Comparison of explicit and implicit difference schemes for parabolic functional differential equations

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Abstract. Initial-boundary value problems of Dirichlet type for parabolic functional differential equations are considered. Explicit difference schemes of Euler type and implicit difference methods are investigated. The following theoretical aspects of the methods are presented. Sufficient conditions for the convergence of approximate solutions are given and comparisons of the methods are presented. It is proved that the assumptions on the regularity of the given functions are the same for both methods. It is shown that the conditions on the mesh for explicit difference schemes are more restrictive than the suitable assumptions for implicit methods. There are implicit difference schemes which are convergent while the corresponding explicit difference methods are not convergent. Error estimates for both methods are constructed.

1. Introduction. Difference methods for parabolic functional differential equations (with initial-boundary conditions) are obtained by replacing partial derivatives with difference operators. Moreover, since differential equations contain functional variables which are elements of the set of continuous functions defined on a subset of a finite-dimensional space, some interpolating operators are needed. This leads to functional difference equations of Volterra type. The stability of functional difference schemes is investigated by using comparison techniques.

In recent years, a number of papers concerning explicit difference methods for parabolic problems have been published. Difference approximations of nonlinear equations with initial-boundary conditions of Dirichlet type were considered in [8], [10], [22]. Numerical treatment of functional differential equations with initial-boundary conditions of Neumann type can be found in [3], [12]. The convergence results for a general class of difference schemes related to parabolic problems with solutions defined on unbounded domains can be found in [16], [29]. The papers [4], [6], [9], [13], [14], [17] concern

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implicit difference schemes for parabolic problems. Error estimates implying the convergence of implicit difference schemes are obtained in those papers by using difference inequalities or simple theorems on recurrent inequalities.

Monotone iterative methods and implicit difference schemes for computing approximate solutions to parabolic equations with time delay were analyzed in [15], [18], [19], [30].

All the above results on numerical methods for parabolic functional differential equations have the following property: the authors have assumed that solutions of initial-boundary value problems are defined on intervals $[0, a] \times [-c, c] \subset \mathbb{R}^{1+n}$. It is clear that the sets of the form $[0, a] \times Q$ where $Q \subset \mathbb{R}^n$ is a bounded set are natural domains on which solutions of mixed problems for parabolic functional differential equations are considered. We start investigations of difference schemes for nonlinear parabolic functional equations with solutions defined on $[0, a] \times Q$.

Note that cylindrical domains appear in theoretical results on the existence and uniqueness of solutions to parabolic functional differential problems ([11], [20], [26], [27]).

In this paper we investigate theoretical questions for explicit and implicit difference schemes generated by initial-boundary value problems of Dirichlet type for functional differential equations with solutions considered on cylindrical domains.

Now we formulate our functional differential problems. For any metric spaces $X$ and $Y$ we denote by $C(X, Y)$ the class of all continuous functions from $X$ into $Y$. We use vectorial inequalities to mean that the same inequalities hold between their corresponding components.

Let $Q \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial Q$. Write $E = [0, a] \times \overline{Q}$, $E_0 = [-b_0, 0] \times \overline{Q}$ where $\overline{Q}$ is the closure of $Q$. For each $(t, x) \in E$ we define

$$D[t, x] = \{(\tau, y) \in \mathbb{R}^{1+n} : \tau \leq 0, (t + \tau, x + y) \in E_0 \cup E\}.$$  

For a function $z : E \cup E \to \mathbb{R}$ and $(t, x) \in E$ we define a function $z_{(t,x)} : D[t, x] \to \mathbb{R}$ by $z_{(t,x)}(\tau, y) = z(t + \tau, x + y)$ for $(\tau, y) \in D[t, x]$. Thus $z_{(t,x)}$ is the restriction of $z$ to the set $(E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n)$, shifted to the set $D[t, x]$.

There is $[c, d] \subset \mathbb{R}^n$ such that $D[t, x] \subset [-b_0 - a, 0] \times [c, d]$ for $(t, x) \in E$. Write $I = [-b_0 - a, 0]$, $B = I \times [c, d]$. Let $M_{n \times n}$ be the class of all $n \times n$ symmetric matrices with real elements. Set $\Omega = E \times C(B, \mathbb{R}) \times \mathbb{R}^n \times M_{n \times n}$, $\partial_0 E = [0, a] \times \partial Q$ and suppose that $F : \Omega \to \mathbb{R}$, $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ are given functions. Let $z$ be an unknown function of the variables $(t, x)$, $x = (x_1, \ldots, x_n)$. We consider the problem consisting of the functional differential equation

\begin{align}
\partial_t z(t, x) &= F(t, x, z_{(t,x)}), \partial_x z(t, x), \partial_{xx} z(t, x)) \\
(1.1)
\end{align}
and the initial-boundary condition
\[
(1.2) \quad z(t, x) = \varphi(t, x) \quad \text{on } E_0 \cup \partial_0 E
\]
where \( \partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z) \), \( \partial_{xx} = [\partial_{x_i x_j} z]_{i,j=1}^n \). We will say that \( F \) satisfies condition \((V)\) if for each \((t, x, q, s) \in E \times \mathbb{R}^n \times M_{n \times n}\) and for \( w, \tilde{w} \in C(B, \mathbb{R}) \) such that \( w(\tau, y) = \tilde{w}(\tau, y) \) for \((\tau, y) \in D[t, x] \) we have \( F(t, x, w, q, s) = F(t, x, \tilde{w}, q, s) \). Condition \((V)\) means that the value of \( F \) at \((t, x, w, q, s) \in \Omega\) depends on \((t, x, q, s)\) and on the restriction of \( w \) to the set \( D[t, x] \). We assume that \( F \) satisfies condition \((V)\) and we consider classical solutions to (1.1), (1.2).

We prove that there are explicit and implicit difference schemes for (1.1), (1.2) which are convergent, and we compare them.

Two types of of assumptions are needed in theorems on the convergence of functional difference problems generated by (1.1), (1.2). The first type conditions concern the regularity of \( F \). Assumptions on the regularity of the given functions are the same for explicit and for implicit difference methods. We prove that error estimates are the same for both methods.

The second type conditions concern the mesh. We show that we need strong assumptions on the mesh for explicit difference methods and we do not need assumptions on the mesh for implicit difference schemes. We show that there are implicit difference methods which are convergent, while the corresponding explicit difference schemes are not. We present error estimates for both methods.

The authors of [1], [3], [4], [8], [10], [12], [13], [16], [22] have assumed that the given functions satisfy the Lipschitz condition or satisfy nonlinear estimates of Perron type with respect to function variables and that these conditions are global with respect to function variables. Our assumptions are more general. We assume that nonlinear estimates of Perron type are local with respect to function variables. It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates of Perron type hold and global inequalities are not satisfied.

Relations between explicit and implicit difference methods for quasilinear functional differential equations are given in [5]. The present paper is a continuation of [5].

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of parabolic functional differential equations can be found in [2], [11], [21], [23]–[26], [31].

Differential equations with deviated arguments and differential integral problems can be derived from (1.1), (1.2) by specializing given operators. Information on applications of functional differential equations can be found in [31].
2. Discretization of functional differential equations. For any spaces \( X \) and \( Y \) we denote by \( F(X,Y) \) the class of all functions defined on \( X \) and taking values in \( Y \). We will denote by \( \mathbb{N} \) and \( \mathbb{Z} \) the sets of natural numbers and integers respectively.

We define a mesh on \( E_0 \cup E \) in the following way. Let \( (h_0, h') = (h_1, \ldots, h_n) \), stand for steps of the mesh with respect to \( t \) and \( x \) respectively. Set \( h = (h_0, h') \) and \( t(r) = r h_0, r \in \mathbb{Z} \). For \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \) we put \( x^{(m)} = (m_1 h_1, \ldots, m_n h_n) \) and

\[
\mathbb{R}_h^n = \{ x^{(m)} : m \in \mathbb{Z}^n \}, \quad Q_h = Q \cap \mathbb{R}_h^n, \quad \overline{Q}_h = \overline{Q} \cap \mathbb{R}_h^n.
\]

Write \( J = \{(i, j) : i, j = 1, \ldots, n, i \neq j\} \) and \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n \) with 1 in the \( i \)th position, \( 1 \leq i \leq n \). Suppose that we have defined the sets \( J_+, J_- \subset J \) such that \( J_+ \cup J_- = J, J_+ \cap J_- = \emptyset \). We assume that \( (j, i) \in J_+ \) if \( (i, j) \in J_+ \). In particular, it may happen that \( J_+ = \emptyset \) or \( J_- = \emptyset \). Relations between \( J_+, J_- \) and equation (1.1) are given in Section 3. For \( x^{(m)} \in \mathbb{R}_h^n \) we put

\[
\theta_1^{(m)} = \{ x^{(m+e_i)} : i = 1, \ldots, n \} \cup \{ x^{(m-e_i)} : i = 1, \ldots, n \},
\]

\[
\theta_2^{(m)} = \{ x^{(m+e_i+e_j)} : (i, j) \in J \} \cup \{ x^{(m-e_i-e_j)} : (i, j) \in J \}
\]

and \( \theta^{(m)} = \theta_1^{(m)} \cup \theta_2^{(m)} \). Write

\[
\text{Int} \ Q_h = \{ x^{(m)} \in Q_h : \theta^{(m)} \subset \overline{Q}_h \}, \quad \partial_0 Q_h = Q_h \setminus \text{Int} \ Q_h.
\]

We will approximate the partial derivatives \( \partial_x = (\partial_{x_1}, \ldots, \partial_{x_n}) \) and

\[
\partial_{xx} = [\partial_{x_ix_j}]^{n}_{i,j=1}
\]

with difference operators \( \delta = (\delta_1, \ldots, \delta_n) \) and

\[
\delta^{(2)} = [\delta_{ij}]^{n}_{i,j=1}
\]

respectively. We will calculate the difference expressions \( \delta z(t^{(r)}, x^{(m)}) \) and \( \delta^{(2)} z(t^{(r)}, x^{(m)}) \) for each point \( (t^{(r)}, x^{(m)}) \in [0, a] \times Q_h \). Then we need additional mesh points on the set \( \partial Q \). For each \( x^{(m)} \in \partial_0 Q_h \) and \( i, j = 1, \ldots, n, i \neq j \), we define

\[
\lambda^{(m)}_{i+} = \max \{ \lambda \in (0, 1) : x^{(m)} + \lambda h_i e_i \in \overline{Q} \},
\]

\[
\lambda^{(m)}_{i-} = \max \{ \lambda \in (0, 1) : x^{(m)} - \lambda h_i e_i \in \overline{Q} \},
\]

\[
\lambda^{(m)}_{i+j} = \max \{ \lambda \in (0, 1) : x^{(m)} + \lambda (h_i e_i + h_j e_j) \in \overline{Q} \},
\]

\[
\lambda^{(m)}_{i-j} = \max \{ \lambda \in (0, 1) : x^{(m)} + \lambda (-h_i e_i + h_j e_j) \in \overline{Q} \},
\]

\[
\lambda^{(m)}_{i+j} = \max \{ \lambda \in (0, 1) : x^{(m)} + \lambda (h_i e_i - h_j e_j) \in \overline{Q} \},
\]

\[
\lambda^{(m)}_{i-j} = \max \{ \lambda \in (0, 1) : x^{(m)} - \lambda (h_i e_i + h_j e_j) \in \overline{Q} \}.
\]
It is clear that these numbers depend on $x^{(m)} \in \partial Q_h$. For simplicity of notation we write $\lambda_{i+}, \lambda_{i-}, \lambda_{i+j+}, \lambda_{i+j-}, \lambda_{i-j+}, \lambda_{i-j-}$ instead of $\lambda_{i+}^{(m)}$, etc. Set

$$S_h^{(1)} = \{x \in \partial Q : \text{there are } x^{(m)} \in \partial Q_h \text{ and } i \in \{1, \ldots, n\} \text{ such that } x = x^{(m)} + \lambda_{i+} h_i e_i \text{ or } x = x^{(m)} - \lambda_{i-} h_i e_i\},$$

$$S_h^{(2)} = \{x \in \partial Q : \text{there are } x^{(m)} \in \partial Q_h \text{ and } (i, j) \in J \text{ such that } x = x^{(m)} + \lambda_{i+j+} (h_i e_i + h_j e_j) \text{ or } x = x^{(m)} + \lambda_{i+j-} (h_i e_i - h_j e_j) \text{ or } x = x^{(m)} - \lambda_{i-j-} (h_i e_i + h_j e_j)\}$$

and $S_h = S_h^{(1)} \cup S_h^{(2)}$. Write $X_h = Q_h \cup S_h$. Then $X_h$ is the set of all mesh points in $Q$.

Denote by $\Delta$ the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{C}$ and $c_0 > 0$ satisfying $K_0 h_0 = b_0$ and $h_i h_{i+1}^{-1} \leq c_0$ for $(i, j) \in J$. Let $K \in \mathbb{N}$ be defined by $Kh_0 < a < (K+1)h_0$. Write

$$E_h = \{(t^{(r)}, x) : 0 \leq r \leq K, x \in X_h\},$$

$$E_{0,h} = \{(t^{(r)}, x) : -K_0 \leq r \leq 0, x \in X_h\},$$

$$\partial_0 E_h = \{(t^{(r)}, x) : 0 \leq r \leq K, x \in S_h\},$$

$$E_{h}' = \{(t^{(r)}, x) : 0 \leq r \leq K-1, x \in Q_h\},$$

$$\theta_h = \{t^{(r)} : 0 \leq r \leq K\}$$

where $h \in \Delta$. For functions $z : E_{0,h} \cup E_h \rightarrow \mathbb{R}$, $\chi : X_h \rightarrow \mathbb{R}$, $\omega : \theta_h \rightarrow \mathbb{R}$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ for $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$ and $\chi^{(m)} = \chi(x^{(m)})$ for $x^{(m)} \in Q_h$ and $\omega^{(r)} = \omega(t^{(r)})$ for $t^{(r)} \in \theta_h$.

If $\mu, \nu \in [-1, 1]$ and $(t^{(r)}, x^{(m)} + \mu h_i e_i + \nu h_j e_j) \in [-b_0, a] \times X_h$ then we put $z^{(r,m+\mu e_i+\nu e_j)} = z(t^{(r)}, x^{(m)} + \mu h_i e_i + \nu h_j e_j)$ and $\chi^{(m+\mu e_i+\nu e_j)} = \chi(x^{(m)}+\mu h_i e_i+\nu h_j e_j)$.

Solutions of difference functional equations are elements of the space $\mathbf{F}(E_{0,h} \cup E_h, \mathbb{R})$. Equation (1.1) contains the function variable $z(t,x)$ which is an element of the space $C(D[t,x], \mathbb{R})$. Thus we need an interpolating operator $T_h : \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R}) \rightarrow C(E_0 \cup E, \mathbb{R})$. In Section 3 we adopt additional assumptions on $T_h$. For $z \in \mathbf{F}(E_{0,h} \cup E_h, \mathbb{R})$ and $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$ we write $(T_h z)_{[r,m]}$ instead of $(T_h z)_{(t^{(r)}, x^{(m)})}$. Set

$$\mathbf{F}_{ex,h}[z]^{(r,m)} = F(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, \delta z^{(r,m)}, \delta^2 z^{(r,m)})$$

and

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} [z^{(r+1,m)} - z^{(r,m)}].$$
We will approximate classical solutions of (1.1), (1.2) with solutions of the difference functional equation
\[ \delta_0 z^{(r,m)} = F_{\text{ex},h}[z]^{(r,m)} \]
with the initial-boundary condition
\[ z(t^{(r)}, x) = \varphi_h(t, x) \quad \text{on } E_{0,h} \cup \partial_0 E_h \]
where \( \varphi_h : E_{0,h} \cup \partial_0 E_h \to \mathbb{R} \) is a given function. Problem (2.1), (2.2) is considered as an explicit difference scheme of Euler type for (1.1), (1.2).

Then we consider the case when
\[ \delta z = \delta_0 z^{(r,m)} = F_{\text{im},h}[z]^{(r,m)} \]
with the initial-boundary condition (2.2) is considered as an implicit difference scheme for (1.1), (1.2).

The above numerical methods have the following properties: the difference operators \( \delta z \) and \( \delta^{(2)} z \) are calculated at the point \( (t^{(r)}, x^{(m)}) \) in (2.1) and at \( (t^{(r+1)}, x^{(m)}) \) in (2.3). The function variable \( (T_h)[r,m] \) appears in a classical sense in both methods.

The definition of the difference operators
\[ \delta z = (\delta_1 z, \ldots, \delta_n z), \quad \delta^{(2)} z = [\delta_{ij} z]_{i,j=1}^n \]
falls naturally into two steps. In the first step we assume that \( x^{(m)} \in \text{Int } Q_h \). Then we consider the case when \( x^{(m)} \in \partial_0 Q_h \). For \( (t^{(r)}, x^{(m)}) \in \theta_h \times \text{Int } Q_h \) we write
\[ \delta^+ z^{(r,m)} = \frac{1}{h_i} [z^{(r,m+e_i)} - z^{(r,m)}], \quad \delta^- z^{(r,m)} = \frac{1}{h_i} [z^{(r,m)} - z^{(r,m-e_i)}], \quad 1 \leq i \leq n, \]
and
\[ \delta_i z^{(r,m)} = \frac{1}{2} [\delta^+ z^{(r,m)} + \delta^- z^{(r,m)}], \quad \delta_{ij} z^{(r,m)} = \delta^+ \delta^- z^{(r,m)}, \quad 1 \leq i \leq n. \]
The difference expressions \( \delta_{ij} z^{(r,m)} \) for \( (i, j) \in J \) are defined by
\[ \delta_{ij} z^{(r,m)} = \frac{1}{2} [\delta^+ \delta_j z^{(r,m)} + \delta^- \delta_j z^{(r,m)}] \quad \text{for } (i, j) \in J_-, \]
\[ \delta_{ij} z^{(r,m)} = \frac{1}{2} [\delta_i \delta^+ z^{(r,m)} + \delta_i \delta^- z^{(r,m)}] \quad \text{for } (i, j) \in J_+. \]
We now define the difference operators (2.4) for \( (t^{(r)}, x^{(m)}) \in \theta_h \times \partial_0 Q_h \). We put first, for \( 1 \leq i \leq n, \)
\[ \delta_i z^{(r,m)} = \frac{1}{h_i (\lambda_{i+} + \lambda_{i-})} [z^{(r,m+\lambda_{i+} e_i)} - z^{(r,m-\lambda_{i-} e_i)}], \]
For \((i, j) \in J_-\) we write

\[
\delta_{ij}z^{(r,m)} = \frac{1}{h_i h_j} \left\{ A_-^{(m)} z^{(r,m)} + B_-^{(m)} z^{(r,m+\lambda_+ e_i)} + C_-^{(m)} z^{(r,m-\lambda_- e_i)} + D_-^{(m)} z^{(r,m+\lambda_+ e_j)} + E_-^{(m)} z^{(r,m-\lambda_- e_j)} + F_-^{(m)} z^{(r,m-\lambda_- j (e_i-e_j))} + G_-^{(m)} z^{(r,m+\lambda_+ j (e_i-e_j))} \right\}
\]

where

\[
A_-^{(m)} = \frac{1}{\lambda_+ j_+ \lambda_+ j_-} - \frac{1}{\lambda_+ \lambda_- j_-} - \frac{1}{\lambda_+ \lambda_- j_+},
\]

\[
B_-^{(m)} = \frac{1}{\lambda_+ (\lambda_+ + \lambda_-)},
\]

\[
D_-^{(m)} = \frac{1}{\lambda_+ (\lambda_+ + \lambda_-)},
\]

\[
F_-^{(m)} = \frac{-1}{\lambda_- j_+ (\lambda_- j_+ + \lambda_- j_-)},
\]

\[
C_-^{(m)} = \frac{1}{\lambda_- (\lambda_+ + \lambda_-)},
\]

\[
E_-^{(m)} = \frac{1}{\lambda_- (\lambda_+ + \lambda_-)},
\]

\[
G_-^{(m)} = \frac{-1}{\lambda_- j_+ (\lambda_- j_+ + \lambda_- j_-)}.
\]

For \((i, j) \in J_+\) we write

\[
\delta_{ij}z^{(r,m)} = \frac{1}{h_i h_j} \left\{ A_+^{(m)} z^{(r,m)} + B_+^{(m)} z^{(r,m+\lambda_+ e_i)} + C_+^{(m)} z^{(r,m-\lambda_- e_i)} + D_+^{(m)} z^{(r,m+\lambda_+ e_j)} + E_+^{(m)} z^{(r,m-\lambda_- e_j)} + F_+^{(m)} z^{(r,m+\lambda_+ j (e_i+e_j))} + G_+^{(m)} z^{(r,m-\lambda_- j (e_i+e_j))} \right\}
\]

where

\[
A_+^{(m)} = \frac{-1}{\lambda_+ j_+ \lambda_+ j_-} + \frac{1}{\lambda_+ \lambda_- j_-} + \frac{1}{\lambda_+ \lambda_- j_+},
\]

\[
B_+^{(m)} = \frac{-1}{\lambda_+ (\lambda_+ + \lambda_-)},
\]

\[
D_+^{(m)} = \frac{-1}{\lambda_+ (\lambda_+ + \lambda_-)},
\]

\[
F_+^{(m)} = \frac{1}{\lambda_- j_+ (\lambda_- j_+ + \lambda_- j_-)},
\]

\[
C_+^{(m)} = \frac{-1}{\lambda_- (\lambda_+ + \lambda_-)},
\]

\[
E_+^{(m)} = \frac{-1}{\lambda_- (\lambda_+ + \lambda_-)},
\]

\[
G_+^{(m)} = \frac{1}{\lambda_- j_+ (\lambda_- j_+ + \lambda_- j_-)}.
\]

The vector \(\delta z^{(r,m)}\) and the matrix \(\delta^{(2)} z^{(r,m)}\) are defined by (2.5)–(2.11). The vector \(\delta z^{(r+1,m)}\) and the matrix \(\delta^{(2)} z^{(r+1,m)}\) appear in (2.3).
In the same way we define the difference expressions
\[
\delta \chi^{(m)} = (\delta_1 \chi^{(m)}, \ldots, \delta_n \chi^{(m)}) \quad \text{and} \quad \delta^{(2)} \chi^{(m)} = [\delta_{ij} \chi^{(m)}]_{i,j=1}^n
\]
where \( \chi : X_h \to \mathbb{R} \).

The above definitions have the following properties. Put \( \lambda_i = \lambda_{i+1} = 1 \) for \( i = 1, \ldots, n \) and \( \lambda_{i+j} = \lambda_{i+j+1} = 1 \) for \( (i, j) \in J_+ \) and \( \lambda_{i-j} = \lambda_{i-j+1} = 1 \) for \( (i, j) \in J_- \). Then the definitions (2.8)–(2.11) are equivalent to (2.5)–(2.7) respectively. It follows that we can use formulas (2.8)–(2.11) for all points \( (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h \). Note that the numbers \( z^{(r+1,m+\mu h_1+\nu h_2)} \) where \( \mu, \nu \in [-1,1] \) appear in (2.3). It follows that (2.3), (2.2) is an implicit difference method for (1.1), (1.2).

We will prove that under natural assumptions on the given functions and on the mesh there exists exactly one solution \( u_h : E_{0,h} \cup E_h \to \mathbb{R} \) to (2.1), (2.2) and there is exactly one solution \( v_h : E_{0,h} \cup E_h \to \mathbb{R} \) to (2.3), (2.2). Solutions of the above difference functional problems are approximate solutions to (1.1), (1.2). We give sufficient conditions for the convergence of the sequences of approximate solutions to a classical solution to (1.1), (1.2).


We first construct estimates for solutions to (1.1), (1.2). A function \( z : E_0 \cup E \to \mathbb{R} \) will be called of class \( C^{1,2} \) if \( z \in C(E_0 \cup E, \mathbb{R}) \) and \( z(\cdot, x) : [-b_0, a] \to \mathbb{R} \) is of class \( C^1 \) for \( x \in \overline{Q} \) and \( z(t, \cdot) : \overline{Q} \to \mathbb{R} \) is of class \( C^2 \) for \( t \in [-b_0, a] \). For \( z \in C(E_0 \cup E, \mathbb{R}) \), \( u \in F(E_{0,h} \cup E_h) \) we define the seminorms
\[
\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in E_0 \cup E, \tau \leq t\}, \quad 0 \leq t \leq a,
\]
\[
\|z\|_{h,r} = \max\{|u(\tau, x)| : (\tau, x) \in E_{0,h} \cup E_h, \tau \leq t^{(r)}\}, \quad 0 \leq r \leq K,
\]
For \( w \in C(B, \mathbb{R}) \) we put \( \|w\|_B = \max\{|w(\tau, y)| : (\tau, y) \in B\} \).

Assumption H0[\(F, \varphi\)]. The function \( F : \Omega \to \mathbb{R} \) of the variables \((t, x, w, q, s)\), where \( q = (q_1, \ldots, q_n) \), \( s = [s_{ij}]_{i,j=1}^n \), satisfies the conditions:

1) \( F \) is continuous and satisfies condition \((V)\),
2) the partial derivatives
\[
\partial_q F = (\partial_{q_1} F, \ldots, \partial_{q_n} F), \quad \partial_s F = [\partial_{s_{ij}} F]_{i,j=1}^n
\]
exist on \( \Omega \) and the functions \( \partial_q F : \Omega \to \mathbb{R}^n \), \( \partial_s F : \Omega \to M_{n \times n} \) are continuous and bounded,
3) the matrix \( \partial_s F \) is symmetric and
\[
(3.1) \quad \sum_{i,j=1}^n \partial_{s_{ij}} F(p) y_i y_j \geq 0 \quad \text{for} \quad y = (y_1, \ldots, y_n) \in \mathbb{R}^n,
\]
where \( P = (t, x, w, q, s) \in \Omega \),
4) there is \( \varphi : [0,a] \times \mathbb{R} \) such that
exists \[ \tilde{\varphi} \] Suppose for contradiction that (3.5) fails to be true. Then the set \( \omega \) is not empty. Write \( \omega = (0, ..., 0) \in \mathbb{R}^n \) and \( 0_{[n \times n]} \in M_{n \times n} \) is the zero matrix,

5) \( \varphi \in C(E_0 \cup \partial_0 E, \mathbb{R}) \) and \( \tilde{\eta} \in \mathbb{R}_+ \) is defined by the relations

\[ |\varphi(t, x)| \leq \tilde{\eta} \text{ on } E_0 \quad \text{and} \quad |\varphi(t, x)| \leq \omega(t, \tilde{\eta}) \text{ on } \partial_0 E \]

where \( \omega(\cdot, \tilde{\eta}) \) is the maximal solution of (3.2) with \( \eta = \tilde{\eta} \).

**Lemma 3.1.** If Assumption \( H_0[F, \varphi] \) is satisfied and \( \tilde{z} : E_0 \cup E \rightarrow \mathbb{R} \) is a solution to (1.1), (1.2) and \( \tilde{z} \) is of class \( C^{1,2} \) then

\[ |\tilde{z}(t, x)| \leq \omega(t, \tilde{\eta}) \text{ on } E. \]

**Proof.** For \( \varepsilon > 0 \) we denote by \( \omega(\cdot, \tilde{\eta}, \varepsilon) \) the maximal solution of the Cauchy problem

\[ \omega'(t) = g(t, \omega(t)), \quad \omega(0) = \tilde{\eta} + \varepsilon. \]

There exists \( \bar{\varepsilon} > 0 \) such that for \( 0 < \varepsilon < \bar{\varepsilon} \) the function \( \omega(\cdot, \tilde{\eta}, \varepsilon) \) is defined on \([0, a] \) and

\[ \lim_{\varepsilon \to 0} \omega(t, \tilde{\eta}, \varepsilon) = \omega(t, \tilde{\eta}) \quad \text{uniformly on } [0, a]. \]

Write \( \zeta(t) = \|\tilde{z}\|_t \) for \( t \in [0, a] \). We now prove that

\[ \zeta(t) < \omega(t, \tilde{\eta}, \varepsilon) \text{ for } t \in [0, a]. \]

Suppose for contradiction that (3.5) fails to be true. Then the set

\[ \Sigma_+ = \{ t \in [0, a] : \zeta \geq \omega(t, \tilde{\eta}, \varepsilon) \} \]

is not empty. Write \( \tilde{t} = \min \Sigma_+ \). From (3.3) it follows that \( \tilde{t} > 0 \) and there exists \( \tilde{x} \in \bar{Q} \) such that \( \omega(\tilde{t}, \tilde{\eta}, \varepsilon) = \zeta(\tilde{t}) = |\tilde{z}(\tilde{t}, \tilde{x})| \). The condition \( |\tilde{z}(t, x)| < \omega(t, \tilde{\eta}, \varepsilon) \) for \( (t, x) \in \partial_0 E \) implies that \( \tilde{x} \in Q \). Two cases are possible: either

(i) \( \tilde{z}(\tilde{t}, \tilde{x}) = \omega(\tilde{t}, \tilde{\eta}, \varepsilon) \) or (ii) \( \tilde{z}(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon) \). In the first case,

\[ D_+ \zeta(\tilde{t}) \geq \omega'(\tilde{t}, \tilde{\eta}, \varepsilon) \]

where \( D_+ \) is the left-hand lower Dini derivative. Write

\[ A(t, x) = F(t, x, \tilde{z}(t, x), \partial_x \tilde{z}(t, x), \partial_{xx} \tilde{z}(t, x)) - F(t, x, \tilde{z}(t, x), 0_{[n]}, 0_{[n \times n]}), \]

\[ B(t, x) = F(t, x, \tilde{z}(t, x), 0_{[n]}, 0_{[n \times n]}). \]
Then
\[
A(t, x) = \sum_{i=1}^{n} \frac{1}{0} \partial q_i F(P(\tau, t, x)) \, d\tau \partial x_i \tilde{z}(t, x)
+ \sum_{i,j=1}^{n} \frac{1}{0} \partial s_{ij} F(P(\tau, t, x)) \, d\tau \partial x_i x_j \tilde{z}(t, x)
\]
where \(P(\tau, t, x), 0 \leq \tau \leq 1,\) are intermediate points defined by the Hadamard mean value theorem. Since \(\tilde{x} \in Q,\) we have \(\partial x \tilde{z}(\tilde{t}, \tilde{x}) = 0\) and
\[
\sum_{i,j=1}^{n} \partial x_i x_j \tilde{z}(\tilde{t}, \tilde{x}) y_i y_j \leq 0 \quad \text{for } y = (y_1, \ldots, y_n) \in \mathbb{R}^n.
\]
The above relations and (3.1) imply
\[
\sum_{i,j=1}^{n} \frac{1}{0} \partial s_{ij} F(P(\tau, \tilde{t}, \tilde{x})) \, d\tau \partial x_i x_j \tilde{z}(\tilde{t}, \tilde{x}) \leq 0
\]
and consequently \(A(\tilde{t}, \tilde{x}) \leq 0.\) Then
\[
D_{-} \zeta(\tilde{t}) \leq \partial t \tilde{z}(\tilde{t}, \tilde{x}) = A(\tilde{t}, \tilde{x}) + B(\tilde{t}, \tilde{x}) \leq g(\tilde{t}, \omega(\tilde{t}, \tilde{\eta}, \varepsilon)) < \omega'(\tilde{t}, \tilde{\eta}, \varepsilon),
\]
which contradicts (3.6). The case \(\tilde{z}(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon)\) can be treated in a similar way. Hence, \(\Sigma_{+}\) is empty and inequality (3.5) is proved. Letting \(\varepsilon\) tend to 0 in (3.5) we obtain (3.4).

**Assumption H\([F, \varphi, \varphi_h]\).** The functions \(F : \Omega \rightarrow \mathbb{R}\) and \(\varphi : E_0 \cup \partial_0 E \rightarrow \mathbb{R}\) satisfy Assumption \(H_0\,[F, \varphi]\) and

1) for \(P = (t, x, w, q, s) \in \Omega\) we have
\[
(3.7) \quad \partial s_{ij} F(P) \geq 0 \text{ for } (i, j) \in J_+, \quad \partial s_{ij} F(P) \leq 0 \text{ for } (i, j) \in J_-,
\]
2) the steps of the mesh satisfy the conditions
\[
(3.8) \quad \frac{1}{h_i} \partial s_{ii} F(P) - \sum_{j=1}^{n} \frac{1}{h_j} \partial s_{ij} F(P) \leq 0,
\]
where \(P \in \Omega, \ i = 1, \ldots, n,\)
3) there is \(\alpha_0 : \Delta \rightarrow \mathbb{R}_+\) such that
\[
|\varphi(t, x) - \varphi_h(t, x)| \leq \alpha_0(h) \text{ on } E_0 \cup \partial_0 E \quad \text{and } \lim_{h \to 0} \alpha_0(h) = 0,
\]
4) the constant \(\tilde{\eta} \in \mathbb{R}_+\) is defined by the relations
\[
(3.9) \quad |\varphi_h(t, x)| \leq \tilde{\eta} \text{ on } E_{0,h} \quad \text{and } |\varphi_h(t, x)| \leq \omega_h(t, \tilde{\eta}) \text{ on } \partial_0 E_h
\]
where \(\omega(\cdot, \tilde{\eta})\) is the maximal solution to (3.2) with \(\eta = \tilde{\eta}.\)
Remark 3.2. We have assumed that the matrix \( \partial_s F \) satisfies the condition: for each \((i, j) \in J\) we have
\[
\partial_{s_{ij}} F(P) \geq 0 \text{ on } \Omega \quad \text{or} \quad \partial_{s_{ij}} F(P) \leq 0 \text{ on } \Omega.
\]
Conditions (3.7) can be considered as definitions of \( J_+ \) and \( J_- \).

Remark 3.3. Suppose that there is \( \bar{c} > 0 \) such that
\[
\partial_{s_{ii}} F(P) - \sum_{j=1}^{n} \left| \partial_{s_{ij}} F(P) \right| \geq \bar{c}, \quad P \in \Omega, \quad i = 1, \ldots, n.
\]
Then condition (3.1) is satisfied (see [28]) and there is \( \varepsilon_0 > 0 \) such that for \( \|h\| < \varepsilon_0 \) and for \( h_1 = \cdots = h_n \) inequalities (3.8) hold.

Assumption \( H[T_h] \). The operator \( T_h : F(E_{0,h} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R}) \) satisfies the conditions:

1) for \( z, \bar{z} \in F(E_{0,h} \cup E_h, \mathbb{R}) \) we have
\[
\| T_h[z] - T_h[\bar{z}] \|_{t(r)} \leq \| z - \bar{z} \|_{h,r}, \quad 0 \leq r \leq K,
\]
2) if \( z : E_0 \cup E \to \mathbb{R}_+ \) is of class \( C^{1,2} \) then there is \( \gamma_* : \Delta \to \mathbb{R}_+ \) such that
\[
\| T_h[z_h] - \bar{z} \|_t \leq \gamma_*(h), \quad 0 \leq t \leq a, \quad \lim_{h \to 0} \gamma_*(h) = 0,
\]
where \( z_h \) is the restriction of \( z \) to \( E_{0,h} \cup E_h \),
3) if \( 0_h \in F(E_{0,h} \cup E_h, \mathbb{R}) \) is given by \( 0_h(t, x) = 0 \) for \((t, x) \in E_{0,h} \cup E_h\) then \( T_h[0_h](t, x) = 0 \) for \((t, x) \in E_0 \cup E\).

Remark 3.4. If \( Q = (-c, c) \) where \((-c, c) \subset \mathbb{R}^n, c = (c_1, \ldots, c_n), c_i > 0\) for \( 1 \leq i \leq n \), then the interpolating operator \( T_h \) given in [17, Chapter VI] satisfies Assumption \( H[T_h] \). The construction of \( T_h \) presented in [17] can be extended to the set \( E_0 \cup E \) considered in this paper.

Suppose that Assumption \( H[F, \varphi, \varphi_h] \) is satisfied. For \( P \in \Omega \) we put
\[
X_0(P) = -2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \lambda_i - \lambda_i^+ \partial_{s_{ii}} F(P)
\]
\[
+ 2h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} \lambda_i - \lambda_i^+ \left| \partial_{s_{ij}} F(P) \right|
\]
\[
+ h_0 \sum_{(i,j) \in J_-} \frac{1}{h_i h_j} \lambda_i - \lambda_i^+ \lambda_i - \lambda_i^+ \partial_{s_{ij}} F(P)
\]
\[
- h_0 \sum_{(i,j) \in J_+} \frac{1}{h_i h_j} \lambda_i - \lambda_i^+ \lambda_i + \lambda_i^+ \partial_{s_{ij}} F(P),
\]
and
\[ X^{(i)}_+(P) = \frac{2h_0}{\lambda_{i+}(\lambda_{i-} + \lambda_{i+})} \left[ \frac{\lambda_{i+}}{2h_i} \partial_{q_i} F(P) \right. \]

\[ + \frac{1}{h_i^2} \partial_{s_{ji}} F(P) - \sum_{j=1}^{n} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)| \left] + \right. \]

\[ X^{(i)}_-(P) = \frac{2h_0}{\lambda_{i-}(\lambda_{i-} + \lambda_{i+})} \left[ -\frac{\lambda_{i-}}{2h_i} \partial_{q_i} F(P) \right. \]

\[ + \frac{1}{h_i^2} \partial_{s_{ji}} F(P) - \sum_{j=1}^{n} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)| \right], \]

where \( i = 1, \ldots, n \) and

\[ Y^{(i,j)}_+(P) = \frac{h_0}{h_i h_j} \frac{1}{\lambda_{i+} j_+} \frac{1}{\lambda_{i-} j_- + \lambda_{i+} j_+} \partial_{s_{ij}} F(P) \quad \text{for} \ (i, j) \in J_+, \]

\[ Z^{(i,j)}_+(P) = \frac{h_0}{h_i h_j} \frac{1}{\lambda_{i-} j_-} \frac{1}{\lambda_{i-} j_- + \lambda_{i+} j_+} \partial_{s_{ij}} F(P) \quad \text{for} \ (i, j) \in J_+, \]

\[ Y^{(i,j)}_-(P) = \frac{-h_0}{h_i h_j} \frac{1}{\lambda_{i-} j_-} \frac{1}{\lambda_{i-} j_- + \lambda_{i+} j_-} \partial_{s_{ij}} F(P) \quad \text{for} \ (i, j) \in J_-, \]

\[ Z^{(i,j)}_-(P) = \frac{-h_0}{h_i h_j} \frac{1}{\lambda_{i+} j_+} \frac{1}{\lambda_{i-} j_- + \lambda_{i+} j_-} \partial_{s_{ij}} F(P) \quad \text{for} \ (i, j) \in J_. \]

Suppose that \( \chi : X_h \rightarrow \mathbb{R} \) and \( x^{(m)} \in Q_h \). Write

\[ (3.11) \]

\[ \Theta^{(m)}_h[\chi, P] = \chi^{(m)} X_0(P) + \sum_{i=1}^{n} \left[ \chi^{(m+\lambda_i e_i)} X^{(i)}_+(P) + \chi^{(m-\lambda_i e_i)} X^{(i)}_-(P) \right] \]

\[ + \sum_{(i, j) \in J_+} \left[ \chi^{(m+\lambda_i j_+ (e_i e_j))} Y^{(i,j)}_+(P) + \chi^{(m-\lambda_i - j_+ (e_i e_j))} Z^{(i,j)}_+(P) \right] \]

\[ + \sum_{(i, j) \in J_-} \left[ \chi^{(m-\lambda_i j_- (e_i e_j))} Y^{(i,j)}_-(P) + \chi^{(m+\lambda_i - j_- (e_i e_j))} Z^{(i,j)}_-(P) \right] \]

where \( P = (t, x, w, q, s) \in \Omega \). Important properties of difference schemes are given in the next lemma.

**Lemma 3.5.** Suppose that Assumption \( H_0[F, \varphi] \) and conditions 1), 2) of Assumption \( H[F, \varphi, \varphi_h] \) are satisfied and \( \chi : X_h \rightarrow \mathbb{R}, x^{(m)} \in Q_h \) and

\[ G^{(m)}_h[\chi, P] = h_0 \sum_{i=1}^{n} \partial_{q_i} F(P) \delta_i \chi^{(m)} \]

\[ + h_0 \sum_{i, j=1}^{n} \partial_{s_{ij}} F(P) \delta_{ij} \chi^{(m)}, \quad P = (t, x, w, q, s) \in \Omega, \]
where $\delta \chi$ and $\delta^{(2)} \chi$ are given by (2.8)–(2.11). Then
\begin{equation}
G^{(m)}_h [\chi, P] = \Theta^{(m)}_h [\chi, P]
\end{equation}
and
\begin{equation}
X^{(i)}_+ (P) \geq 0, \quad X^{(i)}_- (P) \geq 0 \quad \text{for } i = 1, \ldots, n,
\end{equation}
\begin{equation}
Y^{(i,j)}_+ (P) \geq 0, \quad Z^{(i,j)}_+ (P) \geq 0 \quad \text{for } (i, j) \in J_+,
\end{equation}
\begin{equation}
Y^{(i,j)}_- (P) \geq 0, \quad Z^{(i,j)}_- (P) \geq 0 \quad \text{for } (i, j) \in J_-,
\end{equation}
and
\begin{equation}
X_0 (P) + \sum_{i=1}^n [X^{(i)}_+ (P) + X^{(i)}_- (P)] + \sum_{(i,j) \in J_+} [Y^{(i,j)}_+ (P) + Z^{(i,j)}_+ (P)] + \sum_{(i,j) \in J_-} [Y^{(i,j)}_- (P) + Z^{(i,j)}_- (P)] = 0.
\end{equation}

**Proof.** An easy computation shows that (3.12) is a consequence of (2.8)–(2.11). From (3.7), (3.8) we obtain (3.13)–(3.15). Condition (3.16) follows from the formulas for $X_0 (P), X^{(i)}_+ (P), X^{(i)}_- (P), Y^{(i,j)}_+ (P), Y^{(i,j)}_- (P), Z^{(i,j)}_+ (P), Z^{(i,j)}_- (P)$.

**Theorem 3.6.** Suppose that Assumptions $H[T_h]$ and $H[F, \varphi, \varphi_h]$ are satisfied.

**I.** There exists exactly one solution $v_h : E_{0,h} \cup E_h \to \mathbb{R}$ to (2.3), (2.2), and
\begin{equation}
|v_h (t, x)| \leq \omega (t, \bar{\eta}) \quad \text{for } (t, x) \in E_h,
\end{equation}
where $\omega (\cdot, \bar{\eta})$ is the maximal solution of the Cauchy problem (3.2) with $\eta = \bar{\eta}$ and $\bar{\eta}$ is defined by (3.9).

**II.** Assume additionally that the steps of the mesh satisfy the condition
\begin{equation}
1 + X_0 (P) \geq 0, \quad P = (t, x, w, q, s) \in \Omega,
\end{equation}
where $X_0 (P)$ is given by (3.10). Then there is exactly one solution $u_h : E_{0,h} \cup E_h \to \mathbb{R}$ to (2.1), (2.2), and
\begin{equation}
|u_h (t, x)| \leq \omega (t, \bar{\eta}) \quad \text{for } (t, x) \in E_h.
\end{equation}

**Proof.** The proof will be divided into two parts.

**I.** We prove that there exists exactly one solution $v_h : E_{0,h} \cup E_h \to \mathbb{R}$ to (2.3), (2.2). Suppose that $0 \leq r < K$ is fixed and that the solution $v_h$ to (2.3), (2.2) is given on $(E_{0,h} \cup E_h) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the values $v_h (t^{(r+1)}, x)$, $x \in X_h$, exist and they are unique. It is sufficient to show that there exists exactly one solution of the system of equations
\begin{equation}
z^{(r+1,m)} = v^{(r,m)} + h_0 F (t^{(r)}, x^{(m)}), (T_h v_h)_{[r,m]}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}
\end{equation}
where \( x^{(m)} \in Q_h \), and
\[
(3.21) \quad z(t^{(r+1)}, x) = \varphi_h(t^{(r+1)}, x) \text{ for } x \in S_h. 
\]
Set
\[
\Sigma_h = \{ \chi \in F(X_h, \mathbb{R}) : \chi(x) = \varphi(t^{(r+1)}, x) \text{ for } x \in S_h \}. 
\]
For \( \chi, \tilde{\chi} \in \Sigma_h \) we put
\[
||\chi - \tilde{\chi}|| = \max \{ ||(\chi - \tilde{\chi})(x) : x \in X_h \}. 
\]
It follows from Assumption \( H[F, \varphi, \varphi_h] \) that there is \( A_h \in \mathbb{R}_+ \) such that
\[
(3.22) \quad A_h + X_0(P) \geq 0 \quad \text{for } P = (t, x, w, q, s) \in \Omega, 
\]
where \( X_0(P) \) is given by (3.10). Let \( W_h \) be the operator defined on \( \Sigma_h \) by
\[
W_h[\chi]^{(m)} = \frac{1}{1 + A_h}[A_h \chi^{(m)} + v^{(r,m)} \]
\[
+ h_0 F(t^{(r)}, x^{(m)}, (T_h v_h)[r,m], \delta \chi^{(m)}, \delta^{(2)} \chi^{(m)})], \quad x^{(m)} \in Q_h, 
\]
where \( \delta \chi^{(m)} \) and \( \delta^{(2)} \chi^{(m)} \) are defined by (2.8)–(2.11) and
\[
W_h[\chi](x) = \varphi_h(t^{(r+1)}, x) \quad \text{for } x \in S_h. 
\]
Then \( W_h : \Sigma_h \to \Sigma_h \). It is clear that problem (3.20), (3.21) is equivalent to the equation
\[
(3.23) \quad \chi = W_h[\chi]. 
\]
We prove that
\[
(3.24) \quad ||W_h[\chi] - W_h[\tilde{\chi}]|| \leq \frac{A_h}{1 + A_h} ||\chi - \tilde{\chi}|| \quad \text{on} \quad \Sigma_h. 
\]
If \( x^{(m)} \in Q_h \) then there is \( P \in \Omega \) such that
\[
(1 + A_h)[W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}] 
\]
\[
= A_h(\chi - \tilde{\chi})^{(m)} + h_0 \sum_{i=1}^{n} \partial_{q_i} F(P) \delta_i(\chi - \tilde{\chi})^{(m)} + h_0 \sum_{i,j=1}^{n} \partial_{s_{ij}} F(P) \delta_{ij}(\chi - \tilde{\chi})^{(m)}. 
\]
We conclude from Lemma 3.5 that
\[
(1 + A_h)[W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}] = (A_h + X_0(P))(\chi - \tilde{\chi})^{(m)} 
\]
\[
+ \sum_{i=1}^{n} X_+^{(i)}(P)(\chi - \tilde{\chi})^{(m+\lambda_i+e_i)} + \sum_{i=1}^{n} X_-^{(i)}(P)(\chi - \tilde{\chi})^{(m-\lambda_i-e_i)} 
\]
\[
+ \sum_{(i,j) \in J_+} [Y_+^{(i,j)}(P)(\chi - \tilde{\chi})^{(m+\lambda_i+j+(e_i+e_j))} + Z_+^{(i,j)}(P)(\chi - \tilde{\chi})^{(m-\lambda_i-j-(e_i+e_j))}] 
\]
\[
+ \sum_{(i,j) \in J_-} [Y_-^{(i,j)}(P)(\chi - \tilde{\chi})^{(m-\lambda_i-j+(e_i-e_j))} + Z_-^{(i,j)}(P)(\chi - \tilde{\chi})^{(m+\lambda_i+j-(e_i-e_j))}]. 
\]
It follows from (3.13)–(3.16), (3.22) that
\[
|W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}| \leq \frac{A_h}{1 + A_h} |[\chi - \tilde{\chi}]|,
\]
x^{(m)} \in Q_h.

If \( x \in S_h \) then \( W_h[\chi](x) - W_h[\tilde{\chi}](x) = 0 \). The above relations imply (3.24). The Banach fixed point theorem implies that there exists exactly one solution to (3.23). It follows that the values \( v_h(t^{(r)}, x), x \in X_h \), exist and they are unique. The function \( v_h \) is given on \( E_{0,h} \). Then the proof of the existence and uniqueness of a solution to (2.3), (2.2) is completed by induction on \( r \), \( 0 \leq r \leq K \).

Now we prove (3.17). Write \( \omega_h^{(r)} = \|v_h\|_{h,r}, 0 \leq r \leq K \). It follows easily that
\[
\tilde{\omega}_{h}^{(r+1)} \leq \omega_{h}^{(r)} + h_0 g(t^{(r)}, \tilde{\omega}_{h}^{(r)}), \quad 0 \leq r \leq K - 1,
\]
and \( \tilde{\omega}_{h}^{(0)} \leq \bar{\eta} \). The function \( \omega(\cdot, \bar{\eta}) \) satisfies the recurrent inequality
\[
(3.25) \quad \omega(t^{(r+1)}, \bar{\eta}) \geq \omega(t^{(r)}, \bar{\eta}) + h_0 g(t^{(r)}, \omega(t^{(r)}, \bar{\eta})), \quad 0 \leq r \leq K - 1.
\]
From the initial inequality \( \omega_{h}^{(0)} \leq \omega(t^{(0)}, \bar{\eta}) \) we conclude that \( \tilde{\omega}_{h}^{(r)} \leq \omega(t^{(r)}, \bar{\eta}) \)
for \( 0 \leq r \leq K \) and (3.17) follows.

II. It is clear that there exists exactly one solution to (2.1), (2.2). We prove (3.19). It follows from (2.1) that
\[
(3.26) \quad u_{h}^{(r+1,m)} = h_0 F(t^{(r)}, x^{(m)}, (T_h u_h)^{[r,m]}, 0_{[n]}, 0_{[n \times n]}) + A(r,m)
\]
where
\[
A(r,m) = u_h^{(r,m)} + h_0 F(t^{(r)}, x^{(m)}, (T_h u_h)^{[r,m]}, \delta u_h^{(r,m)}, \delta^{(2)} u_h^{(r,m)})
\quad - F(t^{(r)}, x^{(m)}, (T_h u_h)^{[r,m]}, 0_{[n]}, 0_{[n \times n]})].
\]
We conclude from Assumption \( H[F, \varphi, \varphi_h] \) and from Lemma 3.5 that for each \( (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h \) there is \( P \in \Omega \) such that
\[
(3.27) \quad A(r,m) = u_h^{(r,m)} + h_0 \sum_{i=1}^{n} \partial_{t_{i}} F(P) \delta_{i} u_h^{(r,m)} + h_0 \sum_{i,j=1}^{n} \partial_{s_{ij}} F(P) \delta_{ij} u_h^{(r,m)}
\quad = u_h^{(r,m)} + \Theta_h^{(m)} [u_h(t^{(r)}, \cdot), P] = (1 + X_0(P)) u_h^{(r,m)}
\quad + \sum_{i=1}^{n} [X_{+}^{(i)}(P) u_h^{(r,m+\lambda_{i} + e_{i})} + X_{-}^{(i)}(P) u_h^{(r,m+\lambda_{i}-e_{i})}]
\quad + \sum_{(i,j) \in J_{+}} [Y_{+}^{(i,j)}(P) u_h^{(r,m+\lambda_{i}+j + (e_{i}+e_{j}))} + Z_{+}^{(i,j)}(P) u_h^{(r,m+\lambda_{i}-j - (e_{i}+e_{j}))}
\quad + \sum_{(i,j) \in J_{-}} [Y_{-}^{(i,j)}(P) u_h^{(r,m-\lambda_{i}-j + (e_{i}-e_{j}))} + Z_{-}^{(i,j)}(P) u_h^{(r,m+\lambda_{i}+j - (e_{i}-e_{j}))}].
\]
Write \( \omega_h^{(r)} = \| u_h \|_{h.r}, 0 \leq r \leq K \). It follows from (3.12)–(3.16) and from (3.18), (3.26), (3.27) that
\[
\omega_h^{(r+1)} \leq \omega_h^{(r)} + h_0 g(t^{(r)}, \omega_h^{(r)}), \quad 0 \leq r \leq K - 1,
\]
and \( \omega_h^{(0)} \leq \bar{\eta} \). The above relations and (3.25) imply (3.19). This completes the proof of the theorem. ■

4. Convergence of difference schemes. For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( W \in M_{n \times n}, W = [w_{ij}]_{i,j=1}^n \) we put
\[
\| x \| = \sum_{i=1}^n |x_i|, \quad \| W \| = \max \left\{ \sum_{j=1}^n |w_{ij}| : 1 \leq i \leq n \right\}.
\]
Write \( \eta_* = \max\{\bar{\eta}, \bar{\eta} \} \) and \( C = \omega(a, \eta_*) \) where \( \omega(\cdot, \eta_*) \) is the maximal solution to (3.2) with \( \eta = \eta_* \). Set \( \Omega[C] = \{ (t, x, w, q, s) : (t, x, w, q, s) \in \Omega : \| w \|_B \leq C \} \).

**Assumption \( H[F, \sigma] \).** There is \( \sigma : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that
1) \( \sigma \) is continuous and it is nondecreasing with respect to both variables,
2) \( \sigma(t, 0) = 0 \) for \( t \in [0, a] \) and the function \( \bar{\omega}(t) = 0 \) for \( t \in [0, a] \) is the maximal solution of the Cauchy problem
\[
\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = 0,
\]
3) the estimate
\[
|F(t, x, w, q, s) - F(t, x, \bar{w}, q, s)| \leq \sigma(t, \| w - \bar{w} \|_B)
\]
is satisfied on \( \Omega[C] \).

**Remark 4.1.** It is important that we have assumed condition (4.2) for \( \| w \|_B, \| \bar{w} \|_B \leq C \). There are differential equations with deviated variables and differential integral equations such that Assumption \( H[F, \sigma] \) holds and condition (4.2) is not satisfied on \( \Omega \). We will give suitable examples.

Suppose that \( \tilde{F} : E \times \mathbb{R}^n \times M_{n \times n} \rightarrow \mathbb{R} \) and \( \tilde{L} : E \rightarrow \mathbb{R}, \phi : E \rightarrow \mathbb{R}^1 \times n, \phi = (\phi_0, \phi_1, \ldots, \phi_n) \), are given functions. We assume that \( \tilde{F} \) and \( \tilde{L} \), \( \phi \) are continuous and \( \phi_0(t, x) \leq t \) and \( \phi(t, x) \in E \) for \( (t, x) \in E \). Then \( \phi(t, x) - (t, x) \in B \) for \( (t, x) \in E \).

Suppose that the function \( G : \mathbb{R} \rightarrow \mathbb{R} \) satisfies the conditions:
(i) \( G \) is of class \( C^1 \) on \( \mathbb{R} \) and there is \( \bar{C} \in \mathbb{R}_+ \) such that \( |G(p)| \leq \bar{C} \) for \( p \in \mathbb{R} \),
(ii) the function \( \tilde{G}(p) = pG'(p), p \in \mathbb{R} \), is unbounded on \( \mathbb{R} \).

Let \( F : \Omega \rightarrow \mathbb{R} \) be defined by
\[
F(t, x, w, q, s) = \tilde{L}(t, x)w(\phi(t, x) - (t, x))G(w(\phi(t, x) - (t, x))) + \tilde{F}(t, x, q, s).
\]
Then (1.1) reduces to the differential equation with deviated variables
\[ \partial_t z(t, x) = \tilde{L}(t, x)z(\phi(t, x))G(z(\phi(t, x))) + \tilde{F}(t, x, \partial_x z(t, x), \partial_{xx} z(t, x)). \]

Suppose that \( \tilde{\beta}, \tilde{\gamma} > 0 \) are such that
\[ |\tilde{L}(t, x)| \leq \tilde{\beta}, \quad |\tilde{F}(t, x, 0_{[n]}, 0_{[n \times n]})| \leq \tilde{\gamma} \quad \text{for} \ (t, x) \in E. \]

Then
\[ |F(t, x, w, 0_{[n]}, 0_{[n \times n]})| \leq \tilde{\beta} \|w\|_B + \tilde{\gamma} \]
where \((t, x, w) \in E \times C(B, \mathbb{R})\) and the solution of (3.2) is given by
\[ \tilde{\omega}(t, \eta) = \eta \exp[\frac{\tilde{\beta} t}{\tilde{\beta}}] + \exp[\frac{\tilde{\beta} t}{\tilde{\beta}} - 1] \tilde{\beta} \tilde{\gamma}. \]

It follows that the function \( F \) given by (4.3) satisfies the Lipschitz condition with respect to the function variable on \( \Omega[C] \) where \( C = \tilde{\omega}(a, \eta) \) and the global Lipschitz condition is not satisfied.

Now we construct an integral functional equation. For the above \( G : \mathbb{R} \rightarrow \mathbb{R} \) and \( \tilde{F} : E \times \mathbb{R}^n \times M_{n \times n} \rightarrow \mathbb{R} \) we define
\[ F(t, x, w, q, s) = \tilde{L}(t, x) \int_{D[t, x]} w(\tau, y) \, dy \, d\tau \, G\left( \int_{D[t, x]} w(\tau, y) \, dy \, d\tau \right) \]
+ \( \tilde{F}(t, x, q, s) \).

Then (1.1) reduces to the integral differential equation
\[ \partial_t z(t, x) = \tilde{L}(t, x) \int_{D[t, x]} z(\tau, y) \, dy \, d\tau \, G\left( \int_{D[t, x]} z(\tau, y) \, dy \, d\tau \right) \]
+ \( \tilde{F}(t, x, \partial_x z(t, x), \partial_{xx} z(t, x)) \).

It is clear that there are \( \tilde{\beta}, \tilde{\alpha} > 0 \) such that the function \( F \) defined by (4.5) satisfies condition (4.4) where \((t, x, w) \in E \times C(B, \mathbb{R})\). Then there is \( L \geq 0 \) such that Assumption \( H[F, \sigma] \) holds for \( \sigma(t, p) = Lp \) and the global Lipschitz condition is not satisfied.

Note that the function
\[ G(p) = C_0 \sin(C_1 p) + C_2 \cos(C_3 p), \]
where \( C_0, C_1, C_2, C_3 \in \mathbb{R} \), satisfies the above conditions (i), (ii).

**Lemma 4.2.** If \( z : E_0 \cup E \rightarrow \mathbb{R} \) is of class \( C^{1,2} \) then there is \( \alpha_* : \Delta \rightarrow \mathbb{R}_+ \) such that
\[ \left\| \delta z^{(r, m)} - \partial_x z^{(r, m)} \right\| \leq \alpha_*(h), \quad \left\| \delta^{(2)} z^{(r, m)} - \partial_{xx} z^{(r, m)} \right\| \leq \alpha_*(h) \]
where \((t^{(r)}, x^{(m)}) \in \theta_h \times Q_h, \ h \in \Delta, \) and \( \lim_{h \rightarrow 0} \alpha_*(h) = 0. \)

We omit a simple proof of the lemma. We now give sufficient conditions for the convergence of difference schemes (2.1), (2.2) and (2.3), (2.2).
THEOREM 4.3. Suppose that Assumptions \( H[T_h], H[F, \varphi, \varphi_h], H[F, \sigma] \) are satisfied and \( \tilde{z} : E_0 \cup E \to \mathbb{R} \) is a solution to (1.1), (1.2) and \( \tilde{z} \) is of class \( C^{1,2} \).

I. There is \( \alpha : \Delta \to \mathbb{R}_+ \) such that
\[
| (\tilde{z} - v_h) (t, x) | \leq \alpha (h) \text{ on } E_h \quad \text{and} \quad \lim_{h \to 0} \alpha (h) = 0,
\]
where \( v_h : E_{0,h} \cup E_h \to \mathbb{R} \) is a solution to (2.3), (2.2) and \( \tilde{z} \) is the restriction of \( \tilde{z} \) to \( E_{0,h} \cup E_h \).

II. Assume that the steps of the mesh satisfy the condition (3.18) where \( X_0 (P) \) is given by (3.10). Then there is \( \alpha : \Delta \to \mathbb{R}_+ \) such that
\[
| (\tilde{z} - u_h) (t, x) | \leq \alpha (h) \text{ on } E_h \quad \text{and} \quad \lim_{h \to 0} \alpha (h) = 0,
\]
where \( u_h : E_{0,h} \cup E_h \to \mathbb{R} \) is a solution to (2.1), (2.2).

Proof. The proof will be divided into two parts.

I. The existence and uniqueness of the solution \( v_h : E_{0,h} \cup E_h \to \mathbb{R} \) to (2.3), (2.2) follows from Theorem 3.6. Let \( \Gamma^{im,h}_r : E^r_h \to \mathbb{R} \) be defined by
\[
\delta_0 h^r_m = R^{im,h} [\tilde{z}(r,m)] + \Gamma^{(r,m)}_{im,h} \text{ on } E^r_h.
\]
It follows from Lemma 4.2 and from Assumption \( H[T_h] \) that there is \( \gamma : \Delta \to \mathbb{R}_+ \) such that
\[
| \Gamma^{(r,m)}_{im,h} | \leq \gamma (h) \text{ on } E^r_h \quad \text{and} \quad \lim_{h \to 0} \gamma (h) = 0.
\]
Write \( \vartheta_h = \tilde{z} - v_h \). Then
\[
\vartheta_h^{(r+1,m)} = \vartheta_h^{(r,m)} + A_h [\tilde{z}_h, v_h]^{(r,m)} + B_h [\tilde{z}_h, v_h]^{(r,m)}
\]
where
\[
A_h [\tilde{z}_h, v_h]^{(r,m)} = h_0 [R^{im,h} [\tilde{z}_h]^{(r,m)} - F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta z^{(r+1,m)}, \delta z^{(r+1,m)})] + \Gamma^{(r,m)}_{im,h},
\]
\[
B_h [\tilde{z}_h, v_h]^{(r,m)} = h_0 [F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta z^{(r+1,m)}, \delta z^{(r+1,m)}) - R^{im,h} [v_h]^{(r,m)}]).
\]
According to Assumption \( H[F, \varphi, \varphi_h] \) we have
\[
B_h [\tilde{z}_h, v_h]^{(r,m)} = h_0 \sum_{i=1}^{n} \partial_q, F(P) \delta_i [\vartheta_h]^{(r+1,m)} + h_0 \sum_{i,j=1}^{n} \partial_4 F(P) \delta_i [\vartheta_h]^{(r+1,m)}
\]
where \( P \in \Omega \) is an intermediate point. We conclude from Lemma 3.5 that
\[
B_h [\tilde{z}_h, v_h]^{(r,m)} = \Theta_h^{(m)} [\vartheta (t^{(r+1)}), \cdot, P].
\]
Write \( \varepsilon_h^{(r)} = || \vartheta_h ||_{h,r}, 0 \leq r \leq K \). It follows from Assumption \( H[T_h] \) and Lemma 3.1 that
According to Assumption $H[T_h]$ and (3.17) we have

\[ \| (T_h \tilde{z}_h)_{[r,m]} \|_B \leq \| T_h \tilde{z}_h \|_{t(r)} \leq \| \tilde{z}_h \|_{h,r} \leq \omega(t^{(r)}, \bar{\eta}), \quad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h. \]

Thus we see that

\[ \| (T_h \tilde{z}_h)_{[r,m]} \|_B \leq C, \quad \| (T_h v_h)_{[r,m]} \|_B \leq C, \quad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h. \]

It follows from the above estimates and Assumption $H[F, \sigma]$ that

\[ |A_h[\tilde{z}_h, v_h]^{(r,m)}| \leq h_0 \sigma(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h). \]

We conclude from (3.13)–(3.16) and from (4.8)–(4.11) that the function $\varepsilon_h$ satisfies the recurrent inequality

\[ \varepsilon_h^{(r+1)} \leq \varepsilon_h^{(r)} + h_0 \sigma(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h), \quad 0 \leq r \leq K - 1, \]

and $\varepsilon_h^{(0)} \leq \alpha_0(h)$. Let us denote by $\omega_h(\cdot, \gamma, \alpha_0)$ the maximal solution of the Cauchy problem

\[ \omega'(t) = \sigma(t, \omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h). \]

Then

\[ \lim_{h \to 0} \omega_h(t, \gamma, \alpha_0) = 0 \quad \text{uniformly on } [0, a] \]

and

\[ \omega_h(t^{(r+1)}, \gamma, \alpha_0) \geq \omega_h(t^{(r)}, \gamma, \alpha_0) \]

\[ + h_0 \sigma(t^{(r)}, \omega_h(t^{(r)}, \gamma, \alpha_0)) + h_0 \gamma(h), \quad 0 \leq r \leq K - 1. \]

This gives $\varepsilon_h(t^{(r)}) \leq \omega_h(t^{(r)}, \gamma, \alpha_0)$ for $0 \leq r \leq K$. Thus we see that assertion (4.6) is satisfied with $\alpha(h) = \omega_h(a, \gamma, \alpha_0)$.

II. We prove (4.7). Let $\Gamma_{ex,h} : E_h' \to \mathbb{R}$ be defined by the relation

\[ \delta_0 \tilde{z}_h^{(r,m)} = F_{ex,h}[\tilde{z}_h]^{(r,m)} + \Gamma_{ex,h}^{(r,m)} \quad \text{on } E_h'. \]

It follows that there is $\gamma : \Delta \to \mathbb{R}_+$ such that

\[ |\Gamma_{ex,h}^{(r,m)}| \leq \gamma(h) \quad \text{on } E_h', \quad \lim_{h \to 0} \gamma(h) = 0. \]

Write $\tilde{\vartheta}_h = \tilde{z}_h - u_h$. Then we have

\[ \tilde{\vartheta}_h^{(r+1,m)} = \tilde{\vartheta}_h^{(r,m)} + A_h[\tilde{z}_h, u_h]^{(r,m)} + B_h[\tilde{z}_h, u_h]^{(r,m)} \]

where

\[ A_h[\tilde{z}_h, v_h]^{(r,m)} = h_0[F_{ex,h}][\tilde{z}_h]^{(r,m)} \]

\[ - F(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]}, \delta \tilde{z}_h^{(r,m)}, \delta(2) \tilde{z}_h^{(r,m)}) + \Gamma_{ex,h}^{(r,m)}, \]

\[ B_h[\tilde{z}_h, u_h]^{(r,m)} = h_0[F(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]}, \delta \tilde{z}_h^{(r,m)}, \delta(2) \tilde{z}_h^{(r,m)}) - F_{ex,h}[v_h]^{(r,m)}]. \]
It follows from Lemma 3.5 that

\[(4.14) \quad \tilde{\vartheta}_h^{(r,m)} + \tilde{B}_h[\tilde{z}_h, u_h]^{(r,m)} = \tilde{\vartheta}_h^{(r,m)} + \Theta_h^{(m)}[\tilde{\vartheta}(t^{(r)}), \cdot, P]\]

\[
= (1 + X_0(P))\tilde{\vartheta}_h^{(r,m)} + \sum_{i=1}^{n} [X_+(P) \tilde{\vartheta}_h^{(r,m+\lambda_i+e_i)} + X_-(P) \tilde{\vartheta}_h^{(r,m-\lambda_i-e_i)}] \\
+ \sum_{(i,j) \in J_+} [Y_+(P) \tilde{\vartheta}_h^{(r,m+\lambda_{i+j}+\epsilon_i+\epsilon_j)} + Z_+(P) \tilde{\vartheta}_h^{(r,m-\lambda_{i-j}-(\epsilon_i+\epsilon_j))}] \\
+ \sum_{(i,j) \in J_-} [Y_-(P) \tilde{\vartheta}_h^{(r,m-\lambda_{i+j}-(\epsilon_i-\epsilon_j))} + Z_-(P) \tilde{\vartheta}_h^{(r,m+\lambda_{i-j}-(\epsilon_i-\epsilon_j))}] \\
\]

where \(P \in \Omega\) is an intermediate point. Write \(\tilde{\varepsilon}_h^{(r)} = \|\tilde{\vartheta}_h\|_{h,r}, 0 \leq r \leq K\). We conclude from Assumption \(H[T_h]\) and (3.19) that

\[\|(T_h u_h)[r,m]\|_{B} \leq \|(T_h u_h)\|_{t^{(r)}} \leq \|u_h\|_{h,r} \leq \omega(t^{(r)}, \tilde{\eta}), \quad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h.\]

The above relations and (4.10) imply

\[\|(T_h \tilde{z}_h)[r,m]\|_{B} \leq C, \quad \|(T_h u_h)[r,m]\|_{B} \leq C, \quad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h.\]

It follows from the above estimates and Assumption \(H[F, \sigma]\) that

\[(4.15) \quad |\tilde{A}_h[\tilde{z}_h, u_h]^{(r,m)}| \leq h_0 \sigma(t^{(r)}, \tilde{\varepsilon}_h^{(r)}) + h_0 \gamma(h).\]

We conclude from (3.13)-(3.16) and from (3.18), (4.13)-(4.15) that

\[|\tilde{\vartheta}_h^{(r,m)} + \tilde{B}_h[\tilde{z}_h, u_h]^{(r,m)}| \leq \tilde{\varepsilon}_h^{(r)} .\]

Thus we see that the function \(\tilde{\varepsilon}_h\) satisfies the recurrent inequality

\[\tilde{\varepsilon}_h^{(r+1)} \leq \tilde{\varepsilon}_h^{(r)} + h_0 \sigma(t^{(r)}, \tilde{\varepsilon}_h^{(r)}) + h_0 \gamma(h), \quad 0 \leq r \leq K - 1,\]

and \(\tilde{\varepsilon}_h^{(0)} \leq \alpha_0(h)\). Then we obtain (4.7) with \(a(h) = \omega_h(a, \gamma, \alpha_0)\) where \(\omega_h(\cdot, \gamma, \alpha_0)\) is the maximal solution to (4.12) with the above given \(\gamma\). This completes the proof of the theorem. ■

**Remark 4.4.** Relations between \(h_0\) and \(h'\) are required in (3.18). Suppose that the steps \((h_1, \ldots, h_n)\) are given and we have constructed the mesh \(X_h\) on \(\overline{Q}\), the coefficients

\[\lambda_{i-}, \lambda_{i+} \quad \text{for} \quad i = 1, \ldots, n, \quad \lambda_{i-j+}, \lambda_{i+j-} \quad \text{for} \quad (i, j) \in J_-,
\]

\[\lambda_{i-j-}, \lambda_{i+j+} \quad \text{for} \quad (i, j) \in J_+\]

are given and the function

\[\tilde{X}_0(P) = \frac{1}{h_0} X_0(P), \quad P = (t, x, w, q, s) \in \Omega,\]

is bounded. It follows that there is \(\varepsilon_0 > 0\) such that condition (3.18) is satisfied for \(0 < h_0 < \varepsilon_0\).
Remark 4.5. Condition (3.18) shows that the conditions on the mesh for explicit difference schemes are more restrictive than the suitable assumptions for implicit methods.

Remark 4.6. Note that assumption (3.18) for \( (t(r), x(m)) \in \theta_h \times \text{Int } Q_h, (w, q, s) \in C(B, \mathbb{R}) \times \mathbb{R}^n \times M_{n \times n} \) is equivalent to the following inequality:

\[
1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{sii} F(P) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |\partial_{sij} F(P)| \geq 0.
\]

The conditions (3.18) and (4.16) are complicated because we consider functional differential equations with all the derivatives \( \partial_{x_i x_j} z[i,j=1] \). Let us consider the equation

\[
\partial_t z(t, x) = \sum_{i=1}^{n} \partial_{x_i x_i} z(t, x) + f(t, x, z(t,x), \partial_x z(t, x))
\]

where \( f : E \times C(B, \mathbb{R}) \times \mathbb{R}^n \) is a given function. Then condition (3.18) has the form

\[
1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \frac{1}{\lambda_i - \lambda_{i+}} \geq 0.
\]

Lemma 4.7. Suppose that Assumptions \( H[T_h], H[F, \varphi, \varphi_h] \), \( H[F, \sigma] \) are satisfied with \( \sigma(t, p) = Lp \) on \([0, a] \times \mathbb{R}_+ \) and \( \tilde{z} : E_0 \cup E \to \mathbb{R} \) is a solution to (1.1), (1.2) and \( \tilde{z} \) is of class \( C^{1,2} \). Then

\[
|\tilde{z}_h(t, x) - v_h(t, x)| \leq \tilde{\alpha}(h) \quad \text{on } E_h
\]

where \( v_h : E_{0,h} \cup E_h \to \mathbb{R} \) is a solution to (2.3), (2.2) and \( \tilde{z}_h \) is the restriction of \( \tilde{z} \) to \( E_{0,h} \cup E_h \) and

\[
\tilde{\alpha}(h) = \begin{cases} 
\alpha_0(h) e^{La} + \frac{\gamma(h)}{L} (e^{La} - 1) & \text{if } L > 0, \\
\alpha_0(h) e^{La} + a\gamma(h) & \text{if } L = 0.
\end{cases}
\]

If we assume that the steps of the mesh satisfy condition (3.18) then

\[
|\tilde{z}_h(t, x) - u_h(t, x)| \leq \tilde{\alpha}(h) \quad \text{on } E_h
\]

with the above given \( \tilde{\alpha}(h) \) where \( u_h : E_{0,h} \cup E_h \to \mathbb{R} \) is a solution to (2.1), (2.2).

We obtain the above estimates by solving problem (4.12) with \( \sigma(t, p) = Lp \).

Lemma 4.7 shows that we have obtained the same error estimates for implicit and for explicit difference schemes.

Remark 4.8. The results presented in the paper can be extended to weakly coupled functional differential systems.
5. Numerical examples. Write

\[ Q = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \]

\[ E = [0, 0.25] \times Q, \quad E_0 = \{0\} \times Q, \quad \partial_0 E = [0, 0.25] \times \partial Q. \]

We consider initial-boundary value problems for functional differential equations with solutions defined on \( E \). Let us denote by \( z \) an unknown function of the variables \((t, x, y)\).

Implicit difference methods lead to nonlinear systems of algebraic equations. In our experiments we have obtained approximate solutions of suitable nonlinear systems by using the Newton method. We have calculated three Newton iterations.

**Example 5.1.** Consider the differential integral equation

\[
\partial_t z(t, x, y) = 2\partial_{xx}z(t, x, y) + 2\partial_{yy}z(t, x, y) \\
+ \sin[\partial_{xx}z(t, x, y) - \partial_{yy}z(t, x, y) - 4t^2(x^2 - y^2)z(t, x, y)] + \partial_{xy}z(t, x, y) \\
+ t \int_x^y s_z(t, s, y) ds - t \int_x^y s_z(t, x, s) ds \\
- z(t, x, y) \sin z(t, x, y) + f(t, x, y)z(t, x, y) + g(t, x, y)
\]

with the initial-boundary condition

\[ z(t, x, y) = 1 \quad \text{for} \quad (t, x, y) \in E_0 \cup \partial_0 E. \]

where

\[
f(t, x, y) = 1 - x^2 - y^2 + 8t - 4xyt^2 - 8t^2(x^2 + y^2) \\
+ \sin \exp\{t(1 - x^2 - y^2)\},
\]

\[
g(t, x, y) = \frac{1}{2} \exp[t(1 - x^2)] - \frac{1}{2} \exp[t(1 - y^2)].
\]

The solution of the above problem is known: it is

\[ \tilde{z}(t, x, y) = \exp\{t(1 - x^2 - y^2)\}. \]

The following tables show the maximal values of errors for several step sizes.

<table>
<thead>
<tr>
<th>Table 1. Explicit difference method</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_1 = h_2 )</td>
</tr>
<tr>
<td>( 2^{-1} \cdot 10^{-1} )</td>
</tr>
<tr>
<td>( 4 \cdot 10^{-2} )</td>
</tr>
<tr>
<td>( 2^{-5} )</td>
</tr>
</tbody>
</table>

Now we consider the implicit difference schemes with steps of the mesh given in Table 2.
Table 2. Explicit difference method, condition \([3.18]\) violated

<table>
<thead>
<tr>
<th>(h_1 = h_2)</th>
<th>(h_0)</th>
<th>Maximal error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^{-1} 10^{-1})</td>
<td>(10^{-3})</td>
<td>(1.373362 \cdot 10^{11})</td>
</tr>
<tr>
<td>(2^{2} 10^{-2})</td>
<td>(10^{-3})</td>
<td>(5.287499 \cdot 10^{13})</td>
</tr>
<tr>
<td>(2^{-5})</td>
<td>(10^{-3})</td>
<td>(3.140010 \cdot 10^{16})</td>
</tr>
</tbody>
</table>

Table 3. Implicit difference method

<table>
<thead>
<tr>
<th>(h_1 = h_2)</th>
<th>(h_0)</th>
<th>Maximal error</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^{-1} 10^{-1})</td>
<td>(10^{-3})</td>
<td>(9.113438 \cdot 10^{-4})</td>
<td>18 s</td>
</tr>
<tr>
<td>(2^{2} 10^{-2})</td>
<td>(10^{-3})</td>
<td>(7.441051 \cdot 10^{-4})</td>
<td>28 s</td>
</tr>
<tr>
<td>(2^{-5})</td>
<td>(10^{-3})</td>
<td>(5.913871 \cdot 10^{-4})</td>
<td>50 s</td>
</tr>
</tbody>
</table>

Example 5.2. Let us consider the differential equation with deviated variables

\[
\partial_t z(t, x, y) = 2 \partial_{xx} z(t, x, y) + 2 \partial_{yy} z(t, x, y) - \partial_{xy} z(t, x, y) \\
- \cos[\partial_{xx} z(t, x, y) - \partial_{yy} z(t, x, y)] \\
+ z(t, \frac{1}{2}(\sqrt{3}x + y), \frac{1}{2}(x - \sqrt{3}y)) \cos z(t, \frac{1}{2}(x + \sqrt{3}y), \frac{1}{2}(|\sqrt{3}x - y|)) \\
+ f(t, x, y)z(t, x, y) + g(t, x, y),
\]

with the initial-boundary condition

\[z(t, x, y) = 1 \quad \text{for} \quad (t, x, y) \in E_0 \cup \partial_0 E,\]

where

\[
f(t, x, y) = x^2 - y^2 - 1 - 8t - 8t^2(x^2 + y^2) + 4xyt^2 - \cos \exp\{t(x^2 + y^2 - 1)\},
\]

\[g(t, x, y) = \cos\{4t^2(x^2 - y^2)\} \exp[t(x^2 + y^2 - 1)].\]

The solution of the above problem is \(\tilde{z}(t, x, y) = e^{t(x^2+y^2-1)}\).

Table 4. Explicit difference method

<table>
<thead>
<tr>
<th>(h_1 = h_2)</th>
<th>(h_0)</th>
<th>Maximal error</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>(10^{-4})</td>
<td>(1.227446 \cdot 10^{-4})</td>
<td>1 min</td>
</tr>
<tr>
<td>(2^{-1} 10^{-1})</td>
<td>(10^{-5})</td>
<td>(8.719288 \cdot 10^{-5})</td>
<td>3 min</td>
</tr>
<tr>
<td>(2^{-2} 10^{-1})</td>
<td>(2^{-5} 10^{-4})</td>
<td>(4.751054 \cdot 10^{-5})</td>
<td>20 min</td>
</tr>
</tbody>
</table>

Table 5. Explicit difference method, condition \([3.18]\) violated

<table>
<thead>
<tr>
<th>(h_1 = h_2)</th>
<th>(h_0)</th>
<th>Maximal error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^{-1})</td>
<td>(2 \cdot 10^{-3})</td>
<td>(1.312591 \cdot 10^{-4})</td>
</tr>
<tr>
<td>(2^{-1} 10^{-1})</td>
<td>(10^{-4})</td>
<td>(1.952779 \cdot 10^{0})</td>
</tr>
<tr>
<td>(2^{-2} 10^{-1})</td>
<td>(10^{-4})</td>
<td>(1.026864 \cdot 10^{9})</td>
</tr>
</tbody>
</table>
Now we consider implicit difference schemes with steps of the mesh given in Table 5.

Table 6. Implicit difference method

<table>
<thead>
<tr>
<th>$h_1 = h_2$</th>
<th>$h_0$</th>
<th>Maximal error</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>$2 \cdot 10^{-3}$</td>
<td>$1.325853 \cdot 10^{-4}$</td>
<td>2 s.</td>
</tr>
<tr>
<td>$2^{-1} 10^{-1}$</td>
<td>$10^{-4}$</td>
<td>$8.771281 \cdot 10^{-5}$</td>
<td>18 s.</td>
</tr>
<tr>
<td>$2^{-2} 10^{-1}$</td>
<td>$10^{-4}$</td>
<td>$4.757541 \cdot 10^{-5}$</td>
<td>76 s.</td>
</tr>
</tbody>
</table>

Remark 5.3. Note that the right hand sides of the equations considered in this section satisfy the assumptions of Theorem 4.3. The local Lipschitz condition with respect to the function variable holds and the global Lipschitz condition is not satisfied.

Our considerations show that there are the following relations between explicit and implicit difference methods for (1.1), (1.2). Assumptions on the regularity of given functions are the same in the theorems on the convergence of explicit and implicit difference schemes. We need condition (3.18) on the mesh for explicit difference methods, while this condition is not needed in the case of implicit difference methods. Error estimates are the same for both methods. Tables 2, 3 and 5, 6 show that there are implicit difference methods which are convergent, while the corresponding explicit difference schemes are not.

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References


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