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An extension theorem with analytic singularities for generalized (N, k)-crosses

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Abstract. The main result of the paper is a new Hartogs type extension theorem for generalized (N, k)-crosses with analytic singularities for separately holomorphic functions and for separately meromorphic functions. Our result is a simultaneous generalization of several known results, from the classical cross theorem, through the extension theorem with analytic singularities for generalized crosses, to the cross theorem with analytic singularities for meromorphic functions.

1. Introduction. In a recent paper [JP5], Jarnicki and Pflug introduced the so-called (N, k)-crosses and gave a new Hartogs type extension theorem which generalizes the classical cross theorem (see [AZ], [NZ]). We shall consider more general objects: generalized (N, k)-crosses, and prove for them an extension theorem with analytic singularities (for definitions and the statement of the main result see Section 2), a simultaneous generalization of the extension theorem for generalized crosses with analytic singularities (see [JP2]) and the cross theorem with analytic singularities for meromorphic functions (see [JP6, Theorem 11.2.1], see also [JP4]), which in turn is, in some sense, a generalization of the Rothstein theorem (see [R]). In particular, the main theorem from [JP5] is contained in our result.

The paper is organized as follows. Section 2 brings the definition of generalized (N, k)-cross and the statement of the main result. We write down some results about crosses and (N, k)-crosses there, as well. Section 3 contains some useful facts. Finally, in Sections 4 and 5, we give the proof of the main result.

The natural objects treated in this article are Riemann regions. For a background on this topic see [JP1]. In the present paper $\mathcal{PLP}(X)$ stands for the family of all pluripolar sets of a Riemann region (X, p); furthermore, $\mathcal{O}(X)$ is the space of all holomorphic functions on X, and $\mathcal{M}(X)$ is the space

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of all meromorphic functions on X. Moreover, for an $f \in \mathcal{M}(X)$ we denote by $\mathcal{S}(f)$ its singular set (as is well known, it is either empty or an analytic set of pure codimension one).

2. Generalized (N,k)-crosses and the statement of the main result. Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $\emptyset \neq A_j \subset D_j$ for $j = 1, \dots, N$, $N \geq 2$. For $k \in \{1, \dots, N\}$ let $I(N,k) := \{\alpha = (\alpha_1, \dots, \alpha_N) \in \{0,1\}^N : |\alpha| = k\}$, where $|\alpha| := \alpha_1 + \dots + \alpha_N$. Put

$$\mathcal{X}_{\alpha,j} := \begin{cases} D_j & \text{if } \alpha_j = 1, \\ A_j & \text{if } \alpha_j = 0, \end{cases} \qquad \mathcal{X}_\alpha := \prod_{j=1}^N \mathcal{X}_{\alpha,j}.$$

For $\alpha \in I(N,k)$ such that $\alpha_{r_1} = \dots = \alpha_{r_k} = 1$, $\alpha_{i_1} = \dots = \alpha_{i_{N-k}} = 0$, where $r_1 < \dots < r_k$ and $i_1 < \dots < i_{N-k}$, put

$$D_\alpha := \prod_{s=1}^k D_{r_s}, \qquad A_\alpha := \prod_{s=1}^{N-k} A_{i_s}.$$

For $a = (a_1, \dots, a_n) \in \mathcal{X}_\alpha$, with α as above, put $a_\alpha^0 := (a_{i_1}, \dots, a_{i_{N-k}}) \in A_\alpha$. Analogously, define $a_\alpha^1 := (a_{r_1}, \dots, a_{r_k}) \in D_\alpha$. For every $\alpha \in I(N,k)$ and every $a = (a_{i_1}, \dots, a_{i_{N-k}}) \in A_\alpha$ define

$$\boldsymbol{i}_{a,\alpha} = (\boldsymbol{i}_{a,\alpha,1}, \dots, \boldsymbol{i}_{a,\alpha,N}) : D_\alpha \to \mathcal{X}_\alpha$$

by

$$\boldsymbol{i}_{a,\alpha,j}(z) := \begin{cases} z_j & \text{if } \alpha_j = 1, \\ a_j & \text{if } \alpha_j = 0, \end{cases} \qquad j = 1, \dots, N,\ z = (z_{r_1}, \dots, z_{r_k}) \in D_\alpha$$

(if $\alpha_j = 0$, then $j \in \{i_1, \dots, i_{N-k}\}$, and if $\alpha_j = 1$, then $j \in \{r_1, \dots, r_k\}$). Similarly, for any $\alpha \in I(N,k)$ and any $b = (b_{r_1}, \dots, b_{r_k}) \in D_\alpha$ define

$$\boldsymbol{l}_{b,\alpha} = (\boldsymbol{l}_{b,\alpha,1}, \dots, \boldsymbol{l}_{b,\alpha,N}) : A_\alpha \to \mathcal{X}_\alpha,$$

$$\boldsymbol{l}_{b,\alpha,j}(z) := \begin{cases} z_j & \text{if } \alpha_j = 0, \\ b_j & \text{if } \alpha_j = 1, \end{cases} \qquad j = 1, \dots, N,\ z = (z_{i_1}, \dots, z_{i_{N-k}}) \in A_\alpha.$$

Definition 2.1. For any $\alpha \in I(N,k)$ let $\Sigma_\alpha \subset A_\alpha$. We define a *generalized (N,k)-cross*

$$\mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in I(N,k)}) = \bigcup_{\alpha \in I(N,k)} \{a \in \mathcal{X}_\alpha : a_\alpha^0 \notin \Sigma_\alpha\}$$

and its *center*

$$\mathfrak{C}(\mathbf{T}_{N,k}) := \mathbf{T}_{N,k} \cap (A_1 \times \dots \times A_N).$$

It is straightforward that

$$\mathfrak{C}(\mathbf{T}_{N,k}) = (A_1 \times \dots \times A_N) \setminus \bigcap_{\alpha \in I(N,k)} \{z \in A_1 \times \dots \times A_N : z_\alpha^0 \in \Sigma_\alpha\},$$

which implies that $\mathfrak{C}(\mathbf{T}_{N,k})$ is non-pluripolar provided that $A_1 \times \cdots \times A_N$ is non-pluripolar and at least one of the Σ_α's is pluripolar (see Proposition 3.2).

Observe that in the case $k = 1$ the set $I(N,k)$ consists of exactly N elements and in this situation we shall use the more convenient notation

$$\Sigma_j := \Sigma_{(\underbrace{0,\dots,0}_{j-1},1,\underbrace{0,\dots,0}_{N-j})}, \qquad j = 1,\dots N.$$

Furthermore, note that if in the definition of generalized (N,k)-cross we take $k = 1$ and $\Sigma_1 = \cdots = \Sigma_N = \emptyset$, then we obtain a *classical N-fold cross* (see [JP3])

$$\mathbf{X} := \mathbb{X}((A_j, D_j)_{j=1}^N) = \mathbb{T}_{N,1}((A_j, D_j)_{j=1}^N, (\emptyset)_{j=1}^N).$$

If we take $k = 1$ (and any Σ_j, $j = 1,\dots,N$), then we get a *generalized N-fold cross* (again see [JP3])

$$\mathbf{T} := \mathbf{T}_{N,1} = \mathbb{T}_{N,1}((A_j, D_j)_{j=1}^N, (\Sigma_j)_{j=1}^N).$$

Finally note that if we take any $k \in \{1,\dots,N\}$ and $\Sigma_\alpha = \emptyset$ for every $\alpha \in I(N,k)$, then we get the (N,k)-*cross* (see [JP5])

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\emptyset)_{\alpha \in I(N,k)}).$$

Definition 2.2 (see [JP6, Chapter 2]). We say that a Riemann region (X,p) over $\mathbb{C}^n$ is *relatively compact* if there exists a Riemann region (X',p') over $\mathbb{C}^n$ such that X is a relatively compact open set in X' and $p = p'|_X$.

Definition 2.3 (see [JP6, Chapter 3]). Let (X,p) be a Riemann region over $\mathbb{C}^n$ and let $A \subset X$. The *relative extremal function of A with respect to X* is the upper semicontinuous regularization $h^\star_{A,X}$ of the function

$$h_{A,X} := \sup\{u : u \in \mathcal{PSH}(X), u \le 1, u|_A \le 0\}.$$

For an open set $Y \subset X$ we put $h_{A,Y} := h_{A\cap Y,Y}$, $h^\star_{A,Y} := h^\star_{A\cap Y,Y}$.

Remark 2.4.

(a) If Y is a connected component of X, then $h_{A,X} = h_{A,Y}$ and $h^\star_{A,X} = h^\star_{A,Y}$ on Y.
(b) $h^\star_{A,X} \in \mathcal{PSH}(X)$.
(c) If $Y_1 \subset Y_2 \subset X$ are open, $A_1 \subset Y_1$ and $A_1 \subset A_2 \subset Y_2$, then $h_{A_2,Y_2} \le h_{A_1,Y_1}$ and $h^\star_{A_2,Y_2} \le h^\star_{A_1,Y_1}$ on Y_1.

Definition 2.5 (see [JP6, Chapter 3]). We say that a set $A \subset X$ is *pluriregular at a point $a \in \overline{A}$* if $h^\star_{A,U}(a) = 0$ for any open neighborhood U of a. Define

$$A^\star = A^{\star,X} := \{a \in \overline{A} : A \text{ is pluriregular at } a\}.$$

We say that A is *locally pluriregular* if $A \neq \emptyset$ and A is pluriregular at each of its points, i.e. $\emptyset \neq A \subset A^\star$.

REMARK 2.6.

(a) If $\emptyset \neq Y \subset X$ is open, then it is locally pluriregular.
(b) If $\emptyset \neq B \subset A \subset X$ and $B \subset Y$ with Y open, then $B^{\star,Y} \subset A^{\star,X} \cap Y = (A \cap Y)^{\star,Y}$.
(c) $h^{\star}_{A,X} = 0$ on $A^{\star}$.
(d) If A is locally pluriregular, then $h^{\star}_{A,X} \equiv h_{A,X} \equiv h^{\star}_{A^{\star},X} \equiv h_{A^{\star},X}$.
(e) $A \setminus A^{\star}$ is a pluripolar set.
(f) $(A \setminus P)^{\star} = A^{\star}$ for any pluripolar set P.
(g) If $A \subset X$ and $B \subset Y$ are locally pluriregular, then so is $A \times B$.
(h) If X is relatively compact (see Definition 2.2), then a set $P \subset X$ is pluripolar iff $h^{\star}_{P,X} \equiv 1$.

DEFINITION 2.7 ([JP5]). Let

$$\widehat{\mathbf{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^{N}) := \Big\{(z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : \sum_{j=1}^{N} h^{\star}_{A_j,D_j}(z_j) < k\Big\}.$$

Note the obvious inclusion $\widehat{\mathbf{X}}_{N,k-1} \subset \widehat{\mathbf{X}}_{N,k}$.

DEFINITION 2.8. Let M be any subset of $\mathbf{T}_{N,k}$. Then for any $\alpha \in I(N,k)$ and any $a \in A_\alpha$ define the *fiber of M at a* to be

$$M_a := \{z \in D_\alpha : \boldsymbol{i}_{a,\alpha}(z) \in M\}.$$

Similarly, for any $b \in D_\alpha$ define the *fiber of M at b* to be

$$M^b := \{z \in A_\alpha : \boldsymbol{l}_{b,\alpha}(z) \in M\}.$$

DEFINITION 2.9. For a relatively closed set $M \subset \mathbf{T}_{N,k}$ (we allow $M = \emptyset$ here) we say that a function $f : \mathbf{T}_{N,k} \setminus M \to \mathbb{C}$ is *separately holomorphic on* $\mathbf{T}_{N,k} \setminus M$ if for every $\alpha \in I(N,k)$ and for every $a \in A_\alpha \setminus \Sigma_\alpha$ the function

$$D_\alpha \setminus M_a \ni z \mapsto f(\boldsymbol{i}_{a,\alpha}(z))$$

is holomorphic. In this case we write $f \in \mathcal{O}_s(\mathbf{T}_{N,k} \setminus M)$.

We denote by $\mathcal{O}^c_s(\mathbf{T}_{N,k} \setminus M)$ the space of all $f \in \mathcal{O}_s(\mathbf{T}_{N,k} \setminus M)$ such that for any $\alpha \in I(N,k)$ and for every $b \in D_\alpha$ the function

$$A_\alpha \setminus (\Sigma_\alpha \cup M^b) \ni z \mapsto f(\boldsymbol{l}_{b,\alpha}(z))$$

is continuous.

DEFINITION 2.10 (see [JP1, Section 3.4]). Let (X,p) be a Riemann region over $\mathbb{C}^n$, let M be a closed subset of X such that

(2.1) for any domain $D \subset X$ the set $D \setminus M$ is connected and dense in X,

and let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X \setminus M)$. We say that a point $a \in M$ is *non-singular with respect to $\mathcal{F}$* (written $a \in M_{ns,\mathcal{F}}$) if there exists an open neighborhood U of

a such that for each $f \in \mathcal{F}$ there exists a function $\widetilde{f} \in \mathcal{O}(U)$ with $\widetilde{f} = f$ on $U \setminus M$.

If $a \in M_{s,\mathcal{F}} := M \setminus M_{ns,\mathcal{F}}$, then we say that a is *singular with respect to* $\mathcal{F}$. If $M_{s,\mathcal{F}} = M$, then we say that M is *singular with respect to* $\mathcal{F}$.

Remark 2.11.

(a) The set $M_{s,\mathcal{F}}$ is closed in M.
(b) For any function $f \in \mathcal{F}$ there exists a holomorphic extension $\widetilde{f} \in \mathcal{O}(X \setminus M_{s,\mathcal{F}})$.
(c) $M_{s,\mathcal{F}}$ is singular with respect to the family $\widetilde{\mathcal{F}} := \{\widetilde{f} : f \in \mathcal{F}\}$.
(d) If $M \neq \emptyset$ is an analytic, singular set, then M is of pure codimension one (cf. [C, Appendix I]).
(e) Let $M \subset X$ be an analytic set of pure dimension $n-1$, and let $M = \bigcup_{i \in I} M_i$ be the decomposition of M into irreducible components (cf. [C, Section 5.4]). Then

$$M_{s,\mathcal{F}} = \bigcup_{i:\, M_i \subset M_{s\mathcal{F}}} M_i = \bigcup_{i:\, M_i \cap \operatorname{Reg} M \cap M_{s,\mathcal{F}} \neq \emptyset} M_i$$

In particular, the set $M_{s,\mathcal{F}}$ is also analytic (cf. [JP1, Proposition 3.4.5]).

Now we are prepared to state the main result.

Theorem 2.12 (Extension theorem for generalized (N,k)-crosses with analytic singularities). *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular for any $j = 1, \dots, N$, let $\Sigma_\alpha \subset \Sigma_\alpha^0 \subset A_\alpha$ with Σ_α^0 pluripolar for any $\alpha \in I(N,k)$. Let $\mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in I(N,k)})$, $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N)$. Let F be an analytic subset of $\widehat{\mathbf{X}}_{N,k}$ with* $\operatorname{codim} F \geq 1$, *and let G be an analytic subset of $\widehat{\mathbf{X}}_{N,k}$ with* $\operatorname{codim} G \geq 1$ *such that $F \subset G$. Assume that for $\alpha \in I(N,k)$ and $a \in A_\alpha \setminus \Sigma_\alpha^0$ the fiber $(\mathbf{T}_{N,k} \cap G)_a$ is thin. Let $\mathcal{G} \subset \mathcal{O}_s^c(\mathbf{T}_{N,k} \setminus G)$ be such that for any $f \in \mathcal{G}, \alpha \in I(N,k)$ and $a \in A_\alpha \setminus \Sigma_\alpha^0$ the function $f_{a,\alpha} := f \circ \boldsymbol{i}_{a,\alpha}$ extends meromorphically to $D_\alpha \setminus (\mathbf{T}_{N,k} \cap F)_a$, i.e. there exists a function $\widetilde{f}_{a,\alpha} \in \mathcal{M}(D_\alpha \setminus (\mathbf{T}_{N,k} \cap F)_a)$ such that $\mathcal{S}(\widetilde{f}_{a,\alpha}) \subset (\mathbf{T}_{N,k} \cap G)_a \setminus (\mathbf{T}_{N,k} \cap F)_a$ and $\widetilde{f}_{a,\alpha} = f_{a,\alpha}$ on $D_\alpha \setminus (\mathbf{T}_{N,k} \cap G)_a$. Denote by $\widehat{F}$ the union of all irreducible components of F of codimension one. Then for any $f \in \mathcal{G}$ there exists an $\widetilde{f} \in \mathcal{M}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{F})$ such that $\mathcal{S}(\widetilde{f}) \cap \mathbf{T}_{N,k} \subset G$ and $\widetilde{f} = f$ on $\mathbf{T}_{N,k} \setminus G$.*

Moreover, if $F = \emptyset$ and $\widehat{G}$ is the union of all irreducible components of G of codimension one, then:

- *for any $f \in \mathcal{F} := \mathcal{O}_s^c(\mathbf{T}_{N,k} \setminus G)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{G})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k} \setminus G$,*
- *$\widehat{G}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,*

- *$\widehat{f}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{G}) \subset f(\mathbf{T}_{N,k} \setminus G)$ for any $f \in \mathcal{F}$ (the Σ_α^0's are redundant here).*

If in addition $G = \emptyset$, then for every $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k}$ and $\widehat{f}(\widehat{\mathbf{X}}_{N,k}) \subset f(\mathbf{T}_{N,k})$. In this case the assumption that the D_j's are domains of holomorphy is not necessary.

Observe that Theorem 2.12 generalizes the following results:

- (Take $G = \emptyset$, $k = 1$, $\Sigma_1 = \cdots = \Sigma_N = \emptyset$)

Theorem 2.13 (Main cross theorem, [AZ], [NZ]). *Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$, let $A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$. Put $\mathbf{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$. Let $f \in \mathcal{O}_s(\mathbf{X})$. Then there exists a uniquely determined $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}})$ such that $\widehat{f} = f$ on $\mathbf{X}$.*

- (Take $G = \emptyset$, $\Sigma_\alpha = \emptyset$, $\alpha \in I(N, k)$)

Theorem 2.14 (Extension theorem for (N, k)-crosses, [JP5]). *For every $f \in \mathcal{O}_s(\mathbf{X}_{N,k})$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{X}_{N,k}$ and $\widehat{f}(\widehat{\mathbf{X}}_{N,k}) \subset f(\mathbf{X}_{N,k})$.*

Theorems 2.13 and 2.14 are formulated for $\mathcal{O}_s$, while there is $\mathcal{O}_s^c$ in the assumptions of Theorem 2.12. However, using argument similar to the one given in Section 4 below, one can show that Theorem 2.12 in the case of $G = \emptyset$ and empty Σ_α's holds true for $\mathcal{F} = \mathcal{O}_s(\mathbf{T}_{N,k})$.

- (Take $G = \emptyset$, $k = 1$)

Theorem 2.15 (Extension theorem for generalized crosses, [JP3]). *Assume that D_j is a Riemann domain over $\mathbb{C}^{n_j}$, and $A_j \subset D_j$ is locally pluriregular. Assume additionally that $\Sigma_j \subset A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times \cdots \times A_N$ is pluripolar for $j = 1, \ldots, N$. Let $\mathbf{X}$ and $\mathbf{T}$ be as in Definition 2.1. Then for every $f \in \mathcal{O}_s^c(\mathbf{T})$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}})$ such that $\widehat{f} = f$ on $\mathbf{T}$.*

- (Take $F = \emptyset, k = 1$)

Theorem 2.16 (Extension theorem for generalized crosses with analytic singularities, [JP2]). *Assume that D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A_1 \times \cdots \times A_{j-1} \times A_{j+1} \times \cdots \times A_N$ is pluripolar for $j = 1, \ldots, N$. Let $\mathbf{X}$ and $\mathbf{T}$ be as in Definition 2.1. Let $S := \mathbf{T} \cap G$, where G is an analytic subset of an open neighborhood $U \subset \widehat{\mathbf{X}}$ of $\mathbf{T}$ with $\operatorname{codim} G \geq 1$. Let*

$$\mathcal{F} := \begin{cases} \mathcal{O}_s(\mathbf{X} \setminus S) & \text{if } \Sigma_1 = \cdots = \Sigma_N = \emptyset, \\ \mathcal{O}_s^c(\mathbf{T} \setminus S) & \text{otherwise.} \end{cases}$$

Then there exist an analytic set $\widehat{S} \subset \widehat{\mathbf{X}}$ and an open neighborhood $U_0 \subset U$ of $\mathbf{T}$ such that:

- $\widehat{S} \cap U_0 \subset G$,
- *for any* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}} \setminus \widehat{S})$ *such that* $\widehat{f} = f$ *on* $\mathbf{T} \setminus S$,
- $\widehat{S}$ *is singular with respect to the family* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if* $U = \widehat{\mathbf{X}}$, *then* $\widehat{S}$ *is the union of all components of* G *of codimension one.*
- (Take $k = 1$)

THEOREM 2.17 (Cross theorem with singularities for meromorphic functions, [JP6, Theorem 11.2.1], see also [JP4]). *Let* $D_j, A_j, \Sigma_j, \mathbf{X}, \mathbf{T}$ *be as in Theorem* 2.16, $j = 1, \ldots, N$. *Let* $\Sigma_j \subset \Sigma_j^0 \subset A_1 \times \ldots \times A_{j-1} \times A_{j+1} \times \ldots \times A_N$ *be pluripolar,* $j = 1, \ldots, N$. *Let* $M \subset S \subset \mathbf{T}$ *be relatively closed and assume that for every* $j \in \{1, \ldots, N\}$ *and every* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j^0 := ((A_1 \times \cdots \times A_{j-1}) \times (A_{j+1} \times \cdots \times A_N)) \setminus \Sigma_j^0$, *the fiber* $S_{(a'_j, \cdot, a''_j)}$ *is thin. Let*

$$\mathcal{G} \subset \begin{cases} \mathcal{O}_s(\mathbf{X} \setminus S) & \text{if } \Sigma_1 = \cdots = \Sigma_N = \emptyset, \\ \mathcal{O}_s^c(\mathbf{T} \setminus S) & \text{otherwise} \end{cases}$$

be such that for any $f \in \mathcal{G}$, $j \in \{1, \ldots, N\}$, *and* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j^0$, *the function* $f(a'_j, \cdot, a''_j)$ *extends meromorphically to* $D_j \setminus M_{(a'_j, \cdot, a''_j)}$, *i.e. there exists a function* $\widetilde{f}_{j,(a'_j, a''_j)} \in \mathcal{M}(D_j \setminus M_{(a'_j, \cdot, a''_j)})$ *with* $\mathcal{S}(\widetilde{f}_{j,(a'_j, a''_j)}) \subset S_{(a'_j, \cdot, a''_j)} \setminus M_{(a'_j, \cdot, a''_j)}$ *and* $\widetilde{f}_{j,(a'_j, a''_j)} = f(a'_j, \cdot, a''_j)$ *on* $D_j \setminus S_{(a'_j, \cdot, a''_j)}$. *Let* $\widehat{M} \subset \widehat{\mathbf{X}}$ *be constructed via Theorem* 10.2.9 *from* [JP6] *with respect to the family*

$$\mathcal{F} := \begin{cases} \mathcal{O}_s(\mathbf{X} \setminus M) & \text{if } \Sigma_1 = \cdots = \Sigma_N = \emptyset, \\ \mathcal{O}_s^c(\mathbf{T} \setminus M) & \text{otherwise.} \end{cases}$$

Then there exists a generalized cross $\mathbf{T}' := \mathbb{T}_{N,1}((A_j, D_j)_{j=1}^N, (\Sigma'_j)_{j=1}^N)$, *with* $\Sigma_j^0 \subset \Sigma'_j$ *and* Σ'_j *pluripolar,* $j = 1, \ldots, N$, *such that for any* $f \in \mathcal{G}$ *there exists an* $\widetilde{f} \in \mathcal{M}(\widehat{\mathbf{X}} \setminus \widehat{M})$ *with:*

- $\widehat{M} \cap \mathbf{T}' \subset M$,
- $\mathcal{S}(\widetilde{f}) \cap \mathbf{T}' \subset S$,
- $\widetilde{f} = f$ *on* $\mathbf{T}' \setminus S$.

REMARK 2.18. Note that if we have $M = \mathbf{T} \cap F$, where F is an analytic subset of an open neighborhood U_F of $\mathbf{T}$ contained in $\widehat{\mathbf{X}}$, then Theorem 2.17 holds true with $\widehat{M}$ constructed via Theorem 2.16 (with the data $(G, S) = (F, M)$).

3. Prerequisites. In this section we recall some definitions and results which will be needed later.

LEMMA 3.1 ([JP6, Lemma 2.1.14]). *Let* K *be a Riemann domain, let* $L \subset K$ *be a subdomain, and let* $A_0 \subset A \subset K$ *be such that* $A_0 \subset L$. *Assume that* A_0 *is not analytically thin at a point* $a_0 \in L \cap \overline{A_0}$ (*see Definition* 1.4.3

in [JP6]). *For a family* $\mathcal{F}$ *of functions* $f : A \to \mathbb{C}$ *consider the following conditions:*

(i) *for every* $f \in \mathcal{F}$ *there exists an* $\widehat{f} \in \mathcal{O}(L)$ *such that* $\widehat{f} = f$ *on* A_0 ($\widehat{f}$ *is uniquely determined*),

(ii) *for every* $f \in \mathcal{F}$ *and for every* $a \in \mathbb{C} \setminus f(A)$ *the function* $1/(f-a)$ *belongs to* $\mathcal{F}$,

(iii) *for every* $f \in \mathcal{F}$ *and for every* $a \in \mathbb{C}$ *with* $|a| > \|f\|_A$ *the function* $1/(f-a)$ *belongs to* $\mathcal{F}$.

Then:

(a) *If* (i) *and* (ii) *are satisfied, then* $\widehat{f}(L) \subset f(A)$ *for every* $f \in \mathcal{F}$ (*in particular,* $\|\widehat{f}\|_D \le \|f\|_A$ *for every* $f \in \mathcal{F}$).

(b) *If* (i) *and* (iii) *are satisfied, then* $\|\widehat{f}\|_L \le \|f\|_A$ *for every* $f \in \mathcal{F}$.

Proposition 3.2 ([JP6, Proposition 2.3.31]). *Let* $(X,p),(Y,q)$ *be Riemann regions over* $\mathbb{C}^n, \mathbb{C}^m$, *respectively, countable at infinity. Then:*

(a) *If* $A \subset X \times Y$ *is pluripolar, then*

$$\{z \in X : A_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \in \mathcal{PLP}(X),$$

where

$$A_{(z,\cdot)} := \{w \in Y : (z,w) \in A\}.$$

(b) *Let* $Q \subset X \times Y$ *be such that* $Q_{(a,\cdot)} \in \mathcal{PLP}(Y)$, $a \in X$. *Let* $C \subset X \times Y$ *be such that*

$$\{z \in X : C_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \notin \mathcal{PLP}(X).$$

Then $C \setminus Q \notin \mathcal{PLP}(X \times Y)$.

Lemma 3.3 ([JP6, Example 3.2.6]). *Let* $\varphi : (X,p) \to (\widehat{X}, \widehat{p})$ *be the maximal holomorphic extension and let* u *be a plurisubharmonic function on* $\widehat{X}$ *such that* $u \le C$ *on* $\varphi(X)$. *Then* $u \le C$ *on* $\widehat{X}$.

Proposition 3.4 ([JP6, Proposition 3.2.25]). *Let* $X_k \nearrow X$ *with* X_k *open for every* $k \in \mathbb{N}$, *and* $A_k \subset X_k$ *with* $A_k \nearrow A$. *Then* $h^{\star}_{A_k^{\star},X_k} \searrow h^{\star}_{A^{\star},X}$.

Proposition 3.5 ([JP6, Proposition 3.2.27]). *Let* X *be a relatively compact Riemann domain. Let* $A \subset X$ *be non-pluripolar and let* $0 < \varepsilon < 1$. *Put*

$$\varDelta(\varepsilon) := \{z \in X : h^{\star}_{A,X}(z) < \varepsilon\}.$$

Then $h^{\star}_{A,\varDelta(\varepsilon)} = (1/\varepsilon)h^{\star}_{A,X}$ *on* $\varDelta(\varepsilon)$. *In particular,* $h^{\star}_{A,\varDelta(\varepsilon)}(z) < 1$, $z \in \varDelta(\varepsilon)$, *which implies that* $A \cap S$ *is non-pluripolar for any connected component* S *of* $\varDelta(\varepsilon)$.

In the case where A *is locally pluriregular, we do not need to assume that* X *is relatively compact.*

LEMMA 3.6 ([JP5]). *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$. Then*

$$h^{\star}_{\widehat{\mathbf{X}}_{N,k-1},\widehat{\mathbf{X}}_{N,k}}(z) = \max\Big\{0, \sum_{j=1}^{N} h^{\star}_{A_j,D_j}(z_j) - k + 1\Big\},$$
$$z = (z_1, \ldots, z_N) \in \widehat{\mathbf{X}}_{N,k}.$$

4. Proof of Theorem 2.12 in the case $F = \emptyset$. Recall that here $\widehat{G}$ equals the union of all irreducible components of G of codimension one.

Proof of Theorem 2.12 in the case $F = \emptyset$. Observe that the inclusion $\widehat{f}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{G}) \subset f(\mathbf{T}_{N,k} \setminus G)$ for every $f \in \mathcal{F}$ is a consequence of Lemma 3.1 with

$$(K, L, A_0, A, \mathcal{F}) = (\widehat{\mathbf{X}}_{N,k} \setminus \widehat{G}, \widehat{\mathbf{X}}_{N,k} \setminus \widehat{G}, \mathbf{T}_{N,k} \setminus G, \mathbf{T}_{N,k} \setminus G, \mathcal{F}).$$

We divide the proof into three steps.

STEP 1: We prove the theorem in the case $G = \emptyset$ (recall that here we do not assume that the D_j's are domains of holomorphy).

In view of Proposition 7.2.6 in [JP6], we may assume that each D_j is a Riemann domain of holomorphy. Indeed, note that by virtue of the continuity of the operation of extension, the class $\mathcal{O}^c_s$ is stable under taking the envelope of holomorphy. Furthermore, using Proposition 3.4 we may assume that D_j is relatively compact (cf. Definition 2.2) and $A_j \subset\subset D_j$ for $j = 1, \ldots, N$.

We apply induction on N. There is nothing to prove in the case $N = k$. Moreover, the case $k = 1$ is solved by Theorem 2.15. Thus, the conclusion holds true for $N = 2$. Suppose it holds true for $N - 1 \geq 2$. Now, we apply induction on k. For $k = 1$ this is just Theorem 2.15. Suppose that the conclusion is true for $k - 1$ with $2 \leq k \leq N - 1$.

Fix an $f \in \mathcal{F}$. Define

$$Q := Q_N = \{z_N \in A_N : \exists \alpha \in I_0(N, k) : (\Sigma_\alpha)_{(\cdot, z_N)} \notin \mathcal{PLP}\},$$

where $I_0(N, k) := I(N, k) \cap \{\alpha : \alpha_N = 0\}$. Proposition 3.2 implies that $Q \in \mathcal{PLP}$. For an $a_N \in A_N \setminus Q$ put

$$\mathbf{T}_{N-1,k}(a_N) := \mathbb{T}_{N-1,k}((A_j, D_j)_{j=1}^{N-1}, ((\Sigma_{(\beta,0)})_{(\cdot,a_N)})_{\beta \in I(N-1,k)}).$$

Consider also the generalized $(N - 1, k - 1)$-cross

$$\mathbf{T}_{N-1,k-1} := \mathbb{T}_{N-1,k-1}((A_j, D_j)_{j=1}^{N-1}, (\Sigma_{(\beta,1)})_{\beta \in I(N-1,k-1)}).$$

It can be easily seen that for a fixed $a_N \in A_N \setminus Q$ we have

$$(\mathbf{T}_{N,k})_{(\cdot,a_N)} = \mathbf{T}_{N-1,k}(a_N) \cup \mathbf{T}_{N-1,k-1},$$

where $(\mathbf{T}_{N,k})_{(\cdot,a_N)}$ is the fiber of the set $\mathbf{T}_{N,k}$ over a_N. Define

$$\mathbf{Y}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j, D_j)_{j=1}^{N-1}), \qquad \mathbf{Y}_{N-1,k-1} := \mathbb{X}_{N-1,k-1}((A_j, D_j)_{j=1}^{N-1}).$$

For any $a_N \in A_N \setminus Q$ we have $f(\cdot, a_N) \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k}(a_N))$. Then, by the inductive assumption, for any $a_N \in A_N \setminus Q$ there exists an $\widehat{f}_{a_N} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k})$ such that $\widehat{f}_{a_N} = f(\cdot, a_N)$ on $\mathbf{T}_{N-1,k}(a_N)$.

Define a 2-fold classical cross

$$\mathbf{Z} = \mathbb{X}(B_1, B_2; E_1, E_2) := \mathbb{X}((B_j, E_j)_{j=1}^2),$$

where $B_1 = \mathbf{T}_{N-1,k-1}$, $B_2 = A_N \setminus Q$, $E_1 = \widehat{\mathbf{Y}}_{N-1,k}$, $E_2 = D_N$. Clearly

$$\mathbf{Z} = (\mathbf{T}_{N-1,k-1} \times D_N) \cup (\widehat{\mathbf{Y}}_{N-1,k} \times (A_N \setminus Q)).$$

(Jarnicki and Pflug in [JP5] considered a 2-fold cross of the form $(\widehat{\mathbf{Y}}_{N-1,k-1} \times D_N) \cup (\widehat{\mathbf{Y}}_{N-1,k} \times A_N)$. This is also possible here. However, our new method simplifies the further reasoning.)

Observe that $\widehat{\mathbf{Z}} = \widehat{\mathbf{X}}_{N,k}$. Indeed, in view of the inductive assumption, $\mathbf{T}_{N-1,k-1}$ is connected (this follows from the connectedness of $\widehat{\mathbf{Y}}_{N-1,k-1}$ and the inclusion $\widehat{f}(\widehat{\mathbf{Y}}_{N-1,k-1}) \subset f(\mathbf{T}_{N-1,k-1}), f \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k-1})$). Take an $\varepsilon \in (0,1)$ and let $U(\varepsilon)$ be the connected component of the open set

$$\{z \in \widehat{\mathbf{Y}}_{N-1,k-1} : h^\star_{\mathbf{T}_{N-1,k-1}, \widehat{\mathbf{Y}}_{N-1,k-1}}(z) < \varepsilon\}$$

containing $\mathbf{T}_{N-1,k-1}$. Then $\widehat{\mathbf{Y}}_{N-1,k-1}$ is the envelope of holomorphy of $U(\varepsilon)$. To see this, take a $g \in \mathcal{O}(U(\varepsilon))$. Then $g|_{\mathbf{T}_{N-1,k-1}} \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k-1})$ and by the inductive assumption there is a $\widehat{g} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k-1})$ with $\widehat{g} = g$ on $\mathbf{T}_{N-1,k-1}$. Since $\mathbf{T}_{N-1,k-1}$ is not pluripolar, we get $\widehat{g} = g$ on $U(\varepsilon)$. Hence any $g \in \mathcal{O}(U(\varepsilon))$ admits a holomorphic extension to $\widehat{\mathbf{Y}}_{N-1,k-1}$, which is a pseudoconvex domain (cf. [JP5]). Using Lemma 3.3 we get $h^\star_{\mathbf{T}_{N-1,k-1}, \widehat{\mathbf{Y}}_{N-1,k-1}} = 0$. Hence $h^\star_{\mathbf{T}_{N-1,k-1}, \widehat{\mathbf{Y}}_{N-1,k}} = h^\star_{\widehat{\mathbf{Y}}_{N-1,k-1}, \widehat{\mathbf{Y}}_{N-1,k}}$ and it suffices to use Lemma 3.6 and pluripolarity of Q.

Let $F_f : \mathbf{Z} \to \mathbb{C}$ be given by the formula

$$F_f(z', z_N) := \begin{cases} \widehat{f}_{z_N}(z') & \text{if } (z', z_N) \in \widehat{\mathbf{Y}}_{N-1,k} \times (A_N \setminus Q), \\ f(z', z_N) & \text{if } (z', z_N) \in \mathbf{T}_{N-1,k-1} \times D_N. \end{cases}$$

Fix an $a_N \in A_N \setminus Q$. It is obvious that $f(\cdot, a_N) \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k-1})$. Then there exists a $\widehat{g}_{a_N} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k-1})$ satisfying $\widehat{g}_{a_N} = f(\cdot, a_N)$ on $\mathbf{T}_{N-1,k-1}$. We shall show that we have the equality $\widehat{f}_{a_N} = \widehat{g}_{a_N}$ on $\widehat{\mathbf{Y}}_{N-1,k-1}$. Since both $\widehat{f}_{a_N}$ and $\widehat{g}_{a_N}$ are extensions of $f(\cdot, a_N)$, we only need to prove the existence of some non-pluripolar set $B \subset \mathbf{T}_{N-1,k}(a_N) \cap \mathbf{T}_{N-1,k-1}$ and use the identity principle. Observe that the set

$$B := \mathfrak{C}(\mathbf{T}_{N-1,k}(a_N)) \cap \mathfrak{C}(\mathbf{T}_{N-1,k-1})$$

is good for our purpose. In particular, $\widehat{f}_{a_N}(z') = f(z', a_N)$, $z' \in \mathbf{T}_{N-1,k-1}$, which shows that F_f is well-defined.

It is clear that $F_f \in \mathcal{O}_s(\mathbf{Z})$. Thus, using Theorem 2.13 we get the existence of a function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{Z}})$ with $\widehat{f} = F_f$ on $\mathbf{Z}$.

We have to verify that $\widehat{f} = f$ on $\mathbf{T}_{N,k}$. Take a point $a \in \mathbf{T}_{N,k}$. The conclusion is obvious if $a \in \mathbf{T}_{N-1,k-1} \times D_N \subset \mathbf{Z}$. Suppose, without losing generality, that $a = (a_1, \dots, a_k, a_{k+1}, \dots, a_N) \in D_\alpha \times (A_\alpha \setminus \Sigma_\alpha)$, where $\alpha = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{N-k})$. Observe that

$$\mathcal{T} := \bigcup_{z_N \in A_N \setminus Q} \mathbf{T}_{N-1,k}(z_N) \times \{z_N\} \subset \widehat{\mathbf{Y}}_{N-1,k} \times (A_N \setminus Q) \subset \mathbf{Z}.$$

We also have

$$\mathcal{T} \subset \bigcup_{z_N \in A_N \setminus Q} (\mathbf{T}_{N,k})_{(\cdot, z_N)} \times \{z_N\} \subset \mathbf{T}_{N,k}.$$

Thus, if $b = (b', b_N) \in \mathcal{T}$, then $\widehat{f}(b) = F_f(b) = \widehat{f}_{b_N}(b') = f(b)$. Bearing this in mind, we easily see that it suffices to find a sequence

$$(b^\nu)_{\nu=1}^\infty \subset \mathcal{T} \cap \{(a_1, \dots, a_k)\} \times (A_\alpha \setminus \Sigma_\alpha)$$

such that $b^\nu \to a$, and then continuity of $f(a_1, \dots, a_k, \cdot)$ will end the proof.

Since Q is pluripolar, there exists a sequence (b_N^ν) convergent to a_N such that $(b_N^\nu) \subset A_N \setminus Q$. Put $P := \bigcup_{\nu=1}^\infty (\Sigma_\alpha)_{(\cdot, b_N^\nu)}$, which is a pluripolar set. This guarantees the existence of a sequence $((b_{k+1}^\nu, \dots, b_{N-1}^\nu)) \subset (A_{k+1} \times \dots \times A_{N-1}) \setminus P$, convergent to $(a_{k+1}, \dots, a_{N-1})$. Finally we put $b^\nu := (a_\alpha^1, b_{k+1}^\nu, \dots, b_{N-1}^\nu)$. It is obvious that $b^\nu \to a$ and that for every $\nu \in \mathbb{N}, b^\nu \in \mathbf{T}_{N-1,k}(b_N^\nu) \times \{b_N^\nu\} \subset \mathcal{T}$.

Step 2: We prove that for every $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus G)$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k} \setminus G$.

As in Step 1, we may assume that D_j is relatively compact and $A_j \subset\subset D_j$ for $j = 1, \dots, N$.

We apply induction on N. There is nothing to prove in the case $N = k$. The case $k = 1$ is solved in Theorem 2.16. Thus, the conclusion holds true for $N = 2$. Suppose it holds true for $N - 1 \geq 2$. Now, we apply induction on k. As we observed (cf. Theorem 2.16), for $k = 1$ the result is known. Suppose that the conclusion is true for $k - 1$ with $2 \leq k \leq N - 1$.

Define

$$\widetilde{Q} := Q_N = \{z_N \in A_N : \exists \alpha \in I_0(N, k) : (\Sigma_\alpha)_{(\cdot, z_N)} \notin \mathcal{PLP}\},$$

where $I_0(N, k)$ is as in Step 1, and

$$R := R_N = \{z_N \in A_N : G_{(\cdot, z_N)} \notin \mathcal{PLP}\}, \qquad Q := \widetilde{Q} \cup R.$$

Proposition 3.2 implies that $Q \in \mathcal{PLP}$. For an $a_N \in A_N \setminus Q$ let $\mathbf{T}_{N-1,k}(a_N)$

be as in Step 1. Let $\mathbf{T}_{N-1,k-1}, \mathbf{Y}_{N-1,k}, \mathbf{Y}_{N-1,k-1}$ be as in Step 1. Put

$$\mathbf{Z} = \mathbb{X}(B_1, B_2; E_1, E_2) := \mathbb{X}((B_j, E_j)_{j=1}^2),$$

where $B_1 = \mathbf{Y}_{N-1,k-1}, B_2 = A_N \setminus Q, E_1 = \widehat{\mathbf{Y}}_{N-1,k}, E_2 = D_N$. Recall that

$$\mathbf{Z} = (\mathbf{Y}_{N-1,k-1} \times D_N) \cup (\widehat{\mathbf{Y}}_{N-1,k} \times (A_N \setminus Q))$$

and $\widehat{\mathbf{Z}} = \widehat{\mathbf{X}}_{N,k}$.

For any fixed $f \in \mathcal{F}$ and for any $a_N \in A_N \setminus Q$ we have

$$f(\cdot, a_N) \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)}).$$

The set $G_{(\cdot,a_N)} \cap \widehat{\mathbf{Y}}_{N-1,k}$ is analytic in $\widehat{\mathbf{Y}}_{N-1,k} \subset (\widehat{\mathbf{X}}_{N,k})_{(\cdot,a_N)}$. Observe that in view of the definition of the set Q we have $\operatorname{codim} G_{(\cdot,a_N)} \geq 1$. From the inductive assumption we conclude that for any $\varphi \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)})$ there exists an $h_{a_N}^{\varphi} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,a_N)})$ such that $h_{a_N}^{\varphi} = \varphi$ on $\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)}$. In particular, there exists an $h_{a_N}^{f(\cdot,a_N)} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,a_N)})$ with $h_{a_N}^{f(\cdot,a_N)} = f(\cdot, a_N)$ on $\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)}$.

Define a function $F_f : \mathbf{Z} \setminus G \to \mathbb{C}$ by

$$F_f(z', z_N) := \begin{cases} f(z', z_N) & \text{if } (z', z_N) \in (\mathbf{Y}_{N-1,k-1} \setminus G_{(\cdot,z_N)}) \times D_N, \\ h_{z_N}^{f(\cdot,z_N)}(z') & \text{if } (z', z_N) \in (\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,z_N)}) \times (A_N \setminus Q). \end{cases}$$

We will show that F_f is well-defined. Observe that for any $a_N \in D_N$ we have

$$f(\cdot, a_N) \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,a_N)}).$$

The set $G_{(\cdot,a_N)} \cap \widehat{\mathbf{Y}}_{N-1,k-1}$ is analytic in $\widehat{\mathbf{Y}}_{N-1,k-1} \subset (\widehat{\mathbf{X}}_{N,k})_{(\cdot,a_N)}$. Two cases have to be considered:

Case 1: $\operatorname{codim} G_{(\cdot,a_N)} \geq 1$. Then from the inductive assumption we conclude that for any $\varphi \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,a_N)})$ there is a $g_{a_N}^{\varphi} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k-1} \setminus G_{(\cdot,a_N)})$ such that $g_{a_N}^{\varphi} = \varphi$ on $\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,a_N)}$. In particular, there exists a function $g_{a_N}^{f(\cdot,a_N)} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k-1} \setminus G_{(\cdot,a_N)})$ with $g_{a_N}^{f(\cdot,a_N)} = f(\cdot, a_N)$ on $\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,a_N)}$.

Case 2: $\operatorname{codim} G_{(\cdot,a_N)} = 0$ (this may happen for $a_N \in P_N \subset D_N$, where P_N is pluripolar). In this case $\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,a_N)} = \emptyset$ (and even $G_{(\cdot,a_N)} = \widehat{\mathbf{Y}}_{N-1,k-1}$ in view of the connectedness of $\widehat{\mathbf{Y}}_{N-1,k-1}$ and of the identity principle for analytic sets) and there is nothing to prove here.

In this situation, it suffices to show that for any $a_N \in A_N \setminus Q$ we have the equality

$$g_{a_N}^{f(\cdot,a_N)} = h_{a_N}^{f(\cdot,a_N)} \quad \text{on } (\widehat{\mathbf{Y}}_{N-1,k-1} \setminus G_{(\cdot,a_N)}) \cap (\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,a_N)}).$$

Using the definition of the set R and the inclusion $\widehat{\mathbf{Y}}_{N-1,k-1} \subset \widehat{\mathbf{Y}}_{N-1,k}$ we see that we only need to show the equality on the set $\widehat{\mathbf{Y}}_{N-1,k-1} \setminus G_{(\cdot,a_N)}$.

Since $h_{a_N}^{f(\cdot,a_N)}$ and $g_{a_N}^{f(\cdot,a_N)}$ are extensions of $f(\cdot,a_N)$, we are done if we show that there exists a non-pluripolar set $B\subset(\mathbf{T}_{N-1,k}(a_N)\cap\mathbf{T}_{N-1,k-1})\setminus G_{(\cdot,a_N)}$. Similarly to Step 1, the set

$$B:=(\mathfrak{C}(\mathbf{T}_{N-1,k}(a_N))\cap\mathfrak{C}(\mathbf{T}_{N-1,k-1}))\setminus G_{(\cdot,a_N)}$$

is good for our purpose.

It is obvious that $F_f\in\mathcal{O}_s(\mathbf{Z}\setminus G)$. Then, applying Theorem 2.16, we conclude that for any function $\Psi\in\mathcal{O}_s(\mathbf{Z}\setminus G)$ there exists a function $\widehat{\psi}\in\mathcal{O}(\widehat{\mathbf{Z}}\setminus G)$ such that $\widehat{\psi}=\Psi$ on $\mathbf{Z}\setminus G$. In particular, there exists an $\widehat{f}\in\mathcal{O}(\widehat{\mathbf{Z}}\setminus G)$ with $\widehat{f}=F_f$ on $\mathbf{Z}\setminus G$.

We have to verify that $\widehat{f}=f$ on $\mathbf{T}_{N,k}\setminus G$. Take a point $a\in\mathbf{T}_{N,k}\setminus G$. The conclusion is obvious if $a\in(\mathbf{T}_{N-1,k-1}\times D_N)\setminus G\subset\mathbf{Z}\setminus G$. Suppose, without losing generality, that $a=(a_1,\dots,a_k,a_{k+1},\dots,a_N)\in(D_\alpha\times(A_\alpha\setminus\Sigma_\alpha))\setminus G$, where $\alpha=(\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{N-k})$. Observe that

$$\mathcal{T}:=\bigcup_{z_N\in A_N\setminus Q}(\mathbf{T}_{N-1,k}(z_N)\setminus G_{(\cdot,z_N)})\times\{z_N\}\subset(\widehat{\mathbf{Y}}_{N-1,k}\times(A_N\setminus Q))\setminus G\subset\mathbf{Z}\setminus G,$$

as well as

$$\mathcal{T}\subset\Big(\bigcup_{z_N\in A_N\setminus Q}(\mathbf{T}_{N,k})_{(\cdot,z_N)}\times\{z_N\}\Big)\setminus G\subset\mathbf{T}_{N,k}\setminus G.$$

Thus, if $b=(b',b_N)\in\mathcal{T}$, then $\widehat{f}(b)=F_f(b)=h_{b_N}^{f(\cdot,b_N)}(b')=f(b)$. Bearing this in mind, we easily see that it suffices to find a sequence

$$(b^\nu)_{\nu=1}^\infty\subset\mathcal{T}\cap\{(a_1,\dots,a_k)\}\times(A_\alpha\setminus\Sigma_\alpha)$$

such that $b^\nu\to a$, and use the continuity of $f(a_1,\dots,a_k,\cdot)$.

Since Q is pluripolar, there exists a sequence $(c_N^\nu)\subset A_N\setminus Q$ which is convergent to a_N. Put $P:=\bigcup_{\nu=1}^\infty(\Sigma_\alpha)_{(\cdot,c_N^\nu)}$, which is a pluripolar set. This guarantees the existence of a sequence $((c_{k+1}^\nu,\dots,c_{N-1}^\nu))\subset(A_{k+1}\times\dots\times A_{N-1})\setminus P$ convergent to $(a_{k+1},\dots,a_{N-1})$. Observe that $a_1,\dots,a_k$ have to be such that $G_{(a_1,\dots,a_k,\cdot)}$ is pluripolar. Finally we observe that now it suffices to take

$$((b_{k+1}^\nu)^s,\dots,(b_N^\nu)^s)\subset(((A_{k+1}\times\dots\times A_{N-1})\setminus P)\times(A_N\setminus Q))\setminus G_{(a_1,\dots,a_k,\cdot)}$$

convergent to $((c_{k+1}^\nu,\dots,c_N^\nu))$ for $\nu\in\mathbb{N}$, and we are done.

Step 3. Put $\widehat{N}:=G_{s,\widehat{\mathcal{F}}}$, where $\widehat{\mathcal{F}}:=\{\widehat{f}:f\in\mathcal{F}\}$. Then $\widehat{N}$ is an analytic set contained in G, singular with respect to the family $\widehat{\mathcal{F}}$. We show that $\widehat{N}=\widehat{G}$. Consider two cases:

Case 1: $\widehat{G}\neq\emptyset$. Observe that the set $\widehat{\mathbf{X}}_{N,k}\setminus\widehat{G}$ is a domain of holomorphy (cf. [JP5]). Thus there exists a non-continuable function $g\in\mathcal{O}(\widehat{\mathbf{X}}_{N,k}\setminus\widehat{G})$

(see [JP1, Proposition 1.8.11]). Then $f := g|_{\mathbf{T}_{N,k}\setminus G} \in \mathcal{O}_s^c(\mathbf{T}_{N,k} \setminus G)$, which implies that there exists a function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{N})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k} \setminus G \subset \widehat{\mathbf{X}}_{N,k} \setminus (\widehat{N} \cup \widehat{G})$. Using non-pluripolarity of the set $\mathbf{T}_{N,k} \setminus G$ we obtain $\widehat{f} = g$ on $\widehat{\mathbf{X}}_{N,k} \setminus (\widehat{N} \cup \widehat{G})$, from which it follows that $\widehat{G} \subset \widehat{N}$. Thus, $\widehat{N}$ is of pure codimension one (cf. Remark 2.11), which implies the equality $\widehat{N} = \widehat{G}$.

Case 2: $\widehat{G} = \emptyset$. Then, if $\widehat{N}$ is non-empty, it has to be of pure codimension one, which clearly contradicts the fact that $\operatorname{codim} G \geq 2$. ∎

5. Proof of Theorem 2.12 in the general case. We start with the following

Proposition 5.1. *If in Theorem 2.17 we have $M = \mathbf{T} \cap F$, where F is an analytic subset of an open neighborhood U_F of $\mathbf{T}$ contained in $\widehat{\mathbf{X}}$ and $S = \mathbf{T} \cap G$, where G is an analytic subset of an open neighborhood U_G of $\mathbf{T}$ contained in $\widehat{\mathbf{X}}$ such that $F \subset G$, then in the conclusion therein one can take $\mathbf{T}' = \mathbf{T}$.*

Proof. For each $f \in \mathcal{G}$ the function $\widetilde{f}_{|\mathbf{T}'\setminus S}$ is in $\mathcal{O}_s^c(\mathbf{T}' \setminus S)$. Hence it extends holomorphically to the function $\widetilde{\widetilde{f}}$ on $\widehat{\mathbf{X}} \setminus \widehat{S}$, where $\widehat{S}$ is constructed via Theorem 2.16. Similarly, f extends holomorphically to the function $\widehat{f}$ on $\widehat{\mathbf{X}} \setminus \widehat{S}$ and we have $\widetilde{\widetilde{f}} = \widehat{f}$ on $\widehat{\mathbf{X}} \setminus \widehat{S}$. We also have $\mathbf{T} \setminus S \subset \widehat{\mathbf{X}} \setminus \widehat{S}$, hence $\widetilde{f} = f$ on $\mathbf{T} \setminus S$ and $\mathcal{S}(\widetilde{f}) \cap \mathbf{T} \subset S$. Indeed, observe that $\widetilde{f}$ is defined on $\widehat{\mathbf{X}} \setminus \widehat{M}$ and $\widehat{M} \cap \mathbf{T} \subset M \subset S$ (the first inclusion follows from Remark 2.18), which implies $\mathbf{T} \setminus S \subset \widehat{\mathbf{X}} \setminus \widehat{M}$ and $\widetilde{\widetilde{f}} = \widehat{f} = f$ on $\mathbf{T} \setminus S$. Moreover, $\widetilde{\widetilde{f}} = \widetilde{f} = f$ on $\mathbf{T}' \setminus S$, from which follows $\widetilde{f} = f$ on $\mathbf{T} \setminus S$. The condition $\mathcal{S}(\widetilde{f}) \cap \mathbf{T} \subset S$ is a consequence of $\widehat{S} \cap \mathbf{T} \subset S$. ∎

Proof of Theorem 2.12 in the general case. We may assume that for any $j = 1, \ldots, N$, D_j is relatively compact and $A_j \subset\subset D_j$.

We apply induction on N. There is nothing to prove in the case $N = k$. The case $k = 1$ is solved in Theorem 2.17. Thus, the conclusion holds true for $N = 2$. Suppose it holds true for $N - 1 \geq 2$. Now, we apply induction on k. As we observed, for $k = 1$ the result is known. Suppose that the conclusion is true for $k - 1$ with $2 \leq k \leq N - 1$.

Define

$$S_N := \{z_N \in A_N : \exists \alpha \in I_0(N, k) : (\Sigma_\alpha^0)_{(\cdot, z_N)} \notin \mathcal{PLP}\},$$
$$R_N := \{z_N \in A_N : G_{(\cdot, z_N)} \notin \mathcal{PLP}\}, \qquad Q_N := S_N \cup R_N.$$

We know that $Q_N \in \mathcal{PLP}$. For any $a_N \in A_N \setminus Q_N$ let $\mathbf{T}_{N-1,k}(a_N)$ be as in

Section 4. Also, let $\mathbf{T}_{N-1,k-1}, \mathbf{Y}_{N-1,k}, \mathbf{Y}_{N-1,k-1}$ be as in Section 4. Put

$$\mathbf{Z} = \mathbb{X}(B_1, B_2; E_1, E_2) := \mathbb{X}((B_j, E_j)_{j=1}^2),$$

where $B_1 = \mathbf{T}_{N-1,k-1}$, $B_2 = A_N \setminus Q_N$, $E_1 = \widehat{\mathbf{Y}}_{N-1,k}$, $E_2 = D_N$. Recall that

$$\mathbf{Z} = (\mathbf{T}_{N-1,k-1} \times D_N) \cup (\widehat{\mathbf{Y}}_{N-1,k} \times (A_N \setminus Q_N))$$

and $\widehat{\mathbf{Z}} = \widehat{\mathbf{X}}_{N,k}$.

For any fixed $f \in \mathcal{G}$ and any $a_N \in A_N \setminus Q_N$ we see that

$$f(\cdot, a_N) \in \mathcal{O}_s^c(\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)})$$

(in particular, it admits a holomorphic extension to $\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,a_N)}$). The sets $G_{(\cdot,a_N)} \cap \widehat{\mathbf{Y}}_{N-1,k}$, $F_{(\cdot,a_N)} \cap \widehat{\mathbf{Y}}_{N-1,k}$ are analytic in $\widehat{\mathbf{Y}}_{N-1,k} \subset (\widehat{\mathbf{X}}_{N,k})_{(\cdot,a_N)}$. Observe that we have $\operatorname{codim} G_{(\cdot,a_N)} \geq 1$, $\operatorname{codim} F_{(\cdot,a_N)} \geq 1$. Moreover, for each $\beta \in I(N-1,k)$ there exists a pluripolar set $(\Sigma^0_{(\beta,0)})_{(\cdot,a_N)} \subset \Sigma_\beta^{a_N}$ such that for each $a' \in (A_{(\beta,0)})_{(\cdot,a_N)} \setminus \Sigma_\beta^{a_N}$ the fiber $(G_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'}$ is thin and the function $(f(\cdot,a_N))_{\beta,a'}$ extends meromorphically to $D_{(\beta,0)} \setminus (F_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'}$, i.e. there exists a function $(\widetilde{f(\cdot,a_N)})_{\beta,a'} \in \mathcal{M}(D_{(\beta,0)} \setminus (F_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'})$ such that $\mathcal{S}((\widetilde{f(\cdot,a_N)})_{\beta,a'}) \subset (G_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'} \setminus (F_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'}$ and $(\widetilde{f(\cdot,a_N)})_{\beta,a'} = (f(\cdot,a_N))_{\beta,a'}$ on $D_{(\beta,0)} \setminus (G_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N))_{a'}$. From the inductive assumption we conclude that for any $f \in \mathcal{G}$ there exists a function $\widetilde{f}_{a_N} \in \mathcal{M}(\widehat{\mathbf{Y}}_{N-1,k} \setminus F_{(\cdot,a_N)})$ such that $\mathcal{S}(\widetilde{f}_{a_N}) \cap \mathbf{T}_{N-1,k}(a_N) \subset G_{(\cdot,a_N)} \cap \mathbf{T}_{N-1,k}(a_N)$ and $\widetilde{f}_{a_N} = f(\cdot,a_N)$ on $\mathbf{T}_{N-1,k}(a_N) \setminus G_{(\cdot,a_N)}$ (in particular, $\widetilde{f}_{a_N} \in \mathcal{O}(\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,a_N)})$).

For any $f \in \mathcal{G}$ define a function $F_f : \mathbf{Z} \setminus G \to \mathbb{C}$ by

$$F_f(z', z_N) := \begin{cases} f(z', z_N) & \text{if } (z', z_N) \in (\mathbf{T}_{N-1,k-1} \setminus G_{(\cdot,z_N)}) \times D_N, \\ \widetilde{f}_{z_N}(z') & \text{if } (z', z_N) \in (\widehat{\mathbf{Y}}_{N-1,k} \setminus G_{(\cdot,z_N)}) \times (A_N \setminus Q_N). \end{cases}$$

As in Section 4 we verify F_f is well-defined and $F_f \in \mathcal{O}_s(\mathbf{Z} \setminus G)$, which implies there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{Z}} \setminus \widehat{G})$ such that $\widehat{f} = F_f$ on $\mathbf{Z} \setminus G$ and $\widehat{f} = f$ on $\mathbf{T}_{N,k} \setminus G$.

Observe that there exists a pluripolar set $\Sigma_1 \subset \mathbf{T}_{N-1,k-1}$ such that for any $a' \in \mathbf{T}_{N-1,k-1} \setminus \Sigma_1$ the fiber $G_{(a',\cdot)}$ is thin. Similarly, for each $a_N \in A_N \setminus Q_N$ the fiber $G_{(\cdot,a_N)}$ is thin. Moreover, for each $a_N \in A_N \setminus Q_N$ the function $F_f(\cdot, a_N)$ extends meromorphically to $\widehat{\mathbf{Y}}_{N-1,k} \setminus F_{(\cdot,a_N)}$. It is easy to see that there exists a pluripolar set $\Sigma_1' \subset \mathbf{T}_{N-1,k-1}$ such that for any $a' \in \mathbf{Y}_{N-1,k-1} \setminus \Sigma_1'$ the function $F_f(a', \cdot)$ extends meromorphically to $D_N \setminus F_{(a',\cdot)}$. Hence from Theorem 2.17 we get the existence of a function $\widetilde{F_f} \in \mathcal{M}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{F})$ such that $\mathcal{S}(\widetilde{F_f}) \cap \mathbf{Z} \subset G \cap \mathbf{Z}$ and $\widetilde{F_f} = F_f$ on $\mathbf{Z} \setminus G$ (in particular, $\widetilde{F_f} \in \mathcal{O}(\widehat{\mathbf{Z}} \setminus \widehat{G})$). Then we have the equalities

$$\begin{aligned} \widetilde{F_f} = F_f = \widehat{f} &\quad \text{on } \mathbf{Z} \setminus G, \\ \widetilde{F_f} = \widehat{f} &\quad \text{on } \widehat{\mathbf{X}}_{N,k} \setminus \widehat{G}, \\ \widetilde{F_f} = f &\quad \text{on } \mathbf{T}_{N,k} \setminus G \end{aligned}$$

and $\mathcal{S}(\widetilde{F_f}) \cap \mathbf{T}_{N,k} \subset G$. So it is enough to put $\widetilde{f} := \widetilde{F_f}$. ∎

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