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On generalized topological spaces I

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Abstract. We begin a systematic study of the category **GTS** of generalized topological spaces (in the sense of H. Delfs and M. Knebusch) and their strictly continuous mappings. We reformulate the axioms. Generalized topology is found to be connected with the concept of a bornological universe. Both **GTS** and its full subcategory **SS** of small spaces are topological categories. The second part of this paper will also appear in this journal.

1. Introduction. Generalized topology in the sense of H. Delfs and M. Knebusch is a rather unknown chapter of general topology. In fact, it is a generalization of the classical concept of topology. The aim of this work (consisting of this paper and [P2]) is to start the systematic study of generalized topology in this sense (not to be confused with other notions of "generalized topology" appearing in the literature).

This concept was defined in [DK] and helped to develop a semialgebraic version of homotopy theory. In 1991, M. Knebusch [K2] suggested that his theory of locally semialgebraic spaces (developed in [DK] together with H. Delfs) and weakly semialgebraic spaces (developed in [K1]) could be generalized to the o-minimal context. (Here locally definable and weakly definable spaces allow one to perform constructions analogous to those known from the traditional homotopy theory.)

The successful generalization in [P1] of the above mentioned theory of Delfs and Knebusch to the case of o-minimal expansions of fields prompts the question whether similar homotopy theories can be developed by the use of generalized topology and other ideas from [DK, K1]. Moreover, even if in many cases a full-fledged homotopy theory may not be achievable, the use of locally definable and weakly definable spaces over various structures will be important. And even on the purely topological level, generalized topology is an interesting notion, related to the notion of a bornological universe

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(see [B] or [V] for the definition). This aspect does not appear in the literature. The paper [BO] does not give the reader enough understanding of the generalized topology (see [P1] for a discussion) despite the heavy use of the theory of locally semialgebraic spaces from [DK]. The $\mathcal{T}$-spaces of [EP] have natural generalized topologies, and it is convenient to see them as locally small spaces.

The category **GTS** of generalized topological spaces and their strictly continuous mappings may be seen as an alternative to the standard category **Top** of topological spaces and continuous mappings. It originates from the categorical concept of Grothendieck topology, and contains **Top** as a full subcategory. In general, the third order concept of a generalized topological space is much more difficult to study than the second order concept of a topological space. Only the use of **GTS** allowed infinite families of definable sets (considered with their natural o-minimal topologies) to be glued together to produce locally definable spaces in [DK] and [P1] and weakly definable spaces in [K1] and [P1]. Many of the proofs of [DK] and [K1] are purely topological (in the sense of generalized topology), and it is important to extract the relevant pieces of information on particular levels of structure. The theory of infinite gluings of definable sets can be reconstructed to a large extent in the setting of *structures with topologies* (to be defined in [P2]).

The present work (consisting of this paper and [P2]) is a continuation of [P1], which was devoted to extending the semialgebraic homotopy theory contained in [DK, K1] to the case of spaces over o-minimal expansions of fields. This work is much more general and gives the origin, a nice axiomatization, and the basic theory of the category **GTS**. The new (relative to traditional topology) concept of admissibility is explained, and we deal with main (generalized) topological concepts such as: small sets, bases, connectedness, and various concepts of discreteness (in this paper) as well as completeness, paracompactness, Lindelöfness, and separation axioms (in [P2]). We prove here that both **GTS** and its full subcategory **SS** are topological categories (as constructs).

The work contains eight subsections devoted to the categories: **GTS**, **SS** in this paper as well as **LSS**, **WSS** and related categories, $\mathbf{GTS}(M)$, $\mathbf{ADS}(\mathcal{M},\sigma)$ and $\mathbf{DS}(\mathcal{M},\sigma)$, $\mathbf{LDS}(\mathcal{M},\sigma)$, $\mathbf{WDS}(\mathcal{M},\sigma)$ in [P2]. In each subsection the following aspects (if applicable and relevant) are considered: definitions, strong or generated topology, special (e.g. constructible) sets, examples, admissibility, small sets, generation and bases, admissible union, description of morphisms, subspaces, limit or colimit properties, interesting functors, special properties of spaces and mappings, open questions. The separation axioms (in the "definable" case) and completeness (in all cases) are treated separately.

The author hopes that from now on generalized topology (hidden in the language of locally semialgebraic spaces of [DK], and weakly semialgebraic spaces of [K1]) will be developed without constraints.

Notation. For families of sets $\mathcal{U}, \mathcal{V}$, we will use the usual set-theoretic operations $\cup, \cap, \setminus$ on (families considered as) sets, and analogous operations on families of sets, for example

$$\mathcal{U} \cap_1 \mathcal{V} = \{U \cap V : U \in \mathcal{U},\, V \in \mathcal{V}\}, \qquad \mathcal{U} \times_1 \mathcal{V} = \{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

If $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_i\}_{i\in I}$, then we will also use another version of family union

$$\mathcal{U} \cup_1^* \mathcal{V} = \{U_i \cup V_i : i \in I\}.$$

In particular, we will write $V \cap_1 \mathcal{U} = \{V\} \cap_1 \mathcal{U}$ for a set V. Occasionally, we will use operations on families of families, for example

$$\Phi \cap_2 \Psi = \{\mathcal{U} \cap_1 \mathcal{V} : \mathcal{U} \in \Phi, \mathcal{V} \in \Psi\} \text{ or } \Phi \cap_2 F = \{\mathcal{U} \cap_1 F : \mathcal{U} \in \Phi\}.$$

2. Generalized topological spaces

2.1. Grothendieck topology. In this subsection we recall what a Grothendieck topology is. The reader may consult books like [BW, KS, MM].

Let $\mathcal{C}$ be a small category (i.e. a category whose objects form a set, and not a proper class). Consider the category of presheaves of sets on $\mathcal{C}$, denoted by $\mathrm{Psh}(\mathcal{C})$ or $\hat{\mathcal{C}}$, which is the category of contravariant functors from $\mathcal{C}$ to the category **Set** of sets and functions. Let us recall the fundamental fact:

Fact 2.1.1 (Yoneda lemma, weak version). *The functor $\mathcal{C} \ni C \mapsto \mathrm{Hom}(-, C) \in \hat{\mathcal{C}}$ is full and faithful, so we can consider $\mathcal{C}$ to be a full subcategory of $\hat{\mathcal{C}}$.*

To define a Grothendieck topology the following notion is usually used: a *sieve* S on an object C is a subobject of $\mathrm{Hom}(-, C)$ as an object of $\hat{\mathcal{C}}$. Since a sieve on C is a presheaf of sets of morphisms with common codomain C, the above definition may be translated (with a minor abuse of language) into: S is a set of morphisms with codomain C such that $f \in S$ implies $f \circ g \in S$, if only $f \circ g$ is meaningful. The largest sieve on C is $\mathrm{Hom}(-, C)$, the smallest is $\emptyset$.

A *Grothendieck topology* J on $\mathcal{C}$ is a function $C \mapsto J(C)$, with $J(C)$ a set of sieves on C such that the following axioms hold:

- (*identity/non-emptiness*) $\mathrm{Hom}(-, C) \in J(C)$ for each C;
- (*stability/base change*) if $S \in J(C)$ and $f : D \to C$ is a morphism of $\mathcal{C}$, then $f^*S = \{g \mid f \circ g \in S\} \in J(D)$;
- (*transitivity/local character*) if $S \in J(C)$ and R is a sieve on C such that $f^*R \in J(D)$ for each $f \in S$, $f : D \to C$, then $R \in J(C)$.

Elements of $J(C)$ are called *covering sieves*. The pair $(\mathcal{C}, J)$ is called a *Grothendieck site*. The above axioms imply the conditions (see Section III.2

of [MM]):

- (*saturation*) if $S \in J(C)$ and R is a sieve containing S, then $R \in J(C)$;
- (*intersection*) if $R, S \in J(C)$, then $R \cap S \in J(C)$.

It follows that each $J(C)$ is a filter (not necessarily proper) on the lattice $\mathrm{Sub}_{\hat{\mathcal{C}}} \mathrm{Hom}(-, C)$ of sieves on C.

If the category $\mathcal{C}$ *has pullbacks* (i.e. fibre products), then instead of covering sieves, we can speak about *covering families* of morphisms (generating respective sieves), so the axioms may be reformulated as:

- (*identity/isomorphism*) for each C, $\{\mathrm{id}_C\}$ is a covering family (also stated as: for each isomorphism $f : D \to C$, $\{f\}$ is a covering family);
- (*base change*) if $\{f_i : U_i \to U\}_i$ is a covering family, and $g : W \to U$ any morphism, then $\{\pi_{2i} : U_i \times_U W \to W\}_i$ is a covering family;
- (*local character*) if $\{f_i : U_i \to U\}_i$ is a covering family, and $\{g_{ij} : V_{ij} \to U_i\}_j$ are covering families, then $\{f_i \circ g_{ij} : V_{ij} \to U\}_{ij}$ is a covering family;

and usually the following is added (see [KS, Definition 16.1.2]):

- (*saturation*) if $\{f_i : U_i \to U\}_i$ is a covering family, and each f_i factorizes through an element of $\{g_j : V_j \to U\}_j$, then $\{g_j : V_j \to U\}_j$ is a covering family.

Alternatively (see [BW, Section 6.7]) one can consider saturated and non-saturated Grothendieck topologies, but for a Grothendieck site saturation is usually assumed.

Grothendieck topology allows one to define sheaves (of sets). A *sheaf* on $\mathcal{C}$ is a presheaf $F \in \hat{\mathcal{C}}$ such that for each covering family $\{U_i \to U\}_i$ in the diagram

$$F(U) \xrightarrow{e} \prod_i F(U_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

the induced morphism e is the equalizer of the standard pair of morphisms p_1, p_2 (cf. [MM, Section III.4]).

A Grothendieck topology is *subcanonical* if every representable presheaf is a sheaf, so in this case we may, by identifying the objects of $\mathcal{C}$ with their respective representable presheaves, consider $\mathcal{C} \subseteq \mathrm{Sh}(\mathcal{C}) \subseteq \mathrm{Psh}(\mathcal{C}) = \hat{\mathcal{C}}$. Grothendieck topologies used in practice are usually subcanonical.

2.2. Generalized topological spaces. This subsection is devoted to introducing a nice axiomatization and basic properties of generalized topological spaces.

For any set X, we have the Boolean algebra $\mathcal{P}(X)$ of all subsets of X, which may be treated as a small category with inclusions as morphisms. In this category fibre products are the same as binary products and the same as binary intersections, so $\mathcal{P}(X)$ has pullbacks. We want to introduce a full subcategory **Op** of $\mathcal{P}(X)$, consisting of "open subsets" of X. Then we want to introduce a subcanonical Grothendieck topology on this category. Subcanonicality means in this setting that for each covering family of morphisms (which may be identified with a family of subsets of a given set, since morphisms are inclusions) the object covered by the family is the supremum of this family in the smaller category **Op**.

The above discussion leads to the notion of a generalized topological space introduced by H. Delfs and M. Knebusch:

Definition 2.2.1 ([DK, p. 1]). A *generalized topological space* (gts) is a set X together with a family $\mathring{\mathcal{T}}(X)$ of subsets of X, called *open sets*, and a family Cov_X of open families, called *admissible* (*open*) *families* or *admissible* (*open*) *coverings*, such that:

- (A1) $\emptyset, X \in \mathring{\mathcal{T}}(X)$ (the empty set and the whole space are open).
- (A2) If $U_1, U_2 \in \mathring{\mathcal{T}}(X)$ then $U_1 \cup U_2, U_1 \cap U_2 \in \mathring{\mathcal{T}}(X)$ (finite unions and finite intersections of open sets are open).
- (A3) If $\{U_i\}_{i\in I} \subset \mathring{\mathcal{T}}(X)$ and I is finite, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$ (finite families of open sets are admissible).

The above three axioms are a strengthening of the identity axiom of a Grothendieck topology. They also ensure that the smaller category has pullbacks. These three axioms may be collectively called the *finiteness* axiom.

- (A4) If $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$ then $\bigcup_{i\in I} U_i \in \mathring{\mathcal{T}}(X)$ (the union of an admissible family is open).

This axiom may be called *co-subcanonicality*. Together with subcanonicality, it means that admissible families are coverings (in the traditional sense) of their unions. This property is imposed by the notation of [DK]: $\{U_i\}_{i\in I} \in \mathrm{Cov}_X(U)$ iff U is the union of $\{U_i\}_{i\in I}$.

Subcanonicality and co-subcanonicality may be collectively called the *naturality* axiom.

- (A5) If $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$, $V \subset \bigcup_{i\in I} U_i$, and $V \in \mathring{\mathcal{T}}(X)$, then $\{V \cap U_i\}_{i\in I} \in \mathrm{Cov}_X$ (the intersections of an admissible family with an open subset of the union of the family form an admissible family).

This is the stability axiom.

- (A6) If $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$ and for each $i \in I$ there is $\{V_{ij}\}_{j\in J_i} \in \mathrm{Cov}_X$ such that $\bigcup_{j\in J_i} V_{ij} = U_i$, then $\{V_{ij}\}_{i\in I,\, j\in J_i} \in \mathrm{Cov}_X$ (all members of admissible coverings of members of an admissible family form together an admissible family).

This is the transitivity axiom.

(A7) If $\{U_i\}_{i\in I} \subset \mathring{\mathcal{T}}(X)$, $\{V_j\}_{j\in J} \in \mathrm{Cov}_X$, $\bigcup_{j\in J} V_j = \bigcup_{i\in I} U_i$, and for each $j \in J$ there exists $i \in I$ such that $V_j \subset U_i$, then $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$ (a coarsening, with the same union, of an admissible family is admissible).

This is the saturation axiom.

(A8) If $\{U_i\}_{i\in I} \in \mathrm{Cov}_X$, $V \subset \bigcup_{i\in I} U_i$ and $V \cap U_i \in \mathring{\mathcal{T}}(X)$ for each i, then $V \in \mathring{\mathcal{T}}(X)$ (if a subset of the union of an admissible family has open intersections with the members of the family then the subset is open).

This axiom may be called *regularity*.

Both saturation and regularity have a smoothing character. Saturation may be achieved by just adding coarsenings (respecting the union) of admissible families. Regularity does not appear in some proofs, but it will be important when a new space is constructed as an admissible union of known spaces.

The above axioms may be restated briefly in the following way.

Definition 2.2.2. A *generalized topological space* is a triple

$$(X, \mathrm{Op}_X, \mathrm{Cov}_X),$$

where X is any set, $\mathrm{Op}_X \subseteq \mathcal{P}(X)$, and $\mathrm{Cov}_X \subseteq \mathcal{P}(\mathrm{Op}_X)$, such that the following axioms are satisfied:

- (*finiteness*) if $\mathcal{U} \in \mathrm{Fin}(\mathrm{Op}_X)$, then $\bigcup \mathcal{U}, \bigcap \mathcal{U} \in \mathrm{Op}_X$, $\mathcal{U} \in \mathrm{Cov}_X$,
- (*stability*) if $V \in \mathrm{Op}_X$, $\mathcal{U} \in \mathrm{Cov}_X$, then $V \cap_1 \mathcal{U} \in \mathrm{Cov}_X$,
- (*transitivity*) if $\Phi \in \mathcal{P}(\mathrm{Cov}_X)$, $\bigcup_1 \Phi \in \mathrm{Cov}_X$, then $\bigcup \Phi \in \mathrm{Cov}_X$,
- (*saturation*) if $\mathcal{U} \in \mathrm{Cov}_X$, $\mathcal{V} \in \mathcal{P}(\mathrm{Op}_X)$, $\mathcal{U} \preceq \mathcal{V}$, then $\mathcal{V} \in \mathrm{Cov}_X$,
- (*regularity*) if $W \in \mathcal{P}(X)$, $\mathcal{U} \in \mathrm{Cov}_X$, $W \cap_1 \mathcal{U} \in \mathcal{P}(\mathrm{Op}_X)$, then $W \cap \bigcup \mathcal{U} \in \mathrm{Op}_X$.

Remark 2.2.3. In the above, $\mathrm{Fin}(\cdot)$ is the family of finite subsets of a given set, and $\mathcal{U} \preceq \mathcal{V}$ means that the two families have the same union and $\mathcal{U}$ is a refinement of $\mathcal{V}$ (so $\mathcal{V}$ is a coarsening of $\mathcal{U}$). Notice that $\mathrm{Op}_X = \bigcup \mathrm{Cov}_X = \bigcup_1 \mathrm{Cov}_X$, hence one can define a generalized topological space as just a pair (X, Cov_X). The naturality axiom does not appear, since our interpretation of the families Op_X (*open sets*) and Cov_X (*admissible families* or *admissible coverings*) is intended to give a Grothendieck site (each member of Cov_X covers its union).

Definition 2.2.4 (cf. [DK, p. 32]). A *strictly continuous mapping* between gts's is a mapping such that the preimage of an admissible family is an admissible family. We then write $f^{-1}(\mathrm{Cov}_Y) \subseteq \mathrm{Cov}_X$, where $f : X \to Y$ is the mapping considered. This condition, in particular, implies that the preimage of an open set should be open (i.e. $f^{-1}(\mathrm{Op}_Y) \subseteq \mathrm{Op}_X$). Mappings satisfying only the latter condition will be called *continuous*.

Strictly continuous mappings may be viewed as morphisms of sites (cf. [BW, Section 6.7] and [KS, Section 17.2]).

DEFINITION 2.2.5. The gts's together with the strictly continuous mappings form a category called here **GTS**. Isomorphisms of **GTS** will be called *strict homeomorphisms*.

DEFINITION 2.2.6 ([AHS, Definition 5.1]). If a category $\mathcal{C}$ together with a faithful functor $U : \mathcal{C} \to \mathbf{Set}$ is given then the pair $(\mathcal{C}, U)$ is called a *construct*. If the functor U is obvious, we speak about the construct $\mathcal{C}$.

REMARK 2.2.7. The category **GTS** has an obvious forgetful functor $U : \mathbf{GTS} \to \mathbf{Set}$ and will be seen as a construct. The monomorphisms are exactly the injective morphisms, since the forgetful functor U is representable (see [AHS, Corollary 7.38]). The epimorphisms are exactly the surjective morphisms, since **GTS** as a construct has indiscrete structures.

DEFINITION 2.2.8. A subset $O \subseteq X$ will be called *weakly open* if it is a union of open subsets of the gts X, and *weakly closed* if its complement is weakly open. (The weakly open subsets form the topology $\tau(\mathrm{Op}_X) = \bigcup_1 \mathcal{P}(\mathrm{Op}_X)$ generated by the family Op_X, called the *generated topology* of the gts X.) The *weak closure* of a subset $Y \subseteq X$ is the closure $\overline{Y}$ of Y in the topology $\tau(\mathrm{Op}_X)$ on X. A mapping $f : X \to Y$ will be called *weakly continuous* if the preimage of any weakly open set is weakly open (i.e. $f^{-1}(\tau(\mathrm{Op}_Y)) \subseteq \tau(\mathrm{Op}_X)$).

DEFINITION 2.2.9. A subset Y of a gts X is *closed* if its complement Y^c is open, and is *locally closed* if it is the intersection of a closed set and an open set. A subset Y of a gts X is *constructible* if it is a Boolean combination of open sets. The family of closed subsets of X will be denoted by Cl_X, and the family of constructible sets by Constr_X.

Recall that each constructible set is a finite union of locally closed sets.

DEFINITION 2.2.10. We will say that a gts has the *closure property* (CPG) if the weak closure of any locally closed set is a closed set.

REMARK 2.2.11. In general, the closure operator on a gts does not exist. If the space X has the closure property (CPG), then the topological closure operator restricted to the class of constructible sets may be treated as the *closure operator* of the generalized topology

$$\bar{\cdot} : \mathrm{Constr}_X \to \mathrm{Cl}_X.$$

FACT 2.2.12. *The preimage of a constructible set under a strictly continuous mapping is constructible.*

Definition 2.2.13. By an *open family* we will understand a subfamily of Op_X. We will say that a family $\mathcal{U}$ is *essentially finite* (resp. *essentially countable*) if some finite (resp. countable) subfamily $\mathcal{U}_0 \subseteq \mathcal{U}$ covers the union of $\mathcal{U}$ (i.e. $\bigcup \mathcal{U}_0 = \bigcup \mathcal{U}$). We will denote by $\mathrm{EssFin}(\mathcal{U})$ the family of essentially finite subfamilies of the family $\mathcal{U}$.

Example 2.2.14. We have the following simple examples of gts's for each $n \in \mathbb{N}$:

1. The space $\mathbb{R}^n_{\mathrm{alg}}$, where the *closed sets* are the algebraic subsets of $\mathbb{R}^n$, and the *admissible families* are the essentially finite open families (this space is just $\mathbb{R}^n$ with the Zariski topology).
2. The space $\mathbb{C}^n_{\mathrm{alg}}$, where the *closed sets* are the algebraic subsets of $\mathbb{C}^n$, and the *admissible families* are the essentially finite open families (this space is $\mathbb{C}^n$ with the Zariski topology).
3. The space $\mathbb{R}^n_{\mathrm{salg}}$, where the *open sets* are the open semialgebraic subsets of $\mathbb{R}^n$, and the *admissible families* are the essentially finite open families.
4. The space $\mathbb{R}^n_{\mathrm{san}}$, where the *open sets* are the open semianalytic subsets of $\mathbb{R}^n$, and the *admissible families* are the open families essentially finite on bounded sets of $\mathbb{R}^n$.
5. The space $\mathbb{R}^n_{\mathrm{suban}}$, where the *open sets* are the open subanalytic subsets of $\mathbb{R}^n$, and the *admissible families* are the open families essentially finite on bounded sets of $\mathbb{R}^n$.
6. The space $\mathbb{R}^n_{\mathrm{top}}$, the usual topological space $\mathbb{R}^n$ (here all open families are *admissible*).
7. The space $\mathbb{R}^n_{\mathrm{ts}}$, where the *open sets* are the sets open in the usual topology, and the *admissible families* are the essentially finite open families.
8. For each topological space (X, τ), we can take $\mathrm{Op}_X = \tau$, and as Cov_X the essentially countable open families.

The function $\sin : \mathbb{R} \to \mathbb{R}$ is an endomorphism of $\mathbb{R}_{\mathrm{top}}$ and of $\mathbb{R}_{\mathrm{san}}$, but not of $\mathbb{R}_{\mathrm{salg}}$.

Example 2.2.15. In cases 3–7 of Example 2.2.14, the weakly open subsets are the open subsets of the natural topology on $\mathbb{R}^n$.

Example 2.2.16. Each topological space may be considered as a gts. An admissible family is understood as any open family. For topological spaces all weakly continuous mappings are continuous and all continuous mappings are strictly continuous.

Example 2.2.17 (non-examples). Given a topological space, we could consider only singletons of open sets as admissible coverings of these sets. This would give the so-called *indiscrete Grothendieck topology*, but not (in general) a gts, because of (A3). On the other hand, if we considered all families of open subsets as coverings of an open set, the resulting *discrete Grothendieck topology* would not (in general) be subcanonical, hence not a generalized topology.

Let us recall two interesting examples of gts's from Remark 23 in [P1].

EXAMPLE 2.2.18 ("the subanalytic site"). If M is a real analytic manifold, then we can declare:

- an *open subset* to be an open subanalytic subset;
- an *admissible family* to be an open family that is essentially finite on compact subsets.

EXAMPLE 2.2.19 ("another site"). In the situation of Example 2.2.18, define:

- an *open subset* to be any subanalytic subset;
- an *admissible family* to be an open family that is essentially finite on compact subsets.

The following three propositions and a remark explain the concept of admissibility. First of all, notice that, for each open family $\mathcal{U}$, saturation implies: $\mathcal{U} \in \mathrm{Cov}_X$ iff $\mathcal{U} \cup \{\emptyset\} \in \mathrm{Cov}_X$ iff $\mathcal{U} \setminus \{\emptyset\} \in \mathrm{Cov}_X$.

PROPOSITION 2.2.20. *If $\mathcal{U}$ and $\mathcal{V}$ are admissible, then:*

(a) *$\mathcal{U} \cup \mathcal{V}$ is admissible,*
(b) *$\mathcal{U} \cup_1 \mathcal{V}$ (and $\mathcal{U} \cup_1^* \mathcal{V}$ if $\mathcal{U}$ and $\mathcal{V}$ are indexed by the same set) are admissible,*
(c) *$\mathcal{U} \cap_1 \mathcal{V}$ is admissible.*

Conversely, if $\bigcup \mathcal{U}$, $\bigcup \mathcal{V}$ are open, $(\bigcup \mathcal{U}) \cap (\bigcup \mathcal{V}) = \emptyset$ and $\mathcal{U} \cup \mathcal{V}$ is admissible, then

(d) *$\mathcal{U}$ and $\mathcal{V}$ are admissible.*

Proof. (a) Let $\Phi = \{\mathcal{U}, \mathcal{V}\}$. Then, by finiteness, $\bigcup_1 \Phi = \{\bigcup \mathcal{U}, \bigcup \mathcal{V}\} \in \mathrm{Fin}(\mathrm{Op}_X) \subseteq \mathrm{Cov}_X$. By transitivity, we get $\bigcup \Phi = \mathcal{U} \cup \mathcal{V} \in \mathrm{Cov}_X$.

(b) Notice that $\mathcal{U} \cup \mathcal{V} \preceq \mathcal{U} \cup_1^* \mathcal{V}$ if $\mathcal{U}$ and $\mathcal{V}$ are indexed by the same set, and $\mathcal{U} \cup \mathcal{V} \preceq \mathcal{U} \cup_1 \mathcal{V}$ in general. Apply saturation.

(c) Consider $\Phi = \{\mathcal{U} \cap_1 V \mid V \in \mathcal{V}\}$. Since $\bigcup_1 \Phi = \{(\bigcup \mathcal{U}) \cap V \mid V \in \mathcal{V}\} = (\bigcup \mathcal{U}) \cap \mathcal{V} \in \mathrm{Cov}_X$, by transitivity we have $\bigcup \Phi = \mathcal{U} \cap_1 \mathcal{V} \in \mathrm{Cov}_X$.

(d) This follows from stability and saturation, since $(\bigcup \mathcal{U}) \cap_1 (\mathcal{U} \cup \mathcal{V}) \preceq \mathcal{U}$, and similarly for $\mathcal{V}$. ∎

PROPOSITION 2.2.21 (omitting admissible unions). *Assume the open families $\mathcal{U}, \mathcal{V}_j$ ($j \in J$) are admissible and the family $\mathcal{U} \cup \bigcup_j (\mathcal{V}_j \cup \{\bigcup \mathcal{V}_j\})$ is admissible. Then $\mathcal{U} \cup \bigcup_j \mathcal{V}_j$ is admissible.*

Proof. Notice that $\mathcal{U} \cup \bigcup_j (\mathcal{V}_j \cup \{\bigcup \mathcal{V}_j\})$ is a refinement of $\mathcal{U} \cup \bigcup_j \{\bigcup \mathcal{V}_j\}$, so the latter family is admissible. By applying transitivity to the family of families $\{\{U\} : U \in \mathcal{U}\} \cup \{\mathcal{V}_j : j \in J\}$, we get the admissibility of the family $\mathcal{U} \cup \bigcup_j \mathcal{V}_j$. ∎

REMARK 2.2.22. Notice that a subfamily of an admissible family may not be admissible, even if they have the same union. Similarly if $\mathcal{U}$ and $\mathcal{V}$ are admissible, then $\mathcal{U} \cap \mathcal{V}$ may not be admissible. For example, consider the space $\mathbb{R}_{\mathrm{salg}}$ of Example 2.2.14. Take $\mathcal{U} = \{(1/n, 1-1/n) : n \geq 3\}$, $\mathcal{V} = \{(0,1)\}$, $\mathcal{W} = \{(0,2)\}$. Then $\mathcal{U} \cup \mathcal{V}$, $\mathcal{U} \cup \mathcal{W} \in \mathrm{Cov}_{\mathbb{R}}$, but $\mathcal{U} = (\mathcal{U} \cup \mathcal{V}) \cap (\mathcal{U} \cup \mathcal{W}) \notin \mathrm{Cov}_{\mathbb{R}}$. On the other hand, an open superfamily (with the same union) of an admissible family is always admissible by saturation.

PROPOSITION 2.2.23. *In any gts, if an open family is essentially finite* (*on its union*), *then it is admissible.*

Proof. First notice that if an open family $\mathcal{U}$ has the largest element U, then $\{U\}$ and $\mathcal{U}$ are refinements of each other. Thus, by saturation, $\mathcal{U}$ is admissible.

Now if an open family $\mathcal{U}$ has a finite subcover $\mathcal{U}_0$ (of its union), then for each $V \in \mathcal{U}_0$ the family $V \cap_1 \mathcal{U}$ has the largest element V, and is admissible. Since $\mathcal{U}_0$ is admissible, so is $\mathcal{U}_0 \cap_1 \mathcal{U}$ by transitivity, and $\mathcal{U}$ by saturation. ∎

COROLLARY 2.2.24. *We always have* $\mathrm{EssFin}(\mathrm{Op}_X) \subseteq \mathrm{Cov}_X \subseteq \mathcal{P}(\mathrm{Op}_X)$.

DEFINITION 2.2.25 (cf. [K1, p. 1]). We call a subset K of a gts X *small* if for each admissible covering $\mathcal{U}$ of any open U, the set $K \cap U$ is covered by finitely many members of $\mathcal{U}$. (We then say that $\mathcal{U}$ is *essentially finite on* K or *on* $K \cap U$.)

PROPOSITION 2.2.26.

(1) *Any subset of a small set is small.*
(2) *The image of a small set under a strictly continuous mapping is small.*

Proof. (1) Take any admissible (open) covering $\mathcal{U}$ of U. Assume $L \subseteq K$ and K is small. Since $\mathcal{U}$ is essentially finite on $K \cap U$, it is also essentially finite on its subset $L \cap U$.

(2) Assume $f : X \to Y$ is strictly continuous, and $K \subseteq X$ is small. Take an admissible open covering $\mathcal{V}$ of V in Y. Then $f^{-1}(\mathcal{V})$ is essentially finite on K, and $f(K) \cap_1 \mathcal{V} = f(K \cap_1 f^{-1}(\mathcal{V}))$ is essentially finite. ∎

PROPOSITION 2.2.27. *The usual Euclidean space with the usual topology* (*denoted* $\mathbb{R}^n_{\mathrm{top}}$) *is a gts in which all small subsets are finite. In particular, the compact interval* $[0,1]$ *in* $\mathbb{R}_{\mathrm{top}}$ *is not small.*

Proof. A small subset K of the (topological) space $\mathbb{R}^n_{\mathrm{top}}$ is quasi-compact and all of its subsets are also quasi-compact. But $\mathbb{R}^n_{\mathrm{top}}$ is Hausdorff. Since all subsets of K are compact, K is a discrete set, and so finally a finite set. ∎

REMARK 2.2.28. Notice that compact sets are small in Examples 2.2.18 and 2.2.19, but not necessarily small in Example 2.2.16 (because an open subset of a compact set is not usually compact).

Let us denote the family of all small subsets of X by Sm_X, and $\mathrm{Smop}_X = \mathrm{Sm}_X \cap \mathrm{Op}_X$.

DEFINITION 2.2.29 ([H-N]). A *bornology* $\mathcal{B}$ on a set X is an ideal in $\mathcal{P}(X)$ containing every singleton. The pair $(X, \mathcal{B})$ is then called a *bornological set*, and each member of $\mathcal{B}$ a *bounded set*. A *bornological universe* is a triple $(X, \tau, \mathcal{B})$, where τ is a topology, and $\mathcal{B}$ is a bornology. A mapping between bornological sets is called *bounded* if it maps bounded sets onto bounded sets.

We always have $\mathrm{Fin}(X) \subseteq \mathrm{Sm}_X \subseteq \mathcal{P}(X)$. Moreover,

FACT 2.2.30. *The family* Sm_X *is a bornology on* X.

For an extensive study of bornologies, see [H-N].

FACT 2.2.31. *If* $\{(\mathrm{Op}_\alpha, \mathrm{Cov}_\alpha)\}_{\alpha \in A}$ *is a family of generalized topologies on a set* X*, then their intersection* $(\bigcap_\alpha \mathrm{Op}_\alpha, \bigcap_\alpha \mathrm{Cov}_\alpha)$ *is a generalized topology.*

REMARK 2.2.32. It is clear that $\bigcup \bigcap_\alpha \mathrm{Cov}_\alpha \subseteq \bigcap_\alpha \bigcup \mathrm{Cov}_\alpha$. The other inclusion follows from the fact that $\{U\}$ is admissible for each open U. The open subsets of $\bigcap_\alpha \mathrm{Cov}_\alpha$ are exactly the members of $\bigcap_\alpha \mathrm{Op}_\alpha$. Hence we can just speak about the intersection $\bigcap_\alpha \mathrm{Cov}_\alpha$.

DEFINITION 2.2.33. For any set X, a family $\mathcal{V} \in \mathcal{P}^2(X)$ and a family of families $\Psi \in \mathcal{P}^3(X)$, the smallest generalized topology Cov such that both $\mathrm{Op} = \bigcup \mathrm{Cov}$ contains $\mathcal{V}$ and Cov contains Ψ will be called the *generalized topology generated by* $\mathcal{V}$ *and* Ψ. We will denote this topology by $\langle \mathcal{V}, \Psi \rangle$, or just $\langle \Psi \rangle$ if the first family is not given.

DEFINITION 2.2.34. A *basis* of a generalized topology is a family of open sets such that any open set is an admissible union of elements from the basis. (The notion of a basis of a topology is a special case of the notion of a basis of a generalized topology.)

EXAMPLE 2.2.35. For each topological space (X, τ), the pair $(\tau, \emptyset)$ generates $\mathrm{EssFin}(\tau)$ (see Proposition 2.2.23). If $\mathcal{B}$ is a basis of τ, then $\mathcal{P}(\mathcal{B})$ as well as $(\mathcal{B}, \mathcal{P}(\mathcal{B}))$ generate $\mathcal{P}(\tau)$.

Fact 2.2.36. *If* $\mathcal{B}$ *is a basis of a generalized topology* (Op, Cov), *then* $(\mathcal{B}, \mathrm{Cov}^{\mathcal{B}})$ *generates* (Op, Cov), *where* $\mathrm{Cov}^{\mathcal{B}} = \mathrm{Cov} \cap \mathcal{P}(\mathcal{B})$.

Proposition 2.2.37. *For any mapping* $f : X \to Y$ *between sets and any family of families* $\Psi \in \mathcal{P}^3(Y)$ *we have* $f^{-1}(\langle \Psi \rangle) \subseteq \langle f^{-1}(\Psi) \rangle$. *If* f *is a bijection, then equality holds.*

Proof. By the axioms of generalized topology, the following rules lead to the construction of $\langle \Psi \rangle$ for any Ψ:

- if $\mathcal{U} \in \mathrm{Fin}(\bigcup \Psi)$, then $\{\bigcup \mathcal{U}\}, \{\bigcap \mathcal{U}\}, \mathcal{U} \in \Psi^+$,
- if $V \in \bigcup \Psi$, $\mathcal{U} \in \Psi$, then $V \cap_1 \mathcal{U} \in \Psi^+$,
- if $\Phi \in \mathcal{P}(\Psi)$, $\bigcup_1 \Phi \in \Psi$, then $\bigcup \Phi \in \Psi^+$,
- if $\mathcal{U} \in \Psi$, $\mathcal{V} \in \mathcal{P}(\bigcup \Psi)$, $\mathcal{U} \preceq \mathcal{V}$, then $\mathcal{V} \in \Psi^+$,
- if $W \in \mathcal{P}(X)$, $\mathcal{U} \in \Psi$, $W \cap_1 \mathcal{U} \in \mathcal{P}(\bigcup \Psi)$, then $\{W \cap (\bigcup \mathcal{U})\} \in \Psi^+$.

Indeed, $\langle \Psi \rangle$ is the family of all families that can be obtained from members of the original Ψ in finitely many steps of passing to a next Ψ^+ using the above rules.

If some $\mathcal{V}$ can be constructed this way from Ψ, then $f^{-1}(\mathcal{V})$ can be constructed similarly from $f^{-1}(\Psi)$. Hence $f^{-1}(\langle \Psi \rangle) \subseteq \langle f^{-1}(\Psi) \rangle$. If f is not injective or not surjective, then there are more possibilities to construct new admissible families in the domain. The bijection case is obvious. ∎

Definition 2.2.38. Let $X = \bigcup_\alpha X_\alpha$, and for each α let $\Phi_\alpha \in \mathcal{P}^3(X_\alpha)$. Write

$$\langle \Phi_\alpha \rangle^*_\alpha = \{\mathcal{U} \in \mathcal{P}^2(X) \mid \mathcal{U} \cap_1 X_\alpha \in \Phi_\alpha \text{ for each } \alpha\}.$$

Lemma 2.2.39. *Assume* $f : Y \to X$ *is any mapping, and* $X = \bigcup_\alpha X_\alpha$. *If for each* α *some family of families* $\Phi_\alpha \in \mathcal{P}^3(X_\alpha)$ *is given, then* $\langle f^{-1}(\Phi_\alpha) \rangle^*_\alpha \supseteq f^{-1}(\langle \Phi_\alpha \rangle^*_\alpha)$, *where each* $f^{-1}(\Phi_\alpha)$ *is considered as an element of* $\mathcal{P}^3(f^{-1}(X_\alpha))$.

Proof. Let $\mathcal{U} \in \langle \Phi_\alpha \rangle^*_\alpha$, so $\mathcal{U} \cap_1 X_\alpha \in \Phi_\alpha$ for each α. Then $f^{-1}(\mathcal{U}) \cap_1 f^{-1}(X_\alpha) \in f^{-1}(\Phi_\alpha)$ for each α. This means $f^{-1}(\mathcal{U}) \in \langle f^{-1}(\Phi_\alpha) \rangle^*_\alpha$. ∎

Lemma 2.2.40. *If every* Φ_α *of Definition* 2.2.38 *is a generalized topology on* X_α, *then* $\langle \Phi_\alpha \rangle^*_\alpha$ *is a generalized topology on* X.

Proof. Left to the reader. ∎

Definition 2.2.41. For any gts X and a subset $Y \subseteq X$ we induce a gts on Y by taking the generalized topology generated by $\mathrm{Cov}_X \cap_2 Y$. Then Y will be called a *subspace* of X, since the inclusion $Y \hookrightarrow X$ is a monomorphism of gts's. If moreover:

- the *open subsets* of the gts Y are exactly the traces of open subsets of the gts X on Y (i.e. $\mathrm{Op}_Y = \mathrm{Op}_X \cap_1 Y$),

- the *admissible families* of the gts Y are exactly the traces of admissible families of the gts X on Y (i.e. $\mathrm{Cov}_Y = \mathrm{Cov}_X \cap_2 Y$),

then Y will be called a *strict subspace* of X.

PROPOSITION 2.2.42. *Assume a set* $X = \bigcup_\alpha X_\alpha$ *is given. For any family* $\{(X_\alpha, \mathrm{Cov}_\alpha)\}_{\alpha\in A}$ *of gts's such that each intersection* $X_\alpha \cap X_\beta$ *is an open subspace both of* $(X_\alpha, \mathrm{Cov}_\alpha)$ *and of* $(X_\beta, \mathrm{Cov}_\beta)$, *there is a unique gts* (X, Cov_X) *having every* $(X_\alpha, \mathrm{Cov}_\alpha)$ *as an open subspace and the family* $\{X_\alpha\}_{\alpha\in A}$ *admissible.*

Proof. For existence, put $\mathrm{Cov}_X = \langle \mathrm{Cov}_\alpha\rangle^*_\alpha$. Then Cov_X is a generalized topology by Lemma 2.2.40, each X_α is an open subspace by assumption, and $\{X_\beta\}_{\beta\in A}$ is admissible, being essentially finite on each X_α.

For uniqueness, assume that Cov_X satisfies the required conditions. Notice that $\mathrm{Cov}_X \subseteq \langle \mathrm{Cov}_\alpha\rangle^*_\alpha$ by stability. If $\mathcal{U} \in \langle \mathrm{Cov}_\alpha\rangle^*_\alpha$, then $\bigcup \mathcal{U}$ belongs to Op_X by assumption and regularity. Hence $\{X_\alpha\}_\alpha \cap_1 \bigcup \mathcal{U}$ belongs to Cov_X by assumption and stability. Each $\mathcal{U} \cap_1 X_\alpha$ belongs to $\mathrm{Cov}_\alpha \subseteq \mathrm{Cov}_X$. By transitivity, $\mathcal{U} \cap_1 \{X_\alpha\}_\alpha$ belongs to Cov_X. By saturation, also $\mathcal{U} \in \mathrm{Cov}_X$. ∎

DEFINITION 2.2.43. A gts (X, Cov_X) as in Proposition 2.2.42 is called the *admissible union* of the family $(X_\alpha, \mathrm{Cov}_\alpha)$ of open subspaces, written $X = \overset{a}{\bigcup}_\alpha X_\alpha$. If the X_α are pairwise disjoint, then X is the (*generalized topological*) *direct sum* (or *coproduct*) of $\{X_\alpha\}_{\alpha\in A}$, written $X = \bigoplus_{\alpha\in A} X_\alpha$. (Notice that each X_α is then also closed. Moreover, each family of unions of some X_α's is open by regularity, and admissible by saturation.)

PROPOSITION 2.2.44. *If* $X = \overset{a}{\bigcup}_\alpha X_\alpha$, *then* $\mathrm{Cov}_X = \langle\{\{X_\alpha\}_\alpha\} \cup \bigcup_\alpha \mathrm{Cov}_{X_\alpha}\rangle$.

Proof. Set $\Omega = \langle\{\{X_\alpha\}_\alpha\} \cup \bigcup_\alpha \mathrm{Cov}_{X_\alpha}\rangle$. If $\mathcal{V} \in \mathrm{Cov}_{X_\alpha}$, then $\mathcal{V}$ is an open family in X of the form $\mathcal{U} \cap_1 X_\alpha$ with $\mathcal{U} \in \mathrm{Cov}_X$. By stability, $\mathcal{V} \in \mathrm{Cov}_X$. Since Cov_X is a generalized topology containing $\{X_\alpha\}_\alpha$, it contains Ω.

Notice that $\mathrm{Op}_X = \bigcup \Omega$. Indeed, if $G \in \mathrm{Op}_X$, then $G \cap_1 \{X_\alpha\}_\alpha \subseteq \bigcup_\alpha \mathrm{Op}_{X_\alpha} \subseteq \bigcup \Omega$. By regularity, $G = G \cap \bigcup_\alpha X_\alpha \in \bigcup \Omega$. The other implication is clear.

If $\mathcal{U} \in \mathrm{Cov}_X$, then for each α we have $\mathcal{U} \cap_1 X_\alpha \in \mathrm{Cov}_X \cap_2 X_\alpha = \mathrm{Cov}_{X_\alpha}$. The family of families $\Phi = \{\mathcal{U} \cap_1 X_\alpha\}_\alpha$ is a subfamily of $\bigcup_\alpha \mathrm{Cov}_{X_\alpha}$. The set $\bigcup \mathcal{U}$ belongs to $\bigcup \Omega$. Since $\bigcup_1 \Phi = (\bigcup \mathcal{U}) \cap_1 \{X_\alpha\}_\alpha \in \Omega$, by transitivity $\bigcup \Phi = \mathcal{U} \cap_1 \{X_\alpha\}_\alpha \in \Omega$. Its coarsening $\mathcal{U}$ is a member of Ω by saturation. ∎

The property of being strictly continuous is local in the following sense.

PROPOSITION 2.2.45. *Let* $X = \overset{a}{\bigcup} \mathcal{U}$ *and* Y *be gts's. For a mapping* $f : X \to Y$, *the following conditions are equivalent:*

(a) *f is strictly continuous,*
(b) *$f|U$ is strictly continuous for each $U \in \mathcal{U}$.*

Proof. For any set $V \subseteq X$, all $V \cap U$ with $U \in \mathcal{U}$ are open iff V is open. An open family $\mathcal{V}$ is admissible in X iff all $\mathcal{V} \cap_1 U$ for $U \in \mathcal{U}$ are admissible (in U). The proposition follows. ∎

QUESTION 2.2.46. Are there any analogous propositions for closed families?

PROPOSITION 2.2.47. *Let $X = \overset{a}{\bigcup}\mathcal{U}$ and Y be gts's. For a map $f : Y \to X$, the following conditions are equivalent:*

(a) *f is strictly continuous,*
(b) *the family $f^{-1}(\mathcal{U})$ is admissible and $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is strictly continuous for each $U \in \mathcal{U}$.*

Proof. If f is strictly continuous, then $f^{-1}(\mathcal{U})$ is admissible and each $f^{-1}(\mathcal{V})$ is admissible for an admissible $\mathcal{V}$. But then $f^{-1}(U) \cap_1 f^{-1}(\mathcal{V}) = f^{-1}(U \cap_1 \mathcal{V})$ is admissible by stability for each $U \in \mathcal{U}$. This is enough, since each U is a strict subspace.

If the family $f^{-1}(\mathcal{U})$ is admissible and each $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is strictly continuous for $U \in \mathcal{U}$, then for an admissible $\mathcal{V}$ each $(f|_{f^{-1}(U)})^{-1}(\mathcal{V}) = f^{-1}(U) \cap_1 f^{-1}(\mathcal{V})$ is admissible. Since $f^{-1}(\bigcup \mathcal{V})$ is open and $f^{-1}(\mathcal{U}) \cap_1 f^{-1}(\bigcup \mathcal{V})$ is admissible, we get admissibility of $f^{-1}(\mathcal{U}) \cap_1 f^{-1}(\mathcal{V})$ by transitivity. Now $f^{-1}(\mathcal{U}) \cap_1 f^{-1}(\mathcal{V}) \preceq f^{-1}(\mathcal{V})$, and this last family is admissible by saturation. ∎

FACT 2.2.48. *Each strictly continuous mapping is weakly continuous and bounded* (*with respect to the bornologies of small sets*).

The following examples show that a weakly continuous (and even continuous) mapping which is bounded (in the bornologies of small sets) may not be strictly continuous.

EXAMPLE 2.2.49 ($\mathbb{R}_{\mathbb{Q}}$). Take the real line $\mathbb{R}$ and define the *open sets* as the finite unions of open intervals with endpoints being rational numbers or infinities. Define the *admissible families* as the essentially finite open families. Denote by $\mathbb{R}_{\mathbb{Q}}$ the resulting gts. All subsets are small in this space. The mapping $\mathbb{R}_{\mathbb{Q}} \ni x \mapsto rx \in \mathbb{R}_{\mathbb{Q}}$ for $r \notin \mathbb{Q}$ is weakly continuous and bounded, but not continuous.

EXAMPLE 2.2.50. Consider an uncountable set X, and the topological discrete generalized topology on it (that is, $\mathrm{Cov}_X = \mathcal{P}^2(X)$). Let $Y = X$, and let Cov_Y be the family $\mathrm{EssCount}(\mathcal{P}(Y))$ of essentially countable families of subsets of Y. Then $\mathrm{id} : Y \to X$ is bounded continuous, but not strictly continuous.

PROPOSITION 2.2.51. *Each subspace Y of a gts X forms an initial subobject of X in* **GTS**.

Proof. Recall that the generalized topology $\langle \mathrm{Cov}_X \cap_2 Y\rangle$ is induced on Y. Assume that (Z, Cov_Z) is any gts, and $f : Z \to Y$ is a mapping such that $i_{YX} \circ f : Z \to X$ is strictly continuous. Then $f^{-1}(\mathrm{Cov}_X \cap_2 Y) \subseteq \mathrm{Cov}_Z$, and $\langle f^{-1}(\mathrm{Cov}_X \cap_2 Y)\rangle \subseteq \mathrm{Cov}_Z$. But $\langle f^{-1}(\mathrm{Cov}_X \cap_2 Y)\rangle \supseteq f^{-1}(\langle \mathrm{Cov}_X \cap_2 Y\rangle)$ by Proposition 2.2.37. This means f is strictly continuous, and Y can be identified with an initial subobject of X. ∎

REMARK 2.2.52. The subspaces in our sense may be identified with the extremal subobjects in the category **GTS**. By Theorem 2.2.60 below and Theorem 21.13(4) in [AHS], the class of extremal subobjects is equal to the class of initial subobjects (i.e. the inclusion is an embedding) and to the class of regular subobjects (i.e. the inclusion is an equalizer) in **GTS**.

PROPOSITION 2.2.53. *Each open subset and each small subset are strict subspaces.*

Proof. For an open U in X, the triple

$$(U, \mathrm{Op}_X \cap_1 U, \mathrm{Cov}_X \cap_2 U) = (U, \mathrm{Op}_X \cap \mathcal{P}(U), \mathrm{Cov}_X \cap \mathcal{P}^2(U))$$

is a gts.

For a small set S in X, we have $\mathrm{Cov}_X \cap_2 S = \mathrm{EssFin}(\mathrm{Op}_X \cap_1 S)$ so the triple

$$(S, \mathrm{Op}_X \cap_1 S, \mathrm{Cov}_X \cap_2 S) = (S, \mathrm{Op}_X \cap_1 S, \mathrm{EssFin}(\mathrm{Op}_X \cap_1 S))$$

is a gts (see Corollary 2.3.2). ∎

REMARK 2.2.54. In general, the problem of proving transitivity, saturation and regularity arises.

REMARK 2.2.55. Dually, we may introduce a *generalized cotopology* on any set: define a gts as a triple $(X, \mathrm{Cl}_X, \mathrm{Int}_X)$, where X is any set, Cl_X is a family of subsets of X called *closed sets*, and Int_X is a family of closed families whose members are *admissible* (*closed*) *intersections* such that the following conditions are satisfied:

- (*co-finiteness*) if $\mathcal{F} \in \mathrm{Fin}(\mathrm{Cl}_X)$, then $\bigcup \mathcal{F}, \bigcap \mathcal{F} \in \mathrm{Cl}_X$, $\mathcal{F} \in \mathrm{Int}_X$,
- (*co-stability*) if $G \in \mathrm{Cl}_X$, $\mathcal{F} \in \mathrm{Int}_X$, then $G \cup_1 \mathcal{F} \in \mathrm{Int}_X$,
- (*co-transitivity*) if $\Phi \in \mathcal{P}(\mathrm{Int}_X)$, $\bigcap_1 \Phi \in \mathrm{Int}_X$, then $\bigcup \Phi \in \mathrm{Int}_X$,
- (*co-saturation*) if $\mathcal{F} \in \mathrm{Int}_X$, $\mathcal{G} \in \mathcal{P}(\mathrm{Cl}_X)$, $\mathcal{G} \preceq^* \mathcal{F}$, then $\mathcal{G} \in \mathrm{Int}_X$,
- (*co-regularity*) if $W \in \mathcal{P}(X)$, $\mathcal{F} \in \mathrm{Int}_X$, $W \cup_1 \mathcal{F} \in \mathcal{P}(\mathrm{Cl}_X)$, then $W \cup (\bigcap \mathcal{F}) \in \mathrm{Cl}_X$.

Here $\mathcal{G} \preceq^* \mathcal{F}$ means that $\bigcap \mathcal{G} = \bigcap \mathcal{F}$ and for each $F \in \mathcal{F}$ there is some $G \in \mathcal{G}$ such that $G \subseteq F$.

Proposition 2.2.56. *Each closed subset of a gts X is a strict subspace.*

Proof. If F is a closed subset of X, then $(\mathrm{Cl}_X \cap_1 F, \mathrm{Int}_X \cap_2 F) = (\mathrm{Cl}_X \cap \mathcal{P}(F), \mathrm{Int}_X \cap \mathcal{P}^2(F))$ is a generalized cotopology with the dual generalized topology $(\mathrm{Op}_X \cap_1 F, \mathrm{Cov}_X \cap_2 F)$. ■

Corollary 2.2.57. *Each locally closed subset of a gts X is a strict subspace.*

Definition 2.2.58. For any family (or class) of mappings $f_i : X \to Y_i$, $i \in I$, if each Y_i has a generalized topology Cov_i, then the generalized topology generated by the union of the preimages $f_i^{-1}(\mathrm{Cov}_i)$ (denoted $\langle \bigcup_i f^{-1}(\mathrm{Cov}_i)\rangle$) on X will be called the *initial generalized topology* for the family $(f_i)_{i\in I}$.

Definition 2.2.59 ([AHS, Definitions 21.1, 21.7]). Let $\mathcal{C}, \mathcal{B}$ be categories. A functor $U : \mathcal{C} \to \mathcal{B}$ is a *topological functor* if each U-structured source has a unique U-initial lift. A construct $(\mathcal{C}, U)$ is called *topological* if the (forgetful) functor U is topological.

Theorem 2.2.60. *The construct* **GTS** *is topological.*

Proof. For a source of mappings $f_i : X \to Y_i$ indexed by a class I, we may assume that I is a set. Assume each Y_i has a generalized topology Cov_i. Give X the initial generalized topology for the family $(f_i)_{i\in I}$. For any (Z, Cov_Z) and a mapping $h : Z \to X$, if all $f_i \circ h$ are morphisms, then $\bigcup_i h^{-1}(f^{-1}(\mathrm{Cov}_i)) \subseteq \mathrm{Cov}_Z$. Since Cov_Z is a generalized topology, it also contains $\langle \bigcup_i h^{-1}(f^{-1}(\mathrm{Cov}_i))\rangle$ as well as $h^{-1}(\langle \bigcup_i f^{-1}(\mathrm{Cov}_i)\rangle)$. This means h is a morphism in **GTS**. We have proved that each U-structured source has a U-initial lift. By [AHS, Theorem 21.5] this is enough, since the construct $(\mathbf{GTS}, U)$ is amnestic, which means that no two different objects in the same fibre of U are equivalent (see [AHS, Definition 5.4]). ■

Corollary 2.2.61 ([AHS, Theorem 21.17]). *The category* **GTS** *is complete, co-complete, wellpowered, co-wellpowered, is an* (*Epi, Extremal Mono-Source*)*-category, has regular factorizations and has separators and coseparators.*

Fact 2.2.62 ([AHS, Proposition 21.15]). *The forgetful functor* $U : \mathbf{GTS} \to \mathbf{Set}$ *preserves and uniquely lifts* (*small*) *limits and colimits. In particular, all* (*small*) *limits and colimits in* **GTS** *are concrete.*

Remark 2.2.63. The category **Top** may be treated as a full subcategory of **GTS** by the full embedding $i_t(X,\tau) = (X, \tau, \mathcal{P}(\tau))$, $i_t(f) = f$.

Definition 2.2.64. The functor top : **GTS** → **Top**, which is given by the formulas

$$\mathrm{top}(X) = X_{\mathrm{top}}, \qquad \mathrm{top}(f) = f,$$

where $\mathrm{Op}_{X_{\mathrm{top}}} = \tau(\mathrm{Op}_X)$, $\mathrm{Cov}_{X_{\mathrm{top}}} = \mathcal{P}(\tau(\mathrm{Op}_X))$, will be called the *topologization* functor.

FACT 2.2.65. *The functor* top *is a right adjoint of* i_t*, hence preserves limits.*

EXAMPLE 2.2.66 (topologization of the Boolean algebra of definable sets). Let $\mathcal{M}$ be a first order, one-sorted mathematical structure (i.e. a set M with some distinguished relations, constants and functions). Then for each $A \subseteq M$ and $n \in \mathbb{N}$, the family $\mathrm{Def}_n(\mathcal{M}, A)$ of A-definable subsets of M^n forms a Boolean algebra. Consider $\mathrm{Op} = \mathrm{Def}_n(\mathcal{M}, A)$, and $\mathrm{Cov} = \mathrm{EssFin}(\mathrm{Op})$. Then the generated topology $\tau(\mathrm{Op})$ is the family of subsets of M^n that are $\bigvee$-definable over A, and the complements of members of $\tau(\mathrm{Op})$ (i.e. weakly closed sets) are the subsets of M^n type-definable over A. (In practice, in model theory often bounds are set on the cardinality of the family of open sets forming a $\bigvee$-definable or type-definable set, or on the cardinality of the set of parameters A; see for example [BOPP, p. 304], [EKP, 0.2 Preliminaries] or [PS, Definition 2.1].)

DEFINITION 2.2.67. A gts X will be called *partially topological* if Op_X is a topology, and *topological* if $\mathrm{Cov}_X = \mathcal{P}(\mathrm{Op}_X)$.

FACT 2.2.68. *Each topological gts is partially topological.*

FACT 2.2.69. *Each weakly continuous mapping between partially topological gts's is continuous.*

REMARK 2.2.70. We get an obvious faithful functor bor : $\mathbf{GTS} \to \mathbf{Bor}$ to the category of bornological sets with bounded mappings, and a faithful functor ubor : $\mathbf{GTS} \to \mathbf{UBor}$, with

$$\mathrm{ubor}(X, \mathrm{Op}_X, \mathrm{Cov}_X)) = (X, \tau(\mathrm{Op}_X), \mathrm{Sm}_X)$$

and $\mathrm{ubor}(f) = f$, to the category of bornological universes with continuous bounded mappings.

PROPOSITION 2.2.71. *The restriction of* ubor *to the full subcategory* $\mathbf{GTS}_{\mathrm{pt}}$ *of partially topological gts's has a left adjoint* gts : $\mathbf{UBor} \to \mathbf{GTS}_{\mathrm{pt}}$ *defined by the rules*

$$\mathrm{gts}(X, \tau, \mathcal{B}) = (X, \tau, \mathrm{EF}(\tau, \mathcal{B})) \quad \textit{and} \quad \mathrm{gts}(f) = f, \quad \textit{where}$$
$$\mathrm{EF}(\tau, \mathcal{B}) = \{\mathcal{U} \subseteq \tau \mid \mathcal{U} \cap_1 S \textit{ is essentially finite for each } S \in \mathcal{B}\}.$$

Proof. Notice that the following hom-sets are equal:

$$\mathbf{GTS}_{\mathrm{pt}}((X, \tau, \mathrm{EF}(\tau, \mathcal{B})), (Y, \sigma, \mathrm{Cov}_Y)) = \mathbf{UBor}((X, \tau, \mathcal{B}), (Y, \sigma, \mathrm{Sm}_Y)).$$

Indeed, belonging to the left side means for a mapping $f : X \to Y$ that $f^{-1}(\sigma) \subseteq \tau$ and $f^{-1}(\mathrm{Cov}_Y) \subseteq \mathrm{EF}(\tau, \mathcal{B})$, and belonging to the right side means that $f^{-1}(\sigma) \subseteq \tau$ and $f(\mathcal{B}) \subseteq \mathrm{Sm}_Y$. These two conditions are equivalent due to Lemma 2.2.72. Indeed, if $f(\mathcal{B}) \subseteq \mathrm{Sm}_Y$ (hence $\mathcal{B} \preceq f^{-1}(\mathrm{Sm}_Y)$), then $f^{-1}(\mathrm{Cov}_Y) \subseteq f^{-1}(\mathrm{EF}(\sigma, \mathrm{Sm}_Y)) \subseteq \mathrm{EF}(\tau, f^{-1}(\mathrm{Sm}_Y)) \subseteq \mathrm{EF}(\tau, \mathcal{B})$. And if $f^{-1}(\mathrm{Cov}_Y) \subseteq \mathrm{EF}(\tau, \mathcal{B})$, then, for any $B \in \mathcal{B}$ and any $\mathcal{V} \in \mathrm{Cov}_Y$, we get $f^{-1}\mathcal{V} \cap_1 B \in \mathrm{EssFin}(\tau \cap_1 B)$. This implies $\mathcal{V} \cap_1 f(B) \in \mathrm{EssFin}(\sigma \cap_1 f(B))$, which means that $f(B)$ belongs to Sm_Y. ∎

LEMMA 2.2.72. *Assume* $(X, \tau, \mathrm{Cov}_X)$ *and* $(Y, \sigma, \mathrm{Cov}_Y)$ *are objects of* $\mathbf{GTS}_{\mathrm{pt}}$. *If f is continuous, then*

$$f^{-1}(\mathrm{EF}(\sigma, \mathrm{Sm}_Y)) \subseteq \mathrm{EF}(\tau, f^{-1}(\mathrm{Sm}_Y)).$$

Proof. Assume $\mathcal{V} \in \mathrm{EF}(\sigma, \mathrm{Sm}_Y)$ and $S \in f^{-1}(\mathrm{Sm}_Y)$. Then $f(S) \in \mathrm{Sm}_Y$, hence $\mathcal{V} \cap_1 f(S) \in \mathrm{EssFin}(\sigma \cap_1 f(S))$. Applying f^{-1}, we get $f^{-1}\mathcal{V} \cap_1 f^{-1} fS \in \mathrm{EssFin}(\tau \cap_1 f^{-1} fS)$. In particular, $f^{-1}\mathcal{V} \cap_1 S \in \mathrm{EssFin}(\tau \cap_1 S)$. This means $f^{-1}\mathcal{V} \in \mathrm{EF}(\tau, f^{-1}(\mathrm{Sm}_Y))$. ∎

DEFINITION 2.2.73. A gts X will be called *weakly discrete* if all its singletons are open subsets, *discrete* if all its subsets are open, and *topological discrete* if all families of subsets of X are open and admissible.

EXAMPLE 2.2.74 (strange weakly discrete small spaces). Let X be any infinite gts where the *open sets* are exactly the finite sets or the whole space, and the *admissible families* are exactly the essentially finite open families. Then X is weakly discrete and the generated topology is discrete.

EXAMPLE 2.2.75 (semialgebraic $\mathbb{N}$). The space $\mathbb{R}_{\mathrm{salg}}$ has a small infinite weakly discrete strict subspace $\mathbb{N}$. This subspace $\mathbb{N}$ differs from the space of Example 2.2.74.

EXAMPLE 2.2.76 (infinite discrete small spaces). On any infinite set X, there is a generalized topology making X a discrete space with X small. It is enough to declare an *open subset* to be any subset, and an *admissible family* to be any essentially finite family.

The spaces from Examples 2.2.74 and 2.2.75 are weakly discrete but not discrete. The space from Example 2.2.76 is discrete, but not topological discrete.

FACT 2.2.77. *For a topological discrete space X, we have* $X = \bigoplus_{x \in X} \{x\}$.

DEFINITION 2.2.78. A subset $Y \subseteq X$ will be called *dense* in X if its weak closure (i.e. closure in the generated topology) is equal to X. A gts X is *separable* if there is a countable dense subset of X.

Spaces 1–7 in Example 2.2.14 are all separable.

DEFINITION 2.2.79. We will say that a gts X satisfies the *first axiom of countability* if each point of X has a countable basis of open neighbourhoods (such a family may easily be made admissible), and satisfies the *second axiom of countability* if there is a countable basis of the generalized topology.

Of course, each gts satisfying the second axiom of countability is separable. The space from Example 2.2.49 satisfies the second axiom of countability.

EXAMPLE 2.2.80. The semialgebraic real line $\mathbb{R}_{\mathrm{salg}}$ has its natural topology generated by a countable basis $\{(p,q) : p < q,\ p,q \in \mathbb{Q}\}$ which is not a basis of the generalized topology. More generally, spaces 3–7 from Example 2.2.14 satisfy the first axiom of countability, but not the second.

The separation axioms in **GTS** have weak and strong versions.

DEFINITION 2.2.81. A gts X will be called: (a) *weakly* T_1 if for each $x \in X$ and $y \in X \setminus \{x\}$ there is an open set U such that $x \in U$ and $y \notin U$; (b) *strongly* T_1 if each singleton is a closed subset of X; (c) *weakly Hausdorff* if for any distinct points $x, y \in X$ there are open disjoint sets U, V such that $x \in U$ and $y \in V$; (d) *strongly Hausdorff* if it is weakly Hausdorff and strongly T_1; (e) *weakly regular* if for each $x \in X$ and each subset F of X not containing x and either closed or a singleton there are open disjoint sets U, V such that $x \in U$ and $F \subseteq V$; (f) *strongly regular* if it is strongly T_1 and weakly regular; (g) *weakly normal* if for two disjoint sets F, G, each either closed or a singleton, there are open disjoint sets U, V such that $F \subseteq U$ and $G \subseteq V$; (h) *strongly normal* if it is strongly T_1 and weakly normal.

PROPOSITION 2.2.82. *A gts X is weakly T_1* (*weakly Hausdorff, respectively*) *if and only if the generated topology of X is T_1* (*Hausdorff, respectively*). *Each weakly regular gts has a regular* (*Hausdorff*) *generated topology.*

Proof. The first two cases are obvious. If a gts X is weakly regular, then for a point $x \in X$ and a weakly closed set F not containing x the set F^c is weakly open. Some open U not intersecting F and containing x exists. By weak regularity, there exist: an open V containing x and an open W containing U^c such that V and W are disjoint. In particular, V and W are weakly open. ∎

There is a weakly discrete small space (see Example 2.2.74) which is weakly Hausdorff but not strongly T_1. This space also has a regular (Hausdorff) strong topology, but is not weakly regular.

PROPOSITION 2.2.83. *Each gts with the closure property* (*CPG*) *and a regular* (*Hausdorff*) *generated topology is weakly regular.*

Proof. Consider a point x and a set F (possibly a singleton) in a gts from the assumption separated by weakly open sets $U \ni x$, $V \supseteq F$. There exists an open subset U' containing x. The weak closure $\overline{U'}$ is closed and disjoint with V. ∎

EXAMPLE 2.2.84. The space $\mathbb{R}_{\mathbb{Q}}$ from Example 2.2.49 is weakly normal, but not strongly T_1 (irrational points are not closed).

DEFINITION 2.2.85. The *quasi-component* $\tilde{C}_x$ of x in the space X is the intersection of all clopen subsets of X containing x.

FACT 2.2.86. *All quasi-components are weakly closed.*

DEFINITION 2.2.87. A subset Y of a gts X will be called *connected* if there is no pair U, V of open subsets of X such that $Y \subseteq U \cup V$, $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$, and $Y \cap U \cap V = \emptyset$.

The spaces from 1–7 of Example 2.2.14 are all connected. Also there is a weakly discrete small space (Example 2.2.74) which is connected. Discrete spaces having more than one point are not connected. "The subanalytic site" is connected iff the manifold M is connected. The space from Example 2.2.75 is not connected.

PROPOSITION 2.2.88. *If X is a gts, $x \in X$, and $\{C_\alpha\}_{\alpha \in A}$ is a family of connected subsets of X each containing x, then $\bigcup_{\alpha \in A} C_\alpha$ is connected.*

Proof. Assume that open subsets U, V of X satisfy $\bigcup_{\alpha \in A} C_\alpha \cap U \neq \emptyset$, $\bigcup_{\alpha \in A} C_\alpha \cap V \neq \emptyset$, $\bigcup_{\alpha \in A} C_\alpha \subseteq U \cup V$, $\bigcup_{\alpha \in A} C_\alpha \cap U \cap V = \emptyset$. Assume $x \in U$. Let $y \in C_{\alpha_0} \cap V$ for some $\alpha_0 \in A$. The sets $U \cap C_{\alpha_0}$ and $V \cap C_{\alpha_0}$ are non-empty, and cover C_{α_0}, thus C_{α_0} is not connected, a contradiction. This proves $\bigcup_{\alpha \in A} C_\alpha$ is connected. ∎

DEFINITION 2.2.89. The *connected component* of a point $x \in X$ of a space X is the largest connected set C_x containing x.

REMARK 2.2.90. Since the weak closure of C_x is connected, C_x is always weakly closed. Each connected component is contained in a quasi-component.

FACT 2.2.91. *Each mapping from a topological discrete space to any gts is strictly continuous* (*hence each subset of the codomain may be the image*).

DEFINITION 2.2.92. A strictly continuous mapping between gts's will be called *open* if the image of any open subset of the domain is an open subset of the range. Similarly, a *closed* mapping maps closed subsets onto closed subsets.

Each strict homeomorphism is both an open mapping and a closed mapping.

EXAMPLE 2.2.93. The identity mapping from the infinite discrete small space (see Example 2.2.76) on some (infinite) set X to the topological discrete space on this set is a closed and open strictly continuous bijection, but not a strict homeomorphism.

Definition 2.2.94. A *local strict homeomorphism* is a strictly continuous mapping $f : X \to Y$ for which there is an open admissible covering $X = \bigcup_{\alpha \in A} X_\alpha$ such that each $f|_{X_\alpha}$ is an open mapping and a strict homeomorphism onto its image.

The next example shows that a local strict homeomorphism may not be an open mapping. (We cannot expect closedness even in **Top**.)

Example 2.2.95. A locally semialgebraic covering $p : \mathbb{R}_{\text{loc}} \to \mathbb{S}^1$ of Example 13 (see also Example 3) in [P1] is a local strict homeomorphism but it is neither an open mapping nor a closed mapping. (Here $\mathbb{R}_{\text{loc}}$ may be understood as $(\mathbb{R}_{\text{salg}})_{\text{loc}}$, see Definition 2.1.15 in [P2].) After passing to the strong topologies, we get the mapping $p_{\text{top}} : \mathbb{R}_{\text{top}} \to (\mathbb{S}^1)_{\text{top}}$, which is open but not closed.

Question 2.2.96. Is any subset of a gts a strict subspace?

2.3. Small spaces

Definition 2.3.1. A gts X (i.e. the pair (X, Cov_X)) will be called a *small space* if the set X is small in the space X. The class of small spaces forms a full subcategory **SS** of **GTS**.

Corollary 2.3.2. *If X is a small space, then* $\text{Cov}_X = \text{EssFin}(\text{Op}_X)$.

Definition 2.3.3. A *Noetherian family* of sets is a family whose intersection is the intersection of some finite subfamily. A family of families will be called *Noetherian* if each of its members is a Noetherian family.

Fact 2.3.4. *For each gts X, we have* $\text{Noeth}(\text{Cl}_X) \subseteq \text{Int}_X \subseteq \mathcal{P}(\text{Cl}_X)$, *where* $\text{Noeth}(\text{Cl}_X)$ *is the family of all Noetherian closed families of X.*

Fact 2.3.5. *A gts X is small iff the family* Int_X *of admissible intersections is Noetherian, i.e.* $\text{Int}_X = \text{Noeth}(\text{Cl}_X)$.

Example 2.3.6. Any topological space (X, τ) may be made small (and partially topological) by declaring $\text{Op}_X = \tau$ and $\text{Cov}_X = \text{EssFin}(\text{Op}_X)$.

Example 2.3.7 (tower of closed sets in $\mathbb{N}$). Let us introduce a generalized topology on $\mathbb{N}$ in the following way: X is *open* iff X is a final interval or the empty set, and any open family is *admissible.* Then also all admissible families are essentially finite.

Fact 2.3.8. *Let X, Y be small spaces, and $f : X \to Y$ be any mapping. The following conditions are equivalent:*

(a) *f is strictly continuous,*
(b) *f is continuous.*

Remark 2.3.9. If both X and Y are small, then in condition (b) of Proposition 2.2.47 we can change “admissible” to “open”.

The spaces from Examples 2.2.74, 2.2.75, 2.2.76 are small.

Proposition 2.3.10. *All subsets of a small space are strict subspaces.*

Proof. This follows from Propositions 2.2.26 and 2.2.53. ∎

Theorem 2.3.11. *The construct* **SS** *is topological.*

Proof. This situation is similar to that of Theorem 2.2.60. We get

$$h^{-1}\Big(\Big\langle \bigcup_i f_i^{-1}(\mathrm{EssFin}(\mathrm{Op}_i))\Big\rangle\Big) \subseteq \Big\langle \bigcup_i h^{-1}(f_i^{-1}(\mathrm{EssFin}(\mathrm{Op}_i)))\Big\rangle$$
$$= \mathrm{EssFin}\Big(\bigcup_i h^{-1}(f_i^{-1}(\mathrm{Op}_i))\Big) \subseteq \mathrm{EssFin}(\mathrm{Op}_Z). \ ∎$$

Fact 2.3.12 ([AHS, Theorem 21.17]). *The category* **SS** *is complete, cocomplete, wellpowered, co-wellpowered, is an* (*Epi, Extremal Mono-Source*)*-category, has regular factorizations and has separators and coseparators.*

Remark 2.3.13. The categories **GTS** and **SS** are not regular, since Counterexample 2.4.5 from [Bor] applies. This is also valid for the categories **LSS**, **WSS**, **NWSS** to be considered.

Fact 2.3.14 ([AHS, Proposition 21.15]). *The forgetful functor* V : **SS** $\to$ **Set** *preserves and uniquely lifts* (*small*) *limits and colimits. In particular, all* (*small*) *limits and colimits in* **SS** *are concrete.*

Proposition 2.3.15. *The canonical projections from a binary* (*or finite*) *product of small spaces to its factors are open and closed mappings.*

Proof. The canonical projection $\pi_2 : X \times Y \to Y$ is open by the construction of the product $X \times Y$, where the open sets are exactly the finite unions of open boxes. Similarly, the dual $\pi_2^c : X \times Y \to Y$ is open, where $\pi_2^c(W) = \{y \in Y \mid (x, y) \in W \text{ for each } x \in X\}$. ∎

Definition 2.3.16. The functor sm : **GTS** $\to$ **SS** given by the formulas

$$\mathrm{sm}(X) = X_{\mathrm{sm}}, \quad \mathrm{sm}(f) = f,$$

where $\mathrm{Op}_{X_{\mathrm{sm}}} = \mathrm{Op}_X$ and $\mathrm{Cov}_{X_{\mathrm{sm}}} = \mathrm{EssFin}(\mathrm{Op}_X)$, will be called the *smallification* functor.

Consider now the full and faithful inclusion functor i_s : **SS** $\to$ **GTS**.

Proposition 2.3.17. *The functor* sm *is a left adjoint of* i_s, *hence preserves colimits.*

Proof. Notice that the preimage of an essentially finite family is always essentially finite. For any object (X, Cov_X) of **GTS** and any object

$(Y, \mathrm{EssFin}(\mathrm{Op}_Y))$ of **SS**, we get the following equality of hom-sets:

$$\mathbf{GTS}((X, \mathrm{Cov}_X), (Y, \mathrm{EssFin}(\mathrm{Op}_Y))) = \mathbf{SS}((X, \mathrm{EssFin}(\mathrm{Op}_X)), (Y, \mathrm{EssFin}(\mathrm{Op}_Y))).$$

Both sets contain exactly those mappings $f : X \to Y$ which satisfy the condition $f^{-1}(\mathrm{Op}_Y) \subseteq \mathrm{Op}_X$. This equality is clearly natural in (X, Cov_X) and $(Y, \mathrm{EssFin}(\mathrm{Op}_Y))$. Since sm is a left adjoint, it preserves colimits by the dual of Proposition 18.9 in [AHS]. ∎

PROPOSITION 2.3.18. *The category* $\mathbf{SS}_{\mathrm{pt}}$ *of partially topological small spaces is isomorphic to the category* **Top**.

Proof. The isomorphism is given by the (restrictions of) the known functors sm : $\mathbf{Top} \to \mathbf{SS}_{\mathrm{pt}}$ and top : $\mathbf{SS}_{\mathrm{pt}} \to \mathbf{Top}$. ∎

FACT 2.3.19. *All small topological discrete spaces are finite.*

PROPOSITION 2.3.20. *A finite subspace of a strongly* T_1 *space is a closed subspace and a small topological discrete gts.*

Proof. Closedness is obvious. A finite strongly T_1 space is discrete and both topological and small. ∎

FACT 2.3.21. *Each indiscrete space* (*i.e. with* $\mathrm{Op}_X = \{\emptyset, X\}$) *is both small and topological.*

DEFINITION 2.3.22. We will say that a gts X *satisfies* (AQC) (i.e. has admissible family of quasi-components) if the family of quasi-components of X is open and admissible (i.e. X decomposes into the direct sum of its quasi-components). We will say that a gts X *satisfies* (ACC) (i.e. has admissible family of connected components) if the family of connected components of X is open and admissible (i.e. X decomposes into the direct sum of its connected components).

PROPOSITION 2.3.23. *Each small space satisfying* (AQC) *has a finite number of quasi-components. Each small space satisfying* (ACC) *has a finite number of connected components.*

Proof. Choose one point in any quasi-component (connected component, respectively) of the space. The resulting strict subspace is topological discrete and small, so it is finite. ∎

The small space from Example 2.2.76 has infinitely many connected components.

PROPOSITION 2.3.24. *A local strict homeomorphism from a small space is an open mapping.*

Proof. This follows by decomposing a local strict homeomorphism into a union of finitely many strict homeomorphisms. ∎

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