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Asymptotic stability and sweeping of substochastic semigroups

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Abstract. A new theorem on asymptotic stability and sweeping of substochastic semigroups is proved, and applied semigroups generated by birth-death processes.

1. Introduction. The purpose of this paper is to provide new sufficient conditions for asymptotic stability and sweeping of substochastic semigroups of operators with nontrivial integral parts. Such operators and semigroups are intensively studied because they play a special role in applications [BLPR, LR, PR, R2, RP1, RP2, RTW]. The book of Lasota and Mackey [LM] and the paper [RPT] are excellent surveys of many results on this subject. This problem has been investigated for Markov operators [KT, R1, PR]. In particular in [R1] it was shown that if such an operator has a positive invariant density f_* and has no other periodic points in the set of densities, then the operator is asymptotically stable. In [PR] it was proved that if a partially integral Markov semigroup has only one invariant density f_* and $f_* > 0$ a.e., then the semigroup is asymptotically stable. Some sufficient conditions for sweeping of integral Markov operators were given in [KT]. Namely, if such an operator P has no invariant density and possesses a subinvariant locally integrable and positive function, then P is sweeping.

Our criteria for asymptotic stability and sweeping generalize the results of [PR, R1]. In particular, earlier results concerning asymptotic stability of integral stochastic semigroups which spread or overlap supports given in [BL, BB, M] follow from our main theorem. The proof of this theorem is based on the results concerning properties of Harris operators [F, JO]. Many abstract results concerning Harris operators can be found in [Ne, Nu].

The plan of the paper is as follows. In Section 2 we give some auxiliary definition and results. In the next section we formulate the main result.

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Its proof is given in Section 4. In Section 5 we apply the main theorem to semigroups generated by birth-death processes.

2. Preliminaries. Let (X,Σ,μ) be a σ-finite measure space and let $L^1(X)=L^1(X,\Sigma,\mu)$. A linear operator $P\colon L^1(X)\to L^1(X)$ satisfying $\|P\|\le 1$ and $Pf\ge 0$ for $f\ge 0$ is called a *substochastic operator*. Denote by $D=D(X,\Sigma,\mu)$ the subset of $L^1(X)$ which consists of all *densities*, i.e.

$$D=\{f\in L^1(X)\colon f\ge 0,\ \|f\|=1\}.$$

A linear mapping $P\colon L^1(X)\to L^1(X)$ is called a *stochastic* (*or Markov*) *operator* if $P(D)\subset D$.

Let f be a density with $f>0$ a.e. Define

$$C=\Big\{x\in X\colon \sum_{n=0}^{\infty}P^nf(x)=\infty\Big\}. \tag{2.1}$$

This definition is independent of the choice of f. A substochastic operator $P: L^1(X)\to L^1(X)$ is called *conservative* if $C=X$ and *dissipative* if $C=\emptyset$.

An operator $Q\colon L^1(X)\to L^1(X)$ is called an *integral* or *kernel operator* if there exists a measurable function $k\colon X\times X\to[0,\infty)$ such that

$$Qf(x)=\int_X k(x,y)f(y)\,\mu(dy) \tag{2.2}$$

for every density f. Any substochastic operator P can be written in the form $P=Q+R$, where R is a nonnegative contraction on $L^1(X)$, Q is a kernel operator and there is no kernel K with $K\le R$ and $K\not\equiv 0$. Fix a substochastic operator P and let $P^n=Q_n+R_n$ be the decomposition of P^n into kernel and singular parts.

The operator P is called a *pre-Harris operator* if

$$\int_X\sum_{n=1}^{\infty}k_n(x,y)\,\mu(dy)>0\quad x\text{-a.e.}, \tag{2.3}$$

where k_n is the kernel corresponding to Q_n. If P is a conservative pre-Harris operator and $\mu(X)=1$, then P is called a *Harris operator*. If instead of (2.3) the operator P satisfies the condition

$$\int_X\int_X\sum_{n=1}^{\infty}k_n(x,y)\,\mu(dy)\,\mu(dx)>0, \tag{2.4}$$

then P is called *partially integral*. Let

$$\Sigma_i=\{A\in\Sigma\colon P^*\mathbf{1}_A=\mathbf{1}_A\}, \tag{2.5}$$

where P^* is the adjoint operator of P.

According to [F, Ch. V, Th. F] if P is conservative, partially integral, and Σ_i is trivial, then P is a pre-Harris operator. Now let P be a substochastic operator and let g be a positive density. Define a new measure space $(X,\Sigma,\bar{\mu})$ with $d\bar{\mu} = g d\mu$ and consider the operator

$$\bar{P}f = (1/g)P(f\cdot g). \tag{2.6}$$

Then $\bar{P}$ is also a substochastic operator on $L^1(X,\Sigma,\bar{\mu})$. If P is pre-Harris or conservative operator then $\bar{P}$ is also pre-Harris or conservative, respectively. In particular, if P is a pre-Harris conservative operator then $\bar{P}$ is a Harris operator.

A family $\{P(t)\}_{t\ge 0}$ of substochastic operators such that:

(a) $P(0) = \mathrm{Id}$,
(b) $P(t+s) = P(t)P(s)$ for $s,\, t \ge 0$,
(c) for each $f \in L^1(X)$ the function $t \mapsto P(t)f$ is continuous with respect to the $L^1(X)$ norm,

is called a *substochastic semigroup*. If $\{P(t)\}_{t\ge 0}$ is a family of stochastic operators which satisfies conditions (a)–(c) then it is called a *stochastic* or *Markov semigroup*. A semigroup $\{P(t)\}_{t\ge 0}$ is called *integral* if for each $t > 0$, the operator $P(t)$ is an integral operator. That is, there exists a measurable function $k\colon (0,\infty)\times X\times X \to [0,\infty)$, called a *kernel*, such that

$$P(t)f(x) = \int_X k(t,x,y)f(y)\,\mu(dy) \tag{2.7}$$

for every density f. A semigroup $\{P(t)\}_{t\ge 0}$ is called *partially integral* if for some $t_0 > 0$, the operator $P(t_0)$ is partially integral. Let $P(t_0) = Q(t_0) + R(t_0)$ be the decomposition of $P(t_0)$ into kernel and singular parts, and let $k(t_0)$ be the kernel corresponding to $Q(t_0)$. According to [F, Ch. V, Lemma B] if P is a substochastic operator and Q is an integral operator then the operators PQ and QP are integral operators. This implies that $Q(t)P(\tau) \le Q(t+\tau)$. From this it follows immediately that if $\{P(t)\}_{t\ge 0}$ is a substochastic semigroup and if for some $t_0 > 0$, the operator $P(t_0)$ is partially integral then for each $t \ge t_0$, the operator $P(t)$ is partially integral.

We also need two definitions concerning the asymptotic behaviour of a semigroup. A density f_* is called *invariant* if $P(t)f_* = f_*$ for each $t > 0$. The semigroup $\{P(t)\}_{t\ge 0}$ is called *asymptotically stable* if there is an invariant density f_* such that

$$\lim_{t\to\infty} \|P(t)f - f_*\| = 0 \quad \text{for } f \in D.$$

An operator P is called *sweeping* with respect to the set $A \in \Sigma$ if

$$\lim_{n\to\infty} \int_A P^n f\, d\mu = 0 \quad \text{for } f \in D.$$

The semigroup $\{P(t)\}_{t\ge 0}$ is *sweeping* with respect to A if

$$\lim_{t\to\infty}\int_A P(t)f\,d\mu = 0 \quad \text{for } f\in D.$$

Let a family $\mathcal{A}\subset \Sigma$ be given. We say that the operator P (or the semigroup $\{P(t)\}_{t\ge 0}$ of operators) is sweeping with respect to $\mathcal{A}$ if P (resp. $\{P(t)\}_{t\ge 0}$) is sweeping with respect to each set $A\in\mathcal{A}$. It is easy to check that the substochastic semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable (resp. sweeping) if there exists a $t_0>0$ such that the operator $P(t_0)$ is asymptotically stable (resp. sweeping). A nonnegative function f_* is called *subinvariant* if $Pf_*\le f_*$. For any $f\in L^1(X)$ the *support* of f is defined up to a set of measure zero by the formula

$$\operatorname{supp} f = \{x\in X\colon f(x)\neq 0\}.$$

We need some results concerning asymptotic stability and sweeping.

Theorem 2.1 ([PR]). *Let $\{P(t)\}_{t\ge 0}$ be a partially integral stochastic semigroup. Assume that the semigroup $\{P(t)\}_{t\ge 0}$ has a unique invariant density f_*. If $f_*>0$ a.e., then the semigroup $\{P(t)\}_{t\ge 0}$ is asymptotically stable.*

Corollary 2.2 ([R1]). *Let $P\colon L^1(X,\Sigma,\mu)\to L^1(X,\Sigma,\mu)$ be a pre-Harris stochastic operator. Assume that P has a subinvariant function $f_*>0$ which is integrable on each member of $\mathcal{A}$. If P has no invariant density then the operator P is sweeping with respect to $\mathcal{A}$.*

3. Main result. The main result of the paper is the following

Main Theorem 3.1. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $\{P(t)\}_{t\ge 0}$ be a substochastic semigroup on $L^1(X)$ which has a unique invariant density f_* and $S=\operatorname{supp} f_*$. Assume that $\{P(t)\}_{t\ge 0}$ is a partially integral semigroup with the kernel $k(t,x,y)$ such that*

$$\int_S\int_S k(t_0,x,y)\,\mu(dx)\,\mu(dy)>0 \tag{3.1}$$

for some $t_0>0$. Moreover, assume that for some $t_1>0$ there does not exist a nonempty measurable set $A\subsetneq Z_1$ such that $P^*(t_1)\mathbf{1}_A\ge \mathbf{1}_A$, where $Z_1=C\setminus S$ and C is a conservative part of $P(t_1)$ and for every $i=1,2$ and every $y_0\in Z_i$ there exist $\varepsilon>0$ and a measurable function $\eta_i\ge 0$ such that $\int_{Z_i}\eta_i\,d\mu>0$ and

$$k(t_1,x,y)\ge \eta_i(x) \tag{3.2}$$

for $x\in X\setminus S$ and $y\in B(y_0,\varepsilon)$, where $Z_2=X\setminus C$ and $B(y_0,\varepsilon)$ is the open ball with center y_0 and radius ε. Then for every $f\in D$ there exists a constant

$c(f)$ such that

$$\lim_{t\to\infty} \mathbf{1}_S P(t)f = c(f)f_*$$

and for every compact set $F \in \Sigma$ and $f \in D$ we have

$$\lim_{t\to\infty} \int\limits_{F\cap X\setminus S} P(t)f(x)\,\mu(dx) = 0. \tag{3.3}$$

REMARK 3.2. Theorem 3.1 remains true if the semigroup $\{P(t)\}_{t\ge 0}$ has no invariant density. In this case we set $S=\emptyset$, we omit condition (3.1) and we have $c(f)=0$.

REMARK 3.3. In practical applications of Theorem 3.1 there is no need to determine the conservative part of the operator $P(t_1)$. It is enough to check the stronger condition that there does not exist a nonempty measurable set $A \subsetneq X\setminus S$ such that $P^*(t_1)\mathbf{1}_A \ge \mathbf{1}_A$. Then $Z_1 = X\setminus S$ or $Z_2 = X\setminus S$.

REMARK 3.4. If a substochastic semigroup $\{P(t)\}_{t\ge 0}$ on $L^1(X)$ has the only one invariant density f_* and $\operatorname{supp} f_* = X$ then $\{P(t)\}_{t\ge 0}$ is a stochastic semigroup and if it is partially integral then from Theorem 3.1 it follows that

$$\lim_{t\to\infty} P(t)f = f_*$$

for each density f.

REMARK 3.5. If A is a measurable set such that $P^*(\bar t)\mathbf{1}_A \ge \mathbf{1}_A$ for some $\bar t > 0$ then

$$\int\limits_A P(\bar t)f\,d\mu \ge \int\limits_A f\,d\mu$$

for all $f \in L^1(X)$ with $f \ge 0$. In particular, if $\operatorname{supp} f \subset A$ then we have $\operatorname{supp} P(\bar t)f \subset A$. This means that we can study the asymptotic properties of the sequence $\{P^n(\bar t)\}_{n\ge 0}$ separately on the set A. Further, we check that for every $f \in L^1(X)$ such that $\operatorname{supp} f \subset S$ we have $\operatorname{supp} P(t)f \subset S$ for $t>0$. This means that the semigroup $\{P(t)\}_{t\ge 0}$ can be restricted to the set $L^1(S)$.

4. Proof. We split the proof of Theorem 3.1 into a sequence of lemmas.

LEMMA 4.1. *Assume that f_* is an invariant density with respect to a substochastic semigroup $\{P(t)\}_{t\ge 0}$ and $S = \operatorname{supp} f_*$. Then for every $f \in L^1(X)$ such that $\operatorname{supp} f \subset S$ we have $\operatorname{supp} P(t)f \subset S$ for $t>0$.*

Proof. It is sufficient to check the assertion for nonnegative functions. Fix an $f \in L^1(X)$ such that $f \ge 0$ and $\operatorname{supp} f \subset S$. Then for each positive integer n there exist a sufficiently large $c>0$ and a nonnegative function ε_n such that $f \le cf_* + \varepsilon_n$ and $\|\varepsilon_n\| < 1/n$. Then

$$\mathbf{1}_{X\setminus S}P(t)f \le c\mathbf{1}_{X\setminus S}P(t)f_* + P(t)\varepsilon_n = c\mathbf{1}_{X\setminus S}f_* + P(t)\varepsilon_n = P(t)\varepsilon_n.$$

Since $\|P(t)\varepsilon_n\| < 1/n$ as $n \to \infty$ we get $\mathbf{1}_{X\setminus S}P(t)f \le 0$ and we obtain $\operatorname{supp} P(t)f \subset S$ for $t > 0$. ∎

LEMMA 4.2. *Let P be a substochastic operator on a probability space $L^1(S)$. If P has an invariant density f_* such that $f_* > 0$ a.e., then P is a stochastic operator.*

Proof. We check that $\|Pf\| = \|f\|$ for all $f \in D$. As in the proof of Lemma 4.1 for each positive integer n there exist a constant $c > 0$ and nonnegative functions g and ε_n such that $f + g = cf_* + \varepsilon_n$ and $\|\varepsilon_n\| < 1/n$. Then

$$\|f\| + \|g\| = c\|f_*\| + \|\varepsilon_n\| \quad \text{and} \quad \|Pf\| + \|Pg\| = c\|Pf_*\| + \|P\varepsilon_n\|.$$

Since $\|Pf_*\| = \|f_*\|$, $\|\varepsilon_n\| \to 0$ and $\|P\varepsilon_n\| \to 0$ as $n \to \infty$ we obtain

$$\|Pf\| + \|Pg\| = \|f\| + \|g\|.$$

On the other hand $\|Pf\| \le \|f\|$ and $\|Pg\| \le \|g\|$ because P is a substochastic operator. Thus $\|Pf\| = \|f\|$. ∎

LEMMA 4.3. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $P\colon L^1(X,\Sigma,\mu) \to L^1(X,\Sigma,\mu)$ be a substochastic operator. Assume that the operator P can be written in the form $Pf(x) = \int k(x,y)f(y)\,\mu(dy) + Rf(x)$, where R is a positive contraction on $L^1(X)$ and the kernel k satisfies the following condition: for every $y_0 \in X$ there exist $\varepsilon > 0$ and a measurable function $\eta \ge 0$ such that $\int \eta\,d\mu > 0$ and $k(x,y) \ge \eta(x)$ for $x \in X$ and $y \in B(y_0,\varepsilon)$. If there exists a measurable function f_* such that $0 < f_* < \infty$ and $Pf_* \le f_*$, then f_* is integrable on compact sets.*

Proof. Suppose, on the contrary, that $\int_F f_*(x)\,\mu(dx) = \infty$ for some compact set $F \subset X$. Then for some $y_0 \in F$ we have $\int_{B(y_0,\delta)} f_*(y)\,\mu(dy) = \infty$ for every $\delta > 0$. Then there exist $\varepsilon > 0$ and a measurable function $\eta \ge 0$ such that $\int \eta\,d\mu > 0$ and $k(x,y) \ge \eta(x)$ for $x \in X$ and $y \in B(y_0,\varepsilon)$. Thus

$$Pf_*(x) \ge \int_X k(x,y)f_*(y)\,\mu(dy) \ge \eta(x) \int_{B(y_0,\varepsilon)} f_*(y)\,\mu(dy).$$

Since $Pf_* \le f_*$, we have $f_*(x) = \infty$ for $x \in \operatorname{supp}\eta$, which is impossible. ∎

LEMMA 4.4. *Let X be a metric space and Σ be the σ-algebra of Borel sets. Let $P\colon L^1(X,\Sigma,\mu) \to L^1(X,\Sigma,\mu)$ be a substochastic operator. Assume that the operator P can be written in the form $Pf(x) = \int k(x,y)f(y)\,\mu(dy) + Rf(x)$, where R is a positive contraction on $L^1(X)$ and the kernel k satisfies the following condition: for every $i = 1,2$ and every $y_0 \in X$ there exist $\varepsilon > 0$ and a measurable function $\eta_i \ge 0$ such that $\int_{Z_i} \eta_i\,d\mu > 0$ and $k(x,y) \ge \eta_i(x)$ for $x \in X$ and $y \in B(y_0,\varepsilon)$, where $Z_1 = C$, $Z_2 = X \setminus C$ and C is a conservative part of P. Assume that there does not exist a nonempty* *measurable set* $A \subsetneq C$ *such that* $P^*\mathbf{1}_A \geq \mathbf{1}_A$. *If* P *has no invariant density then* P *is sweeping with respect to the family* $\mathcal{F}$ *of compact sets.*

Proof. Let C be the conservative part of P given by (2.1). Then from [F, Ch. 2, Th. B], $P^*\mathbf{1}_C \geq \mathbf{1}_C$. This implies that if $\operatorname{supp} f \subset C$ then $\operatorname{supp} Pf \subset C$ for any $f \in L^1(X)$. Thus we can restrict the operator P to the space $L^1(C)$. We denote this restriction by P_C. The operator P_C is stochastic on $L^1(C)$. Thus P_C is conservative, partially integral, and $\Sigma_i = \{\emptyset, C\}$. Hence P_C is a pre-Harris operator. Let $g \in L^1(C)$ be a positive density, then the operator $\bar{P}\colon L^1(C) \to L^1(C)$ given by (2.6) is a Harris operator. According to [F, Ch. VI, Th. E] there exists a measurable function h such that $0 < h < \infty$ and $\bar{P}_C h = h$. Set $f_* = hg$. Then $0 < f_* < \infty$, f_* is a measurable function and $P_C f_* = f_*$. According to Lemma 4.3 the function f_* is integrable on members of $\mathcal{F}' = \{F \cap C\colon F \in \mathcal{F}\}$. Moreover P_C has no invariant density. From Corollary 2.2 the operator P_C is sweeping with respect to $\mathcal{F}'$.

Now, let $f \in L^1(X)$, $f > 0$, be a fixed density. Then $\sum_{n=0}^{\infty} P^n f(x) < \infty$ for $x \in X \setminus C$. We define an auxiliary operator $P_{X\setminus C}$ on $L^1(X \setminus C)$. For $g \in L^1(X)$ we put $P_{X\setminus C} g(x) = \mathbf{1}_{X\setminus C} P\tilde{g}(x)$, where $\tilde{g}(x) = 0$ for $x \in C$ and $\tilde{g}(x) = g(x)$ for $x \in X \setminus C$. Set $f_* = \sum_{n=0}^{\infty} P^n_{X\setminus C} f < \infty$. Then f_* is a measurable function and $P_{X\setminus C} f_* \leq f_*$, $f_* > 0$. By Lemma 4.3, f_* is integrable on members of $\mathcal{F}''$, where $\mathcal{F}'' = \{F \cap (X \setminus C)\colon F \in \mathcal{F}\}$. Let $F \in \mathcal{F}$. Then

$$\int\limits_{F\cap(X\setminus C)} \sum_{n=0}^{\infty} P^n_{X\setminus C} f(x)\,\mu(dx) = \sum_{n=0}^{\infty} \int\limits_{F\cap(X\setminus C)} P^n_{X\setminus C} f(x)\,\mu(dx) < \infty. \tag{4.1}$$

Thus

$$\lim_{n\to\infty} \int\limits_{F\cap(X\setminus C)} P^n_{X\setminus C} f(x)\,\mu(dx) = 0 \quad \text{for } F \in \mathcal{F}. \tag{4.2}$$

Set $a_n = \int_C P^n f\,d\mu$. Then the sequence (a_n) is increasing and convergent to some a. Indeed, we have $\int_C g\,d\mu = \int_C P^{n-k} g\,d\mu$ for $g \in L^1(C)$, $n \geq k$. Thus

$$a_k = \int\limits_C \mathbf{1}_C P^k f\,d\mu = \int\limits_C P^{n-k}(\mathbf{1}_C P^k f)\,d\mu \leq \int\limits_C P^n f\,d\mu = a_n.$$

Fix $\varepsilon > 0$. Then $|a - a_k| < \varepsilon$ for some integer k. Set $g = \mathbf{1}_C P^k f$. Then

$$\int\limits_C |P^n f - P_C^{n-k} g|\,d\mu \leq \varepsilon \qquad \text{for } n \geq k. \tag{4.3}$$

The operator P_C is sweeping with respect to the family $\mathcal{F}'$. Thus

$$\lim_{n\to\infty} \int\limits_{F\cap C} P_C^n g\,d\mu = 0 \quad \text{for } F \in \mathcal{F}.$$

From this and (4.2), (4.3) it follows that

$$\lim_{n\to\infty}\int_F P^n f\,d\mu = 0 \quad \text{for } f\in D,\ F\in\mathcal{F}. \ \blacksquare$$

Proof of Theorem 3.1. According to Lemma 4.1 we can restrict the semigroup $\{P(t)\}_{t\ge 0}$ to the space $L^1(S)$; we denote this restriction by $\{P_S(t)\}_{t\ge 0}$. Lemma 4.2 shows that $\{P_S(t)\}_{t\ge 0}$ is a stochastic semigroup. Condition (3.1) guarantees that this semigroup is partially integral. Moreover it has the unique invariant density f_* and $f_*>0$. According to Theorem 2.1, the semigroup $\{P_S(t)\}_{t\ge 0}$ is asymptotically stable, i.e. $\lim_{t\to\infty} P_S(t)f = f_*$ for $f\in L^1(S)\cap D$. Now let $f\in L^1(X)\cap D$. We introduce an auxiliary function $\varphi_f(t) = \int_S P(t)f\,d\mu$. Let $0<s<t$. Lemma 4.2 yields $\int_S g\,d\mu = \int_S P(t-s)g\,d\mu$ for $g\in L^1(S)$. Thus

$$\varphi_f(s) = \int_S \mathbf{1}_S P(s)f\,d\mu = \int_S P(t-s)(\mathbf{1}_S P(s)f)\,d\mu \le \int_S P(t)f\,d\mu = \varphi_f(t).$$

This implies that $\varphi_f(t)$ is a nondecreasing function of t. Set

$$c(f) = \lim_{t\to\infty}\varphi_f(t) = \lim_{t\to\infty}\int_S P(t)f\,d\mu.$$

We check that $\lim_{t\to\infty}\mathbf{1}_S P(t)f = c(f)f_*$. Let $\varepsilon>0$ be given. Then there exists $t_0>0$ such that $\int_S P(t_0)f\,d\mu \ge c(f)-\varepsilon$. Let $g=\mathbf{1}_S P(t_0)f$. Then $\lim_{s\to\infty} P(s)g = \|g\|f_*$ from the asymptotic stability of $\{P_S(t)\}_{t\ge 0}$. Since

$$P(t)f = P(t-t_0)P(t_0)f \ge P(t-t_0)g$$

we have $P(t)f \ge \|g\|f_* - \delta(t)$, where $\lim_{t\to\infty}\|\delta(t)\| = 0$. From the inequality $\|g\|\ge c(f)-\varepsilon$ it follows that

$$P(t)f \ge f_*(c(f)-\varepsilon) - \delta(t),$$

and consequently $\lim_{t\to\infty}\|(\mathbf{1}_S P(t)f - c(f)f_*)^-\| = 0$. But since

$$\lim_{t\to\infty}\int_X 1_S P(t)f\,d\mu = \int_X c(f)f_*\,d\mu$$

we obtain $\lim_{t\to\infty}\mathbf{1}_S P(t)f = c(f)f_*$.

Now we introduce an auxiliary semigroup $\{\tilde P(t)\}_{t\ge 0}$ on $L^1(X\setminus S)$. Let $f\in L^1(X\setminus S)$. Set $\tilde f(x)=0$ for $x\in S$ and $\tilde f(x) = f(x)$ for $x\in X\setminus S$. Define $\tilde P(t)f(x) = \mathbf{1}_{X\setminus S}P(t)\tilde f(x)$. We claim that $\tilde P(t+\tau)f = \tilde P(t)(\tilde P(\tau)f)$. Indeed,

$$\tilde P(t+\tau)f = \mathbf{1}_{X\setminus S}P(t+\tau)\tilde f = \mathbf{1}_{X\setminus S}P(t)(P(\tau)\tilde f).$$

According to Lemma 4.1, $\operatorname{supp} P(t)(\mathbf{1}_S P(\tau)\tilde f)\subset S$ and consequently

$$\mathbf{1}_{X\setminus S}P(t)(\mathbf{1}_S P(\tau)\tilde f) = 0.$$

Since $P(\tau)\tilde f = \mathbf{1}_{X\setminus S}P(\tau)\tilde f + \mathbf{1}_S P(\tau)\tilde f$ we have

$$\begin{aligned}\tilde P(t+\tau)f &= \mathbf{1}_{X\setminus S}P(t)(P(\tau)\tilde f) = \mathbf{1}_{X\setminus S}P(t)(\mathbf{1}_{X\setminus S}P(\tau)\tilde f)\\ &= \mathbf{1}_{X\setminus S}P(t)(\tilde P(\tau)f) = \tilde P(t)\tilde P(\tau)f.\end{aligned}$$

The last equality follows from the fact that $\operatorname{supp}\tilde P(\tau)f \subset X\setminus S$. Thus $\{\tilde P(t)\}_{t\ge 0}$ is a substochastic semigroup on $L^1(X\setminus S)$.

It is easy to show that the semigroup $\{P(t)\}_{t\ge0}$ is sweeping with respect to a set $A\subset X\setminus S$ if the semigroup $\{\tilde P(t)\}_{t\ge0}$ is. Indeed, let $f\in D$. Then $f = \mathbf{1}_{X\setminus S}f + \mathbf{1}_S f$. From Lemma 4.1 we have $\operatorname{supp} P(t)(\mathbf{1}_S f)\subset S$ and consequently $\int_A P(t)(\mathbf{1}_S f)\,d\mu = 0$. This implies

$$\int_A P(t)f\,d\mu = \int_A P(t)(\mathbf{1}_{X\setminus S}f)\,d\mu = \int_A \mathbf{1}_{X\setminus S}P(t)\tilde f\,d\mu = \int_A \tilde P(t)f\,d\mu.$$

Thus if $\lim_{t\to\infty}\int_A \tilde P(t)f\,d\mu = 0$ then $\lim_{t\to\infty}\int_A P(t)f\,d\mu = 0$.

Let $\mathcal F$ be the family of compact sets. In order to prove (3.3) we show that the semigroup $\{\tilde P(t)\}_{t\ge0}$ is sweeping with respect to the family $\mathcal F' = \{F\cap(X\setminus S)\colon F\in\mathcal F\}$. Observe that the semigroup $\{\tilde P(t)\}_{t\ge0}$ has no invariant density. Indeed, if g were an invariant density for $\{\tilde P(t)\}_{t\ge0}$ then $P(t)g\ge g$ for $t\ge0$. But since $\{P(t)\}_{t\ge0}$ is substochastic we would have $P(t)g = g$ for $t\ge 0$, which contradicts the assumption that $\{P(t)\}_{t\ge0}$ has the unique invariant density f_*. Now, define a substochastic operator P on $L^1(X\setminus S, \Sigma,\mu)$ by $P=\tilde P(t_1)$. By Lemma 4.4 the operator P is sweeping with respect to $\mathcal F'$. Thus the semigroup $\{\tilde P(t)\}_{t\ge0}$ is sweeping with respect to $\mathcal F'$ and so (3.3) holds, which completes the proof. ∎

5. Example. Most substochastic semigroups satisfy the Foguel alternative, i.e. they are either asymptotically stable or sweeping from compact sets [KM, RPT]. For example any semigroup generated by a nondegenerate diffusion process is asymptotically stable or sweeping. Now we give a simple example which shows that a semigroup can be "partially" asymptotically stable and "partially" sweeping, i.e. Theorem 3.1 holds with $c(f)$ depending on f.

We consider a birth-death process

$$x_i'(t) = -a_i x_i(t) + b_{i-1}x_{i-1}(t) + d_{i+1}x_{i+1}(t),\quad i\ge 0, \tag{5.1}$$

where $a_i = b_i + d_i$, $b_i\ge 0$, $d_i\ge 0$, and $b_{-1} = d_0 = 0$. We assume additionally that there is a constant $C>0$ such that $b_i\le Ci$ for all $i\ge 0$. The last condition guarantees that the system (5.1) generates a stochastic semigroup $\{P(t)\}_{t\ge0}$ on l^1, the space of absolutely summable sequences. The semigroup $\{P(t)\}_{t\ge0}$ is given by $(P(t)\bar x)_i = x_i(t)$, where $x(t) = (x_i(t))$ is the solution of (5.1) with the initial condition $x(0)=\bar x$, $\bar x\in l^1$. The semigroup $\{P(t)\}_{t\ge0}$

can be written in the form

$$(P(t)x)_i = \sum_{j=0}^{\infty} p_{ij}(t)x_j, \tag{5.2}$$

where $p_{ij}(t)$ are continuous nonnegative functions. Observe that any stochastic operator $P\colon l^1 \to l^1$ is an integral operator and satisfies (3.2). Indeed, $l^1 = L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ with the counting measure $\mu(A)$, where $\mu(A)$ is the number of elements of A and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We have

$$(Px)_i = \sum_{j=0}^{\infty} p_{ij}x_j = \int_{\mathbb{N}} k(i,j)x(j)\,\mu(dj),$$

where we put $k(i,j) = p_{ij}$ and $x(j) = x_j$, which means that P is an integral operator. Since $\mathbb{N}$ is a discrete topological space condition (3.2) follows from

$$\sum_{i=0}^{\infty} p_{ij} = \int_{\mathbb{N}} k(i,j)\,\mu(di) = 1 \quad \text{for each } j \in \mathbb{N}. \tag{5.3}$$

Now assume that there exists $n > 0$ such that $b_n = 0$, $b_i > 0$ for $i \neq n$, and $d_i > 0$ for $i > 0$. Then for each $t > 0$ we have $p_{ij}(t) > 0$ when $i \le j$, $j \le i \le n$, or $n \le j \le i$, and we have $p_{ij}(t) = 0$ if $j \le n < i$.

Now we check when $P^*(t)\mathbf{1}_A \ge \mathbf{1}_A$. We have $P^*(t)\mathbf{1}_A(j) = \sum_{i\in A} p_{ij}(t)$. This means that $P^*(t)\mathbf{1}_A \ge \mathbf{1}_A$ if and only if $\sum_{i\in A} p_{ij}(t) \ge 1$ for all $j \in A$. From condition (5.3) and the above inequalities for $p_{ij}(t)$ we deduce that if $j > n$ and $j \in A$ then $A = \mathbb{N}$ and if $j \le n$ and $j \in A$ then $\{0, 1, \ldots, n\} \subset A$. Thus the only sets A which satisfy $P^*(t)\mathbf{1}_A \ge \mathbf{1}_A$ are $\emptyset$, $\mathbb{N}$, and $\mathbb{N}_n = \{0, 1, \ldots, n\}$. Moreover if the sequence $x^* = (x_i^*)$ is an invariant density and $S = \operatorname{supp} x^*$ then $P^*(t)\mathbf{1}_S \ge \mathbf{1}_S$. It follows that the semigroup has at most one invariant density, because in the opposite case we could find two invariant densities with disjoint supports, which is impossible. Also since $p_{ij}(t) = 0$ for $j \le n < i$, we can restrict the semigroup $\{P(t)\}_{t\ge 0}$ to the space $L^1(\mathbb{N}_n, 2^{\mathbb{N}_n}, \mu)$, and $\{P(t)\}_{t\ge 0}$ is still a stochastic semigroup on this space. The existence of an invariant density follows immediately from the ergodic theorem for Markov chains on a finite space. The invariant density $x^* = (x_i^*)$ can also be found directly by solving the system

$$\begin{aligned}
a_0x_0 &= d_1x_1,\\
a_1x_1 &= b_0x_0 + d_2x_2,\\
a_2x_2 &= b_1x_1 + d_3x_3,\\
&\vdots\\
a_nx_n &= b_{n-1}x_{n-1}.
\end{aligned}$$

We have checked all assumptions of Theorem 3.1. Thus for each $\bar{x} \in l^1$ there exists a constant $c(\bar{x})$ such that the solution of (5.1) with the initial condition $x(0) = \bar{x}$ satisfies

$$\lim_{t\to\infty} x_i(t) = \begin{cases} c(\bar{x})x_i^* & \text{for } i \le n, \\ 0 & \text{for } i > n. \end{cases}$$

Now consider a continuous random walk with an absorbing state at zero, that is, the gambler's ruin problem. This process is described by (5.1) with $a_i = b_i + d_i = 1$ for $i \ge 1$, $b_{-1} = b_0 = d_0 = 0$ and $b_i = b$, $d_i = d$ for $i \ge 1$, i.e., it is given by the following system:

$$\begin{aligned} x_0'(t) &= dx_1(t), \\ x_1'(t) &= -x_1(t) + dx_2(t), \\ x_i'(t) &= -x_i(t) + bx_{i-1}(t) + dx_{i+1}(t), \quad i \ge 2. \end{aligned} \tag{5.4}$$

By b we denote the birth rate (the probability of winning the game). The death rate (the probability of losing) is denoted by d. We assume additionally that $b > d$. The semigroup $\{P(t)\}_{t\ge 0}$ generated by (5.4) has the unique invariant density $x^* = (x_i^*)$, where $x_0^* = 1$ and $x_i^* = 0$ for $i \ge 1$. Let $\bar{x} \in D$. Since in the long run, the probability of losing all of the initial capital i (probability of absorption) is $(d/b)^i$ (see [A, Ch.6.4.3]), we have

$$c(\bar{x}) = \sum_{i=0}^{\infty} \left(\frac{d}{b}\right)^i \bar{x}_i \quad \text{and} \quad \lim_{t\to\infty} (P(t)\bar{x})_0 = c(\bar{x})x_0^*.$$

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