{2} - 1$ with the harmonic half-plane mapping L defined by (1.2) being the extremal function (see [18]). Similarly, f is fully convex in  $|z| < 3 - \sqrt{8}$  if  $f \in \mathcal{S}_H^*$  or  $\mathcal{C}_H$ and the harmonic Koebe function K given by (1.5) shows that this bound is best possible (see [17, 19]). However, the exact radius of full convexity of order  $\alpha$  ( $0 < \alpha < 1$ ) for starlike, convex and close-to-convex mappings in  $\mathcal{S}_H$ is still unknown (see Section 3).

It is worth remarking that the condition (2.2) is sufficient but not necessary for a function  $f \in S_H$  to map  $\mathbb{D}$  onto a convex domain (see [12, Theorem 3]). The next theorem provides a sufficient condition for a sensepreserving harmonic mapping to be fully convex.

THEOREM 2.3. A sense-preserving harmonic function  $f = h + \bar{g}$  is fully convex in  $\mathbb{D}$  if the analytic functions  $h + \epsilon g$  are convex in  $\mathbb{D}$  for all  $|\epsilon| = 1$ .

*Proof.* It suffices to show that f is convex in |z| < r for each  $r \leq 1$ . To see this, fix  $r_0 \in (0, 1]$ . Then the analytic functions  $h + \epsilon g$  are convex in  $|z| < r_0$ . By [2, Theorem 5.7], it follows that f is convex in  $|z| < r_0$ .

However, if  $f = h + \bar{g}$  is fully convex then the functions  $h + \epsilon g$  need not be convex for all  $|\epsilon| = 1$ . Indeed, consider the function  $F(z) = L((\sqrt{2}-1)z)$ ,  $z \in \mathbb{D}$ , where L is given by (1.2). Writing  $F = P + \bar{Q}$ , we see that  $P - Q = k((\sqrt{2}-1)z)$  is not convex; here k is the Koebe function.

Jahangiri [8] gave the following sufficient coefficient condition for a function to be in the class  $\mathcal{FK}_H(\alpha)$ :

LEMMA 2.4 ([8]). Let  $f = h + \overline{g}$ , where h and g are given by (1.1), and let  $0 \leq \alpha < 1$ . If

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \le 1,$$

then  $f \in \mathcal{FK}_H(\alpha)$ .

The analytic description of functions in  $\mathcal{FK}_H(\alpha)$   $(0 \le \alpha < 1)$  is given in the following theorem, which extends [1, Theorem 3, p. 139] for  $0 < \alpha < 1$ .

THEOREM 2.5. Let  $f = h + \bar{g} \in \mathcal{H}$  be sense-preserving and let  $0 \leq \alpha < 1$ . Then  $f \in \mathcal{FK}_H(\alpha)$  if and only if

(2.3) 
$$|zh'(z)|^2 \left[ \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) - \alpha \right] - |zg'(z)|^2 \left[ \operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) + \alpha \right]$$
  
>  $\operatorname{Re}[z^2(zh''(z)g'(z) - 2\alpha h'(z)g'(z) - zh'(z)g''(z))]$ 

for all  $z \in \mathbb{D}$ .

*Proof.* Suppose that  $f \in \mathcal{FK}_H(\alpha)$ . A simple calculation shows that (2.2) reduces to the condition (2.3). Conversely, if f satisfies (2.3), then by the proof of Theorem 3 in [1, p. 139], f maps each circle |z| = r < 1 in a one-to-one manner onto a convex curve satisfying (2.2), so that  $f \in \mathcal{FK}_H(\alpha)$ .

Unlike fully convex mappings, a fully starlike mapping need not be univalent (see [1]). The affine mappings  $f(z) = \alpha z + \beta \overline{z}$ , with  $|\alpha| > |\beta|$ , are fully starlike of order  $(|\alpha| - |\beta|)/(|\alpha| + |\beta|)$ . It is also clear that every fully convex mapping of order  $\alpha$  is fully starlike of order  $\alpha$ . However, the converse is not true, as seen by the following example.



Fig. 1. Image of the subdisk  $|z| < 1/n^{1/(n-1)}$  under  $f_n(z) = z + \overline{z}^n/n$  ( $\alpha = 0$ ) for different values of n.

EXAMPLE 2.6. For  $n \ge 2$  and  $\alpha \in [0, 1)$ , consider the function

$$f_n(z) = z + \frac{1-\alpha}{n+\alpha} \overline{z}^n, \quad z \in \mathbb{D}.$$

By Lemma 2.9 below, the functions  $f_n$   $(n \ge 2)$  are fully starlike of order  $\alpha$ . This can also be seen directly: for  $z = re^{i\theta}$ , we have

$$\frac{\partial}{\partial \theta} \arg f_n(re^{i\theta}) = \operatorname{Re} \frac{z - \frac{n(1-\alpha)}{n+\alpha} \overline{z}^n}{z + \frac{1-\alpha}{n+\alpha} \overline{z}^n} \ge \frac{1 - \frac{n(1-\alpha)}{n+\alpha} r^{n-1}}{1 + \frac{1-\alpha}{n+\alpha} r^{n-1}} > \alpha.$$

Further, observe that

$$\frac{\partial}{\partial \theta} \left( \arg \left\{ \frac{\partial}{\partial \theta} f_n(re^{i\theta}) \right\} \right) = \operatorname{Re} \frac{z + \frac{n^2(1-\alpha)}{n+\alpha} \bar{z}^n}{z - \frac{n(1-\alpha)}{n+\alpha} \bar{z}^n} \ge \frac{1 - \frac{n^2(1-\alpha)}{n+\alpha} r^{n-1}}{1 + \frac{n(1-\alpha)}{n+\alpha} r^{n-1}}.$$

Therefore, it follows that  $f_n$  is fully convex of order  $\alpha$  in  $|z| < 1/n^{1/(n-1)}$ . In



Fig. 2. Image of the subdisk  $|z| < 1/n^{1/(n-1)}$  under  $f_n(z) = z + \bar{z}^n/(2n+1)$  ( $\alpha = 1/2$ ) for different values of n.

particular, this shows that  $f_n$  is not fully convex of order  $\alpha$  in  $\mathbb{D}$  (see Figures 1 and 2).

The condition (2.1) is sufficient but not necessary for a function  $f \in S_H$  to map  $\mathbb{D}$  onto a starlike domain (see [12, Theorem 1]). Similar to Theorem 2.3, the next theorem provides a sufficient condition which guarantees a sense-preserving harmonic mapping to be fully starlike. The proof follows by invoking [7, Theorem 3, p. 10].

THEOREM 2.7. A sense-preserving harmonic function  $f = h + \bar{g}$  is fully starlike in  $\mathbb{D}$  if the analytic functions  $h + \epsilon g$  are starlike in  $\mathbb{D}$  for each  $|\epsilon| = 1$ .

Note that the exact radius of full starlikeness of order  $\alpha$  ( $0 \leq \alpha < 1$ ) for the subclasses  $S_H^*$ ,  $\mathcal{K}_H$  and  $\mathcal{C}_H$  in  $\mathcal{S}_H$  is still unknown. The results in this direction are investigated in the next section. However, if  $\alpha = 0$  then Theorem 2.7 immediately gives COROLLARY 2.8. Suppose that  $f = h + \bar{g} \in S_H$ .

- (i) If  $f \in \mathcal{K}_H$  then f is fully starlike in at least  $|z| < 4\sqrt{2} 5$ .
- (ii) If  $f \in C_H$  then f is fully starlike in at least  $|z| < 3 \sqrt{8}$ .
- (iii) If  $f \in \mathcal{S}_H^*$  then f is fully starlike in at least  $|z| < \sqrt{2} 1$ .

*Proof.* (i) Since  $f \in \mathcal{K}_H$ , the analytic functions  $h + \epsilon g$  are close-to-convex for all  $|\epsilon| = 1$  by [2, Theorem 5.7, p. 15]. Since the radius of starlikeness for the close-to-convex analytic mappings is  $4\sqrt{2} - 5$  (see [11]), the functions  $h + \epsilon g$  are starlike in  $|z| < 4\sqrt{2} - 5$ . By Theorem 2.7, f is fully starlike in  $|z| < 4\sqrt{2} - 5$ . This proves (i).

(ii) follows from the fact that  $f \in C_H$  is fully convex (and hence fully starlike) in  $|z| < 3 - \sqrt{8}$ .

(iii) If  $f \in \mathcal{S}_{H}^{*}$  then by [5, Lemma, p. 108], the function  $F = H + \overline{G}$  belongs to  $\mathcal{K}_{H}$ , where zH'(z) = h(z), zG'(z) = -g(z), and H(0) = G(0) = 0, so that f is fully starlike in at least  $|z| < \sqrt{2} - 1$ .

In terms of coefficients, the next lemma gives a sufficient condition for functions  $f \in \mathcal{H}$  to be in  $\mathcal{FS}^*_H(\alpha)$ .

LEMMA 2.9 ([9]). Let  $f = h + \bar{g}$ , where h and g are given by (1.1). Furthermore, let

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| \le 1$$

and  $0 \leq \alpha < 1$ . Then  $f \in \mathcal{FS}^*_H(\alpha)$ .

Corresponding to Theorem 2.5, the analytic characterization of functions in  $\mathcal{FS}_{H}^{*}(\alpha)$  is given in the following theorem, with the case  $\alpha = 0$  reducing to [1, Theorem 3, p. 139].

THEOREM 2.10. Let  $f = h + \bar{g} \in \mathcal{H}$  be sense-preserving and let  $0 \leq \alpha < 1$ . Then  $f \in \mathcal{FS}^*_H(\alpha)$  if and only if  $f(z) \neq 0$  for 0 < |z| < 1 and

(2.4) 
$$|h(z)|^{2} \left(\operatorname{Re} \frac{zh'(z)}{h(z)} - \alpha\right) - |g(z)|^{2} \left(\operatorname{Re} \frac{zg'(z)}{g(z)} + \alpha\right)$$
$$> \operatorname{Re}(zh(z)g'(z) + 2\alpha h(z)g(z) - zh'(z)g(z))$$

for all  $z \in \mathbb{D}$ .

*Proof.* The necessity part follows immediately by the univalence of f and (2.1). For sufficiency, suppose that  $f(z) \neq 0$  for  $z \neq 0$  and (2.4) holds. Then, by the proof of Theorem 3 in [1, p. 139], f is fully starlike in  $\mathbb{D}$  and satisfies (2.1), hence it is univalent by [12, Theorem 1], so  $f \in \mathcal{FS}^*_H(\alpha)$ .

REMARK. For normalized analytic functions f, conditions (2.3) and (2.4) reduce to (1.3). In general, given  $\alpha \in [0, 1)$  and  $h \in \mathcal{K}(\alpha)$ ,

$$f = h + \epsilon \bar{h} \in \mathcal{FK}_H\left(\frac{1-|\epsilon|}{1+|\epsilon|}\alpha\right) \quad \text{for } |\epsilon| < 1.$$

To see this, note that for  $z = re^{i\theta} \in \mathbb{D} \setminus \{0\}$ , we have

$$\begin{split} \frac{\partial}{\partial \theta} \bigg( \arg \bigg\{ \frac{\partial}{\partial \theta} f(re^{i\theta}) \bigg\} \bigg) &= \operatorname{Re} \frac{zh'(z) + z^2h''(z) + \overline{\epsilon(zh'(z) + z^2h''(z))}}{zh'(z) - \overline{\epsilon zh'(z)}} \\ &= \frac{(1 - |\epsilon|^2)|zh'(z)|^2}{|zh'(z) - \overline{\epsilon zh'(z)}|^2} \operatorname{Re} \bigg( 1 + \frac{zh''(z)}{h'(z)} \bigg) > \frac{1 - |\epsilon|}{1 + |\epsilon|} \alpha. \end{split}$$

In particular, this shows that  $\mathcal{K}(\alpha) \subset \mathcal{FK}_H(\alpha)$ . A similar statement shows that  $\mathcal{S}^*(\alpha) \subset \mathcal{FS}^*_H(\alpha)$ .

We employ a similar calculation carried out in [5, Section 3.5] to examine the full convexity and full starlikeness of order  $\alpha$  ( $0 \le \alpha < 1$ ) of the harmonic half-plane mapping L given by (1.2).

EXAMPLE 2.11. Observe that

$$\frac{\partial}{\partial \theta} \left( \arg \left\{ \frac{\partial}{\partial \theta} L(re^{i\theta}) \right\} \right) = \frac{1 - 6r^2 + r^4 + 12r^2 \cos^2 \theta - 4r(1+r^2) \cos^3 \theta}{1 + (2\cos^2 \theta - 3)[4\cos \theta(1+r^2) - 6r]r + r^4}$$
  
herefore if we set

Therefore if we set

$$p(r,u) = 1 - 6r^{2} + r^{4} + 12r^{2}u^{2} - 4r(1+r^{2})u^{3}$$
$$- \alpha[1 + (2u^{2} - 3)\{4u(1+r^{2}) - 6r\}r + r^{4}]$$

where  $u = \cos \theta$ , then p(r, -1) > 0 and p(r, 1) > 0 for  $r \in (0, 1)$  and  $\alpha \in [0, 1)$ . Also, further analysis shows that p(r, u) has a local minimum at  $u = u_0$ , where  $u_0$  is given by

$$u_0 = \frac{r(1+\alpha) - \sqrt{\alpha(1+2\alpha)(1+r^4) + (1+4\alpha+5\alpha^2)r^2}}{(1+r^2)(1+2\alpha)}.$$

Consequently, L is fully convex of order  $\alpha$  in  $|z| < r_C$ , where  $r_C = r_C(\alpha)$  is the positive root of the equation  $p(r, u_0) = 0$ . In particular,

$$r_C(0) = \sqrt{2} - 1,$$
  

$$r_C\left(\frac{1}{4}\right) = \frac{1}{3}\sqrt{\frac{1}{3}(223 - 70\sqrt{10})} \approx 0.246499$$
  

$$r_C\left(\frac{1}{2}\right) = \frac{1}{\sqrt{26 + 15\sqrt{3}}} \approx 0.138701,$$
  

$$r_C\left(\frac{3}{4}\right) = \sqrt{\frac{5}{681 + 182\sqrt{14}}} \approx 0.0605898.$$



Fig. 3. Images of the subdisks  $|z| < \sqrt{2} - 1$  and  $|z| < \sqrt{(7\sqrt{7} - 17)/2}$  under the mapping L

Regarding the full starlikeness of L, note that

$$\frac{\partial}{\partial \theta} \arg L(re^{i\theta}) = \frac{(1-r^2)[1+(2\cos^2\theta-5)r\cos\theta+3r^2-r^3\cos\theta]}{(1-2r\cos\theta+r^2)^2(\cos\theta-r)^2+(1-\cos^2\theta)(1-r^2)^2}.$$

Considering the function

(2.5) 
$$q(r, u, \alpha) = (1 - r^2)[1 + u(2u^2 - 5)r + 3r^2 - ur^3] - \alpha[(1 - 2ru + r^2)^2(u - r)^2 + (1 - u^2)(1 - r^2)^2]$$

where  $u = \cos \theta$ , we see that for  $\alpha \in (0, 1)$ ,  $q(r, u, \alpha) \ge 0$  for  $-1 \le u \le 1$  if and only if

$$q(r, -1, \alpha) = (1+r)^4 [1-r-\alpha(1+r)^2] \ge 0.$$

This inequality implies that  $r \leq r_S$  where  $r_S = r_S(\alpha)$  is given by

$$r_S(\alpha) = \frac{\sqrt{1+8\alpha} - (1+2\alpha)}{2\alpha}.$$

This shows that L is fully starlike of order  $\alpha$  in  $|z| < r_S$ , provided  $\alpha \in (0, 1)$ . In case  $\alpha = 0$ , (2.5) takes the form

$$q(r, u, 0) = (1 - r^2)[1 + u(2u^2 - 5)r + 3r^2 - ur^3].$$

Observe that

 $q(r, -1, 0) = (1 - r)(1 + r)^4 > 0$  and  $q(r, 1.0) = (1 + r)(1 - r)^4 > 0.$ 

Also, differentiation gives

$$\frac{\partial}{\partial u}q(r,u,0) = (1-r^2)[(6u^2-5)r - r^3],$$

showing that q(r, u, 0) has a local minimum at  $u = \sqrt{(5 + r^2)/6}$  and a local

maximum at  $u = -\sqrt{(5+r^2)/6}$ . Thus  $q(r, u, 0) \ge 0$  for  $-1 \le u \le 1$  if and only if

$$q\left(r,\sqrt{\frac{5+r^2}{6}},0\right) = \frac{1}{9}(1-r^2)\left[9-5\sqrt{6}\,r\sqrt{5+r^2}+27r^2-\sqrt{6}\,r^3\sqrt{5+r^2}\right] \ge 0.$$

This inequality implies that  $r \leq r_S(0) = \sqrt{(7\sqrt{7} - 17)/2} \approx 0.871854$ . Thus *L* is fully starlike in  $|z| < r_S(0)$  (see Figure 3). Note that  $r_C(0) = r_S(1 - 1/\sqrt{2})$ .

REMARK. Given  $f \in \mathcal{FK}_H$ , it will be interesting to determine  $\alpha \in [0, 1)$ for which  $f \in \mathcal{FS}_H^*(\alpha)$ . For instance, the function L(az)/a, where  $a = \sqrt{2} - 1$  and L is given by (1.2), belongs to the class  $\mathcal{FK}_H$  and Example 2.11 shows that  $L(az)/a \in \mathcal{FS}_H^*(1-1/\sqrt{2})$ . This function motivates the following harmonic analogue of the well-known Marx–Strohhäcker inequality:

PROBLEM 2.12. Determine  $\alpha \in [0, 1-1/\sqrt{2}]$  such that  $\mathcal{FK}_H \subset \mathcal{FS}^*_H(\alpha)$ .

The next result deals with the harmonic analogue of Alexander's theorem in the context of fully convex and fully starlike mappings of order  $\alpha$  ( $0 \le \alpha < 1$ ). The proof is similar to [1, Theorem 4, p. 140] and therefore it is omitted.

THEOREM 2.13. Let h, g, H and G be analytic functions in the unit disc  $\mathbb{D}$ , related by

$$zH'(z) = h(z)$$
 and  $zG'(z) = -g(z)$ .

Then  $f = h + \bar{g}$  is fully starlike of order  $\alpha$  if and only if  $F = H + \bar{G}$  is fully convex of order  $\alpha$ , where  $0 \le \alpha < 1$ .

This theorem provides abundant examples of fully convex and fully starlike mappings of order  $\alpha$  ( $0 \leq \alpha < 1$ ). For instance, since the functions  $f_n$  defined in Example 2.6 are fully starlike in  $\mathbb{D}$ , the functions  $F_n(z) = z - [(1 - \alpha)/(n(n + \alpha))]\overline{z}^n$  ( $z \in \mathbb{D}$ ) are fully convex of order  $\alpha$ . Similarly, since the function  $L(r_S z)$  is fully starlike of order  $\alpha$ ,  $r_S = r_S(\alpha)$  being the radius of full starlikeness of order  $\alpha$  for L determined in Example 2.11, the function

$$p(z) = \operatorname{Re} \frac{1}{1 - r_S z} - i \operatorname{arg}(1 - r_S z) \quad (z \in \mathbb{D})$$

is fully convex of order  $\alpha$ .

The next example shows that Theorem 2.13 does not cover the entire classes  $\mathcal{FK}_H(\alpha)$  and  $\mathcal{FS}^*_H(\alpha)$ .

EXAMPLE 2.14. If  $r_C = r_C(\alpha)$  is the radius of full convexity of order  $\alpha$  for the mapping L determined in Example 2.11, then the function  $F(z) = L(r_C z)/r_C = H(z) + \overline{G(z)} \in \mathcal{FK}_H(\alpha)$  and the corresponding function  $f(z) = C(r_C z)/r_C = H(z) + \overline{G(z)} \in \mathcal{FK}_H(\alpha)$ 

 $h(z) + \overline{g(z)}$ , where

$$h(z) = zH'(z) = \frac{z}{(1 - r_C z)^3}$$
 and  $g(z) = -zG'(z) = \frac{r_C z^2}{(1 - r_C z)^3}$ 

is not even locally univalent, since the Jacobian of f, given by

$$J_f(z) = \frac{1 - r_C^2 |z|^2}{|1 - r_C z|^8} (1 + r_C^2 |z|^2 + 4r_C \operatorname{Re} z),$$

vanishes at  $z = (2 - \sqrt{3})/r_C$ . However, by Theorem 2.13, f is fully starlike of order  $\alpha$ . It is easily seen that  $f \in \mathcal{FS}^*_H(\alpha)$  for  $|z| < (2 - \sqrt{3})/r_C$ .

The partial analogue of Theorem 2.13 for the classes  $\mathcal{FK}_H(\alpha)$  and  $\mathcal{FS}_H^*(\alpha)$  is given in the following corollary.

COROLLARY 2.15. If  $f = h + \bar{g} \in \mathcal{FS}^*_H(\alpha)$   $(0 \le \alpha < 1)$  and if H and G are the analytic functions defined by

$$zH'(z) = h(z), \quad zG'(z) = -g(z), \text{ and } H(0) = G(0) = 0,$$
  
then  $F = H + \bar{G} \in \mathcal{FK}_H(\alpha).$ 

Proof. By Theorem 2.13, F is fully convex of order  $\alpha$  and hence univalent. By hypothesis, the Jacobian  $J_F(z) \neq 0$  for each  $z \in \mathbb{D}$ , and since  $J_F(0) = |H'(0)|^2 - |G'(0)|^2 = 1 - |g'(0)|^2 > 0$ , it follows that F is sense-preserving in  $\mathbb{D}$ , so that  $F \in \mathcal{FK}_H(\alpha)$ .

**3. Radii problems.** In this section, we generalize the results given in [10] to fully starlike functions of order  $\alpha$  and fully convex functions of order  $\alpha$  for  $0 \leq \alpha < 1$ . The proofs are easy modifications of the proofs of the corresponding proofs from [10]. For completeness, we include the details. The following identities will be useful:

(3.1) 
$$\frac{r}{(1-r)^2} = \sum_{n=1}^{\infty} nr^n, \qquad \frac{r(r^2+4r+1)}{(1-r)^4} = \sum_{n=1}^{\infty} n^3 r^n, \\ \frac{r(1+r)}{(1-r)^3} = \sum_{n=1}^{\infty} n^2 r^n, \quad \frac{r(1+r)(1+10r+r^2)}{(1-r)^5} = \sum_{n=1}^{\infty} n^4 r^n.$$

THEOREM 3.1. Let h and g have the form (1.1) with  $b_1 = g'(0) = 0$ ,  $0 \le \alpha < 1$  and the coefficients of the series satisfying the conditions (1.4). Then  $f = h + \bar{g}$  is univalent and fully starlike of order  $\alpha$  in the disk  $|z| < r_S$ , where  $r_S = r_S(\alpha)$  is the real root of the equation

(3.2) 
$$2(1-\alpha)(1-r)^4 + \alpha(1-r)^2 - (1+r)^2 = 0$$

in the interval (0,1). Moreover, this result is sharp for each  $\alpha \in [0,1)$ .

*Proof.* The coefficient conditions (1.4) imply that h and g are analytic in  $\mathbb{D}$ . Thus,  $f = h + \bar{g}$  is harmonic in  $\mathbb{D}$ . Let 0 < r < 1. It suffices to show

that  $f_r \in \mathcal{FS}^*_H(\alpha)$ , where  $f_r$  is defined by

(3.3) 
$$f_r(z) = \frac{f(rz)}{r} = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \sum_{n=2}^{\infty} b_n r^{n-1} z^n, \quad z \in \mathbb{D}.$$

Consider the sum

(3.4) 
$$S = \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |a_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\alpha}{1-\alpha} |b_n| r^{n-1}.$$

Using the coefficient bounds (1.4) and simplifying, we have

$$S \le \frac{1}{3(1-\alpha)} \bigg[ 2\sum_{n=2}^{\infty} n^3 r^{n-1} + (1-3\alpha) \sum_{n=2}^{\infty} n r^{n-1} \bigg].$$

According to Lemma 2.9, we need to show that  $S \leq 1$  or equivalently

$$2\sum_{n=2}^{\infty} n^3 r^{n-1} + (1-3\alpha) \sum_{n=2}^{\infty} n r^{n-1} \le 3(1-\alpha).$$

By using the identities (3.1), the last inequality reduces to

$$\frac{(1+r)^2}{(1-r)^4} - \frac{\alpha}{(1-r)^2} \le 2(1-\alpha),$$

or

$$2(1-\alpha)(1-r)^4 + \alpha(1-r)^2 - (1+r)^2 \ge 0.$$

Thus, by Lemma 2.9,  $f_r \in \mathcal{FS}^*_H(\alpha)$  for  $r \leq r_S$  where  $r_S$  is the real root of (3.2) in (0,1). In particular, f is univalent and fully starlike of order  $\alpha$  in  $|z| < r_S$ .

To prove the sharpness, consider the function  $f_0(z) = h_0(z) + \overline{g_0(z)}$  with

$$h_0(z) = 2z - H(z)$$
 and  $g_0(z) = G(z)$   $(z \in \mathbb{D}),$ 

where  $K = H + \overline{G}$  is given by (1.5), so that

$$f_0(z) = z - \frac{1}{6} \sum_{n=2}^{\infty} (n+1)(2n+1)z^n + \frac{1}{6} \sum_{n=2}^{\infty} (n-1)(2n-1)z^n.$$

As  $f_0$  has real coefficients, for  $r \in (0, 1)$  we obtain

$$J_{f_0}(r) = (h'_0(r) + g'_0(r))(h'_0(r) - g'_0(r))$$
  
=  $\frac{(1 - 7r + 6r^2 - 2r^3)(1 - 10r + 11r^2 - 8r^3 + 2r^4)}{(1 - r)^7}.$ 

Note that the roots of the equation (3.2) in (0, 1) are decreasing as functions of  $\alpha \in [0, 1)$ . Consequently,  $r_S(\alpha) \leq r_S(0) \approx 0.112903$  and as  $J_{f_0}(r_S(0)) = 0$ , in view of Lewy's theorem, the function  $f_0$  is not univalent in |z| < r if  $r > r_S(0)$ . Also, since

$$\frac{\partial}{\partial \theta} \arg f_0(re^{i\theta}) \Big|_{\theta=0} = \frac{rh'_0(r) - rg'_0(r)}{h_0(r) + g_0(r)} = \frac{1 - 10r + 11r^2 - 8r^3 + 2r^4}{(1 - r)^2(2r^2 - 4r + 1)}$$

we find that if  $z = r_S$ , where  $r_S$  is the real root of (3.2) in (0, 1), then

$$\left. \frac{\partial}{\partial \theta} \arg f_0(r e^{i\theta}) \right|_{\theta=0, \, r=r_S} = \alpha,$$

showing that the bound  $r_S$  is best possible.

If  $\alpha = 0$  then Theorem 3.1 simplifies to [10, Theorem 1.5]. Also, Theorem 3.1 readily gives the following corollary.

COROLLARY 3.2. Let  $f \in S_H^{*0}$  (resp.  $C_H^0$ ) and  $0 \le \alpha < 1$ . Then f is fully starlike of order  $\alpha$  in at least  $|z| < r_S$ , where  $r_S$  is the real root of (3.2) in (0, 1).

Corollary 2.8 shows that the result in Corollary 3.2 is not sharp if  $\alpha = 0$ . Proceeding as in Theorem 3.1 and invoking Lemma 2.4 instead of Lemma 2.9, we have the following result.

THEOREM 3.3. Under the hypothesis of Theorem 3.1,  $f = h + \overline{g}$  is univalent and fully convex of order  $\alpha$  in the disk  $|z| < r_C$ , where  $r_C = r_C(\alpha)$  is the real root of the equation

(3.5) 
$$2(1-\alpha)(1-r)^5 + \alpha(1+r)(1-r)^2 - (1+r)(r^2+6r+1) = 0$$

in the interval (0,1). In particular, f is univalent and fully convex in  $|z| < r_C(0) \approx 0.0614313$ .

The bound  $r_C$  given by (3.5) is sharp by considering the function  $f_0(z) = 2z - K(z)$  where K is given by (1.5). In fact, as  $f_0$  has real coefficients, we obtain, when  $\theta = 0, r = r_C$ ,

$$\frac{\partial}{\partial \theta} \left( \arg \left\{ \frac{\partial}{\partial \theta} f_0(re^{i\theta}) \right\} \right) = \frac{1 - 17r_C + 13r_C^2 - 21r_C^3 + 10r_C^4 - 2r_C^5}{(1 - r_c)^2(1 - 7r_C + 6r_C^2 - 2r_C^3)} = \alpha.$$

Theorem 3.3 immediately gives

COROLLARY 3.4. Let  $f \in \mathcal{S}_{H}^{*0}$  (resp.  $\mathcal{C}_{H}^{0}$ ) and  $0 \leq \alpha < 1$ . Then f is fully convex of order  $\alpha$  in at least  $|z| < r_{C}$ , where  $r_{C}$  is the real root of (3.5) in (0, 1).

It is clear that the result in Corollary 3.4 is not sharp if  $\alpha = 0$ . Corresponding to Theorem 3.1, the next theorem determines the radius of univalence and full starlikeness of order  $\alpha$  for functions  $f = h + \bar{g} \in \mathcal{H}$ , where the Taylor coefficients of the series of h and g satisfy (1.6).

THEOREM 3.5. Let h and g have the form (1.1) with  $b_1 = g'(0) = 0$ ,  $0 \le \alpha < 1$  and the coefficients of the series satisfying the conditions (1.6).

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Then  $f = h + \bar{g}$  is univalent and fully starlike of order  $\alpha$  in the disk  $|z| < r_S$ , where  $r_S = r_S(\alpha)$  is the real root of the equation

(3.6) 
$$(2-\alpha)(1-r)^3 + \alpha r(1-r)^2 - 1 - r = 0$$

in the interval (0,1). Moreover, this result is sharp for each  $\alpha \in [0,1)$ .

*Proof.* Following the notation and the method of the proof of Theorem 3.1, it suffices to show that  $f_r \in \mathcal{FS}^*_H(\alpha)$ . Considering the sum (3.4) and using the coefficient bounds (1.6), we have on simplification

$$S \le \frac{1}{1 - \alpha} \left[ \sum_{n=2}^{\infty} n^2 r^{n-1} - \alpha \sum_{n=2}^{\infty} r^{n-1} \right].$$

By Lemma 2.9, we need to show that  $S \leq 1$ , or equivalently

$$\sum_{n=2}^{\infty} n^2 r^{n-1} - \alpha \sum_{n=2}^{\infty} r^{n-1} \le 1 - \alpha.$$

Bu using the identities (3.1), the last inequality reduces to

$$\frac{1+r}{(1-r)^3} - \frac{\alpha r}{1-r} \le 2 - \alpha,$$

or

$$(2-\alpha)(1-r)^3 + \alpha r(1-r)^2 - 1 - r \ge 0.$$

Thus, by Lemma 2.9, we deduce that f is univalent and fully starlike of order  $\alpha$  in  $|z| < r_S$ , where  $r_S$  is the real root of (3.6).

The sharpness part of the theorem follows if we consider the function  $f_0 = 2z - L(z)$  ( $z \in \mathbb{D}$ ) where L is given by (1.2) so that

$$f_0(z) = z - \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \sum_{n=2}^{\infty} \frac{n-1}{2} z^n.$$

As  $f_0$  has real coefficients, for  $r \in (0, 1)$  we obtain

$$J_{f_0}(r) = \frac{(1 - 4r + r^2)(1 - 7r + 6r^2 - 2r^3)}{(1 - r)^5}$$

Again observe that the roots of (3.6) in (0,1) are decreasing as functions of  $\alpha \in [0,1)$ . Consequently,  $r_S(\alpha) \leq r_S(0) \approx 0.16487$  and as  $J_{f_0}(r_S(0)) = 0$ , by Lewy's theorem we deduce that the function  $f_0$  is not univalent in |z| < r if  $r > r_S(0)$ . Also,

$$\frac{\partial}{\partial \theta} \arg f_0(re^{i\theta}) \bigg|_{\theta=0, r=r_S} = \frac{1 - 7r_S + 6r_S^2 - 2r_S^3}{(1 - r_S)^2(1 - 2r_S)} = \alpha,$$

showing that the bound  $r_S$  is best possible.

Note that Theorem 3.5 reduces to [10, Theorem 1.9] in case  $\alpha = 0$ . Moreover, Theorem 3.5 quickly yields COROLLARY 3.6. Let  $f \in \mathcal{K}_H^0$  and  $0 \le \alpha < 1$ . Then f is fully starlike of order  $\alpha$  in at least  $|z| < r_S$ , where  $r_S$  is the real root of (3.6) in (0,1).

If  $\alpha = 0$  then the result in Corollary 3.6 is not sharp in view of Corollary 2.8. It is expected that Corollary 3.6 can be further improved, and since the harmonic half-plane mapping L given by (1.2) is extremal in  $\mathcal{K}_{H}^{0}$ , Example 2.11 motivates the following conjecture:

CONJECTURE A. If  $f \in \mathcal{K}_H^0$ , then f is fully starlike of order  $\alpha$   $(0 \le \alpha < 1)$ in  $|z| < r_S$  where  $r_S = r_S(\alpha)$  is given by

$$r_S(\alpha) = \begin{cases} \frac{\sqrt{1+8\alpha} - (1+2\alpha)}{2\alpha} & \text{if } \alpha \in (0,1), \\ \sqrt{\frac{7\sqrt{7} - 17}{2}} & \text{if } \alpha = 0. \end{cases}$$

Using Lemma 2.4 and proceeding as in Theorem 3.5, we obtain the following result:

THEOREM 3.7. Under the hypothesis of Theorem 3.5,  $f = h + \bar{g}$  is univalent and fully convex of order  $\alpha$  in the disk  $|z| < r_C$ , where  $r_C = r_C(\alpha)$  is the real root of the equation

(3.7) 
$$2(1-\alpha)(1-r)^4 + \alpha(1-r)^2 - (r^2 + 4r + 1) = 0$$

in the interval (0,1). In particular, f is univalent and fully convex in  $|z| < r_C(0) \approx 0.0903331$ .

The radius bound  $r_C$  given by (3.7) is sharp for each  $\alpha \in [0, 1)$  by considering the function  $f_0(z) = h_0(z) + \overline{g_0(z)}$ , where

$$h_0(z) = 2z - M(z)$$
 and  $g_0(z) = N(z)$   $(z \in \mathbb{D}),$ 

 $L = M + \overline{N}$  being given by (1.2), and noticing that

$$\frac{\partial}{\partial \theta} \left( \arg \left\{ \frac{\partial}{\partial \theta} f_0(re^{i\theta}) \right\} \right) \Big|_{\theta=0, r=r_C} = \frac{1 - 12r_C + 11r_C^2 - 8r_C^3 + 2r_C^4}{(1 - r_C)^2(1 - 4r_C + 2r_C^2)} = \alpha.$$

An immediate consequence of Theorem 3.7 is

COROLLARY 3.8. If  $f \in \mathcal{K}_{H}^{0}$  and  $0 \leq \alpha < 1$ , then f is fully convex of order  $\alpha$  in  $|z| < r_{C}$ , where  $r_{C}$  is the real root of (3.7).

It is known that the result given in Corollary 3.8 is not sharp if  $\alpha = 0$ . Since the harmonic half-plane mapping L given by (1.2) gives the sharp bound for  $\alpha = 0$ , Example 2.11 motivates the following conjecture:

CONJECTURE B. If  $f \in \mathcal{K}_H^0$ , then f is fully convex of order  $\alpha$  ( $0 \le \alpha < 1$ ) in  $|z| < r_S$  where  $r_C = r_C(\alpha)$  is the positive root of the equation  $p(r, u_0) = 0$  in (0,1) with

$$p(r,u) = 1 - 6r^2 + r^4 + 12r^2u^2 - 4r(1+r^2)u^3 - \alpha[1 + (2u^2 - 3)\{4u(1+r^2) - 6r\}r + r^4]$$

and

$$u_0 = \frac{r(1+\alpha) - \sqrt{\alpha(1+2\alpha)(1+r^4) + (1+4\alpha+5\alpha^2)r^2}}{(1+r^2)(1+2\alpha)}$$

For  $\alpha = 0$ , this conjecture has been confirmed (see [18]).

4. Harmonic convolution. Consider the half-plane mapping L in  $\mathcal{K}_{H}^{0} \subset \mathcal{K}_{H}$  given by (1.2). The coefficients of the product L \* L are too large for this product to be in  $\mathcal{K}_{H}$ . In fact, the image of the unit disk  $\mathbb{D}$  under L \* L is  $\mathbb{C} \setminus (-\infty, -1/4]$ , which is not a convex domain. However,  $L * L \in \mathcal{S}_{H}^{*0}$ , by [4, Theorem 3]. Consider the following example:

EXAMPLE 4.1. We show that L \* L maps the subdisks |z| < r onto convex domains precisely for  $r \leq 2 - \sqrt{3}$ . Since we can write L as

$$L(z) = \frac{1}{2} \frac{z}{1-z} + \frac{1}{2} \frac{z}{(1-z)^2} + \frac{1}{2} \frac{z}{1-z} - \frac{1}{2} \frac{z}{(1-z)^2} \quad (z \in \mathbb{D}),$$

and using the fact that for an analytic function  $\varphi$  with  $\varphi(0) = 0$  we have

$$\frac{z}{1-z} * \varphi(z) = \varphi(z)$$
 and  $\frac{z}{(1-z)^2} * \varphi(z) = z \varphi'(z)$   $(z \in \mathbb{D}),$ 

it follows that

(4.1) 
$$(L * L)(z) = U(z) + \overline{V(z)},$$

where

$$U(z) = \frac{1}{4} \frac{z}{1-z} + \frac{1}{2} \frac{z}{(1-z)^2} + \frac{1}{4} \frac{z(1+z)}{(1-z)^3},$$
  
$$V(z) = \frac{1}{4} \frac{z}{1-z} - \frac{1}{2} \frac{z}{(1-z)^2} + \frac{1}{4} \frac{z(1+z)}{(1-z)^3}.$$

After simplification, we get

$$(L * L)(z) = \frac{1}{2} \operatorname{Re} \frac{z(z^2 - z + 2)}{(1 - z)^3} + i \operatorname{Im} \frac{z}{(1 - z)^2}$$

To prove our assertion, it will be necessary to study the change of the tangent direction

$$\Psi_r(\theta) = \arg\left\{\frac{\partial}{\partial\theta}(L*L)(re^{i\theta})\right\}$$

of the image curve as the point  $z = re^{i\theta}$  moves around the circle |z| = r. Note that

$$\frac{\partial}{\partial \theta}(L * L)(re^{i\theta}) = A(r, \theta) + iB(r, \theta),$$

where

$$|1 - z|^8 A(r, \theta) = -r(1 - r^2)[r^2 \sin 3\theta + r(1 + r^2) \sin 2\theta + (1 - 9r^2 + r^4) \sin \theta]$$
  
and

$$|1 - z|^{6}B(r, \theta) = r(1 - r^{2})[(1 + r^{2})\cos\theta - 2r(1 + \sin^{2}\theta)],$$

so that

$$\tan \Psi_r(\theta) = \frac{B(r,\theta)}{A(r,\theta)} = \frac{(1 - 2r\cos\theta + r^2)[2r(1 + \sin^2\theta) - (1 + r^2)\cos\theta]}{r^2\sin3\theta + r(1 + r^2)\sin2\theta + (1 - 9r^2 + r^4)\sin\theta}$$

A lengthy calculation leads to an expression for the derivative in the form

$$[4r^{2}u^{2} + 2r(1+r^{2})u + (1-10r^{2}+r^{4})]^{2}\frac{\partial}{\partial\theta}\tan\Psi_{r}(\theta) = q(r,u),$$

where  $u = \cos \theta$  and

$$\begin{split} q(r,u) &= 1 + 2u(u^2 - 2)r + 8(1 - 4u^2 + 2u^4)r^2 + 2u(34 - 21u^2 + 4u^4)r^3 \\ &\quad - 2(41 - 24u^2 + 8u^4)r^4 + 2u(34 - 21u^2 + 4u^4)r^5 \\ &\quad + 8(1 - 4u^2 + 2u^4)r^6 + 2u(u^2 - 2)r^7 + r^8. \end{split}$$

Observe that the roots of q(r, u) = 0 in (0, 1) are increasing as functions of  $u \in [-1, 1]$ . Consequently,  $q(r, u) \ge 0$  for  $-1 \le u \le 1$  if and only if

$$q(r,-1) = 1 + 2r - 8r^2 - 34r^3 - 50r^4 - 34r^5 - 8r^6 + 2r^7 + r^8 \ge 0.$$

This inequality implies that  $r \leq 2 - \sqrt{3}$ , which proves that the tangent angle



Fig. 4. Image of the subdisk  $|z| < 2 - \sqrt{3}$  under the mapping L \* L

 $\Psi_r(\theta)$  increases with  $\theta$  if  $r \leq 2 - \sqrt{3}$  but is not monotonic for  $2 - \sqrt{3} < r < 1$ . Thus, the harmonic mapping L \* L sends each disk  $|z| < r \leq 2 - \sqrt{3}$  to a convex region, but the image is not convex when  $2 - \sqrt{3} < r < 1$  (see Figure 4).

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THEOREM 4.2. Let h and g have the form (1.1) with  $b_1 = g'(0) = 0$ ,  $0 \le \alpha < 1$  and the coefficients of the series satisfying the conditions

$$|a_n| \le \left(\frac{n+1}{2}\right)^2$$
 and  $|b_n| \le \left(\frac{n-1}{2}\right)^2$  for all  $n \ge 2$ 

Then  $f = h + \bar{g}$  is univalent and fully starlike of order  $\alpha$  in the disk  $|z| < r_0$ , where  $r_0 = r_0(\alpha)$  is the real root of the equation

(4.2) 
$$2(1-\alpha)(1-r)^4 + \alpha(1-r)^2 - (r^2 + r + 1) = 0$$

in the interval (0,1). In particular, f is univalent and fully starlike in  $|z| < r_0$ , where  $r_0 = r_0(0) \approx 0.129831$  is the root of the biquadratic equation  $2r^4 - 8r^3 + 11r^2 - 9r + 1 = 0$ . Moreover, the result is sharp for each  $\alpha \in [0, 1)$ .

*Proof.* Following the method of the proof of Theorem 3.1, it suffices to show that  $f_r \in \mathcal{FS}^*_H(\alpha)$ , where  $f_r$  is defined by (3.3). Considering the sum (3.4) and using the coefficient bounds, we have

$$S \le \frac{1}{2(1-\alpha)} \left[ \sum_{n=2}^{\infty} n^3 r^{n-1} + (1-2\alpha) \sum_{n=2}^{\infty} n r^{n-1} \right].$$

According to Lemma 2.9, we need to show that  $S \leq 1$ , or equivalently

$$\sum_{n=2}^{\infty} n^3 r^{n-1} + (1-2\alpha) \sum_{n=2}^{\infty} n r^{n-1} \le 2(1-\alpha).$$

By using the identities (3.1), the last inequality reduces to

$$\frac{r^2 + r + 1}{(1 - r)^4} - \frac{\alpha}{(1 - r)^2} \le 2(1 - \alpha),$$

or

$$2(1-\alpha)(1-r)^4 + \alpha(1-r)^2 - (r^2 + r + 1) \ge 0.$$

Thus, by Lemma 2.9,  $f_r \in \mathcal{FS}^*_H(\alpha)$  for  $r \leq r_0$  where  $r_0 = r_0(\alpha)$  is the real root of the equation (4.2) in (0,1). In particular, f is univalent and fully starlike of order  $\alpha$  in  $|z| < r_0$ .

Next, to prove the sharpness, we consider the function

$$f_0(z) = h_0(z) + g_0(z)$$

with

$$h_0(z) = 2z - U(z)$$
 and  $g_0(z) = V(z)$ ,

where  $L * L = U + \overline{V}$  is given by (4.1). Note that

$$f_0(z) = z - \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right)^2 z^n + \sum_{n=2}^{\infty} \left(\frac{n-1}{2}\right)^2 z^n.$$

As  $f_0$  has real coefficients, we see that for  $r \in (0, 1)$ ,

$$J_{f_0}(r) = (h'_0(r) + g'_0(r))(h'_0(r) - g'_0(r))$$
  
=  $\left(2 - \frac{1+r}{(1-r)^3}\right) \left(2 - \frac{1}{2} \frac{1}{(1-r)^2} - \frac{1}{2} \frac{1+4r+r^2}{(1-r)^4}\right)$   
=  $\frac{(1 - 7r + 6r^2 - 2r^3)(1 - 9r + 11r^2 - 8r^3 + 2r^4)}{(1-r)^7}.$ 

Note that the roots of (4.2) in (0, 1) are decreasing as functions of  $\alpha \in [0, 1)$ . Consequently,  $r_0(\alpha) \leq r_0(0) \approx 0.112903$  and as  $J_{f_0}(r_0(0)) = 0$ , Lewy's theorem shows that  $f_0$  is not univalent in |z| < r if  $r > r_0(0)$ . Also, regarding starlikeness, we observe that

$$\frac{\partial}{\partial \theta} \arg f_0(re^{i\theta}) \Big|_{\theta=0} = \frac{rh'_0(r) - rg'_0(r)}{h_0(r) + g_0(r)} = \frac{1 - 9r + 11r^2 - 8r^3 + 2r^4}{(1 - r)^2(2r^2 - 4r + 1)},$$

therefore if we set  $z = r_0$ , where  $r_0$  is the real root of (4.2) in (0, 1), then

$$\frac{\partial}{\partial \theta} \arg f_0(re^{i\theta}) \bigg|_{\theta=0, r=r_0} = \alpha,$$

showing that the bound  $r_0$  is best possible.

The following result is an immediate consequence of Theorem 4.2.

COROLLARY 4.3. Let  $f, g \in \mathcal{K}_H^0$  and  $0 \le \alpha < 1$ . Then f \* g is univalent and fully starlike of order  $\alpha$  in at least  $|z| < r_0$ , where  $r_0 = r_0(\alpha)$  is the real root of (4.2) in (0,1). In particular, f \* g is univalent and fully starlike in  $|z| < r_0(0) \approx 0.129831$ .

Invoking Lemma 2.4 instead of Lemma 2.9 and proceeding in a similar manner to Theorem 4.2, we obtain the following result:

THEOREM 4.4. Under the hypothesis of Theorem 4.2,  $f = h + \bar{g}$  is univalent and fully convex of order  $\alpha$  in the disk  $|z| < s_0$ , where  $s_0 = s_0(\alpha)$  is the real root of the equation

(4.3)  $2(1-\alpha)(1-r)^5 + \alpha(1+r)(1-r)^2 - (1+r)(r^2+4r+1) = 0$ in the interval (0,1). In particular, f is univalent and fully convex in  $|z| < s_0(0) \approx 0.0712543$ .

It is worth remarking that the result regarding the univalence of f in Theorem 4.4 can be further improved to 0.129831 as seen by Theorem 4.2. However, the estimate  $s_0$  given by (4.3) regarding full convexity of order  $\alpha$  is sharp by considering the function  $f_0(z) = 2z - (L * L)(z)$ , where L is given by (1.2). In fact, as  $f_0$  has real coefficients, we obtain

$$\frac{\partial}{\partial \theta} \left( \arg \left\{ \frac{\partial}{\partial \theta} f_0(re^{i\theta}) \right\} \right) \Big|_{\theta=0, r=s_0} = \frac{1 - 15s_0 + 15s_0^2 - 21s_0^3 + 10s_0^4 - 2s_0^5}{(1 - s_0)^2 (1 - 7s_0 + 6s_0^2 - 2s_0^3)} = \alpha.$$

Theorem 4.4 easily gives

COROLLARY 4.5. Let  $f, g \in \mathcal{K}_H^0$  and  $0 \le \alpha < 1$ . Then f \* g is univalent and fully convex of order  $\alpha$  in at least  $|z| < s_0$ , where  $s_0 = s_0(\alpha)$  is the real root of (4.3) in (0,1). In particular, f \* g is univalent and fully convex in  $|z| < s_0(0) \approx 0.0712543$ .

It is expected that Corollary 4.5 can be further improved, and since the function L given by (1.2) is extremal in  $\mathcal{K}_{H}^{0}$ , in view of Example 4.1 we have the following conjecture:

CONJECTURE C. If  $f, g \in \mathcal{K}_H^0$ , then f \* g is univalent and fully convex in  $|z| < 2 - \sqrt{3}$ .

Another similar result regarding convolution of analytic functions is that if  $f_1 \in \mathcal{K}$  and  $f_2 \in \mathcal{S}^*$ , then  $f_1 * f_2 \in \mathcal{S}^*$ . Even this result does not extend to harmonic mappings. To see this, observe that the functions  $f_n = z + \bar{z}^n/n$  $(n \geq 2)$  defined in Example 2.6 (with  $\alpha = 0$ ) are in  $\mathcal{S}_H^{*0}$  and

$$(L*f_n)(z) = z - \frac{n-1}{2n}\overline{z}^n,$$

where L is defined by (1.2). Note that  $L * f_n \in \mathcal{S}_H^{*0}$  if and only if n = 2, 3. Indeed, for |z| = 1, we have

$$\frac{\partial}{\partial \theta} \arg(L * f_n)(z) = \operatorname{Re} \frac{z + \frac{n-1}{2}\bar{z}^n}{z - \frac{n-1}{2n}\bar{z}^n} \ge \frac{n(3-n)}{3n-1}.$$

This observation, together with the fact that  $L * f_n$  is univalent only if n = 2, 3, leads to  $L * f_n \in \mathcal{S}_H^{*0}$  for n = 2, 3.

The next result follows from an easy modification of the proof of Theorem 4.2.

THEOREM 4.6. Let h and g have the form (1.1) with  $b_1 = g'(0) = 0$ ,  $0 \le \alpha < 1$  and the coefficients of the series satisfying the conditions

$$|a_n| \le \frac{1}{12}(n+1)^2(2n+1)$$
 and  $|b_n| \le \frac{1}{12}(n-1)^2(2n-1)$  for all  $n \ge 2$ .

Then  $f = h + \bar{g}$  is univalent and fully starlike of order  $\alpha$  in the disk  $|z| < r_0$ , where  $r_0 = r_0(\alpha)$  is the real root of the equation

(4.4) 
$$12(1-\alpha)(1-r)^5 + \alpha(r^2+3r+6)(1-r)^2 - 6(1+r)^3 = 0$$

in the interval (0,1). In particular, f is univalent and fully starlike in  $|z| < r_0$ , where  $r_0 = r_0(0) \approx 0.0855165$  is the root of the equation  $2r^5 - 10r^4 + 21r^3 - 17r^2 + 13r - 1 = 0$ . Moreover, the result is sharp.

To prove the sharpness in Theorem 4.6, we consider the function

$$f_0(z) = 2z - (L * K)(z) = h_0(z) + g_0(z),$$

where

$$h_0(z) = 2z - \left(\frac{1}{4}\frac{z + \frac{1}{3}z^3}{(1-z)^3} + \frac{1}{4}\frac{z}{(1-z)^2} + \frac{1}{4}\frac{z(1+z)^2}{(1-z)^4} + \frac{1}{4}\frac{z(1+z)}{(1-z)^3}\right)$$
$$g_0(z) = -\left(\frac{1}{4}\frac{z + \frac{1}{3}z^3}{(1-z)^3} - \frac{1}{4}\frac{z}{(1-z)^2} - \frac{1}{4}\frac{z(1+z)^2}{(1-z)^4} + \frac{1}{4}\frac{z(1+z)}{(1-z)^3}\right).$$

Note that

$$f_0(z) = z - \frac{1}{12} \sum_{n=2}^{\infty} (n+1)^2 (2n+1) z^n + \frac{1}{12} \sum_{n=2}^{\infty} (n-1)^2 (2n-1) z^n,$$

and for  $r \in (0, 1)$ , the Jacobian of  $f_0$  is given by

$$J_{f_0}(r) = \frac{(1 - 13r + 17r^2 - 21r^3 + 10r^4 - 2r^5)(1 - 11r + 11r^2 - 8r^3 + 2r^4)}{(1 - r)^9},$$

so that  $J_{f_0}(r_0(0)) = 0$ . Regarding starlikeness, observe that

$$\frac{\partial}{\partial \theta} \arg f_0(re^{i\theta}) \bigg|_{\theta=0, r=r_0} = \frac{6(1-13r_0+17r^2-21r_0^3+10r_0^4-2r_0^5)}{(1-r_0)^2(6-39r_0+35r_0^2-12r_0^3)} = \alpha,$$

where  $r_0$  is the real root of (4.4) in (0,1). Theorem 4.6 gives

COROLLARY 4.7. If  $f \in \mathcal{S}_{H}^{*0}$  and  $g \in \mathcal{K}_{H}^{0}$ , then f \* g is univalent and fully starlike of order  $\alpha$  in the disk  $|z| < r_0$ , where  $r_0 = r_0(\alpha)$  is the real root of (4.4) in (0,1).

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Sumit Nagpal Department of Mathematics University of Delhi Delhi 110 007, India E-mail: sumitnagpal.du@gmail.com V. Ravichandran Department of Mathematics University of Delhi Delhi 110 007, India and School of Mathematical Sciences Universiti Sains Malaysia 11800 USM, Penang, Malaysia E-mail: vravi68@gmail.com

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