ANNALES
POLONICI MATHEMATICI
107.3 (2013)

Growth and fixed points of meromorphic solutions of higher order linear differential equations

by Habib Habib and Benharrat Belaïdi (Mostaganem)

Abstract. We investigate the growth and fixed points of meromorphic solutions of higher order linear differential equations with meromorphic coefficients and their derivatives. Our results extend the previous results due to Peng and Chen.

1. Introduction and statement of results. In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromorphic functions (see [H], [YY]). In addition, we will use $\sigma(f)$, $\sigma_2(f)$ to denote respectively the order and the hyper-order of growth of a meromorphic function $f(z)$, and $\lambda(f)$, $\overline{\lambda}(f)$, $\overline{\tau}(f)$ to denote respectively the exponents of convergence of the zero-sequence, of the sequence of distinct zeros and of the sequence of distinct fixed points of $f(z)$. See [H], [YY], [C2], [WY] for notations and definitions.

Consider the second order linear differential equation

$$f'' + A_1(z)e^{P(z)}f' + A_0(z)e^{Q(z)}f = 0, \tag{1.1}$$

where $P(z), Q(z)$ are nonconstant polynomials and $A_1(z)$, $A_0(z)$ ($\not\equiv 0$) are entire functions such that $\sigma(A_1) < \deg P(z)$, $\sigma(A_0) < \deg Q(z)$. Gundersen [Gu2, p. 419] showed that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [CS], Chen and Shon have investigated the case when $\deg P(z) = \deg Q(z)$ and proved the following results.

Theorem 1.1 ([CS]). *Let $A_j(z)$ ($\not\equiv 0$) ($j = 0, 1$) be meromorphic functions with $\sigma(A_j) < 1$ ($j = 0, 1$), a, b be complex numbers such that $ab \neq 0$*

2010 *Mathematics Subject Classification*: Primary 34M10; Secondary 30D35.
Key words and phrases: linear differential equation, meromorphic solutions, order of growth, fixed points.

DOI: 10.4064/ap107-3-2

and $\arg a \neq \arg b$ *or* $a = cb$ $(0 < c < 1)$. *Then every meromorphic solution* $f(z) \not\equiv 0$ *of the equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \tag{1.2}$$

has infinite order.

In the same paper, Chen and Shon have investigated the fixed points of solutions, their 1st and 2nd derivatives and differential polynomials, and proved

THEOREM 1.2 ([CS]). *Let* $A_j(z)$ $(j = 0, 1)$, a, b, c *satisfy the additional hypotheses of Theorem* 1.1. *Let* d_0, d_1, d_2 *be complex constants that are not all zero. If* $f(z) \not\equiv 0$ *is any meromorphic solution of* (1.2), *then:*

(i) f, f', f'' *all have infinitely many fixed points and satisfy*

$$\overline{\lambda}(f - z) = \overline{\lambda}(f' - z) = \overline{\lambda}(f'' - z) = \infty,$$

(ii) *the differential polynomial*

$$g(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinitely many fixed points and satisfies $\overline{\lambda}(g - z) = \infty$.

Recently in [PC], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and proved the following result.

THEOREM 1.3 ([PC]). *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ *be entire functions with* $\sigma(A_j) < 1$, a_1, a_2 *be complex numbers such that* $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (*suppose that* $|a_1| \leq |a_2|$). *If* $\arg a_1 \neq \pi$ *or* $a_1 < -1$, *then every solution* f $(\not\equiv 0)$ *of the equation*

$$f'' + e^{-z}f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z})f = 0$$

has infinite order and $\sigma_2(f) = 1$.

The main purpose of this paper is to extend the results of Theorem 1.3 to some higher order linear differential equations. We will prove the following results.

THEOREM 1.4. *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_1(z)$ $(\not\equiv 0)$ *and* $B_l(z)$ $(l = 2, \ldots, k-1)$ *be meromorphic functions with*

$$\max\{\sigma(A_j)\ (j = 1, 2),\ \sigma(B_l)\ (l = 1, \ldots, k-1)\} < 1,$$

a_1, a_2 *be complex numbers such that* $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (*suppose that* $|a_1| \leq |a_2|$). *If* $\arg a_1 \neq \pi$ *or* $a_1 < -1$, *then every meromorphic solution* f $(\not\equiv 0)$ *of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_2 f'' + B_1 e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z})f = 0 \tag{1.3}$$

satisfies $\sigma(f) = \infty$.

THEOREM 1.5. *Let* $A_j(z)$ $(j = 1,2)$, $B_l(z)$ $(l = 1,\ldots,k-1)$, a_1, a_2 *satisfy the additional hypotheses of Theorem* 1.4. *If* f $(\not\equiv 0)$ *is any meromorphic solution of* (1.3), *then* f, f' f'' *all have infinitely many fixed points and*

$$\overline{\tau}(f) = \overline{\tau}(f') = \overline{\tau}(f'') = \infty.$$

2. Preliminary lemmas. We define the *linear measure* of a set $E \subset [0,\infty)$ by $m(E) = \int_0^\infty \chi_E(t)\,dt$ and the *logarithmic measure* of a set $F \subset (1,\infty)$ by $lm(F) = \int_1^\infty (\chi_F(t)/t)\,dt$, where χ_H is the characteristic function of a set H.

LEMMA 2.1 ([Gu1]). *Let* f *be a transcendental meromorphic function with* $\sigma(f) = \sigma < \infty$. *Let* $\varepsilon > 0$ *be a given constant, and let* k, j *be integers satisfying* $k > j \geq 0$. *Then there exists a set* $E_1 \subset [-\pi/2, 3\pi/2)$ *of linear measure zero such that if* $\psi \in [-\pi/2, 3\pi/2) \setminus E_1$, *then there is a constant* $R_0 = R_0(\psi) > 1$ *such that for all* z *with* $\arg z = \psi$ *and* $|z| \geq R_0$,

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

LEMMA 2.2 ([CS], [M]). *Consider* $g(z) = A(z)e^{az}$, *where* $A(z) \not\equiv 0$ *is a meromorphic function of order* $\sigma(A) = \alpha < 1$, *and* a *is a complex constant,* $a = |a|e^{i\varphi}$ $(\varphi \in [0, 2\pi))$. *Set* $E_2 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, *so* E_2 *is a finite set. Then for any given* ε $(0 < \varepsilon < 1 - \alpha)$ *there is a set* $E_3 \subset [0, 2\pi)$ *of linear measure zero such that if* $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3)$, *then for* r *sufficiently large we have:*

(i) *If* $\cos(\varphi + \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)r\delta(az,\theta)\} \leq |g(z)| \leq \exp\{(1+\varepsilon)r\delta(az,\theta)\}. \tag{2.2}$$

(ii) *If* $\cos(\varphi + \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)r\delta(az,\theta)\} \leq |g(z)| \leq \exp\{(1-\varepsilon)r\delta(az,\theta)\}, \tag{2.3}$$

where $\delta(az,\theta) = |a|\cos(\varphi+\theta)$.

LEMMA 2.3 ([PC]). *Let* $n \geq 1$ *be a natural number, and* $P_j(z) = a_{jn}z^n + \cdots$ $(j = 1,2)$ *be nonconstant polynomials, where* a_{jq} $(q = 1,\ldots,n)$ *are complex numbers and* $a_{1n}a_{2n} \neq 0$. *Set* $z = re^{i\theta}$, $a_{jn} = |a_{jn}|e^{i\theta_j}$, $\theta_j \in [-\pi/2, 3\pi/2)$, $\delta(P_j,\theta) = |a_{jn}|\cos(\theta_j + n\theta)$. *Then there is a set* $E_4 \subset [-\pi/(2n), 3\pi/(2n))$ *that has linear measure zero such that if* $\theta_1 \neq \theta_2$, *then there exists a ray* $\arg z = \theta$, *with* $\theta \in (-\pi/(2n), \pi/(2n)) \setminus (E_4 \cup E_5)$, *satisfying either*

$$\delta(P_1,\theta) > 0, \qquad \delta(P_2,\theta) < 0, \tag{2.4}$$

or

$$\delta(P_1,\theta) < 0, \qquad \delta(P_2,\theta) > 0, \tag{2.5}$$

where $E_5 = \{\theta \in [-\pi/(2n), 3\pi/(2n)) : \delta(P_j, \theta) = 0\}$ *is a finite set, in particular, it has linear measure zero.*

Remark (see [PC]). In Lemma 2.3, we can obtain the same conclusion if we replace $\theta \in (-\pi/(2n), \pi/(2n)) \setminus (E_4 \cup E_5)$ by $\theta \in (\pi/(2n), 3\pi/(2n)) \setminus (E_4 \cup E_5)$.

Lemma 2.4 ([CS]). *Let $f(z)$ be a transcendental meromorphic function of order $\rho(f) = \alpha < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_6 \subset [-\pi/2, 3\pi/2)$ that has linear measure zero such that if $\theta \in [-\pi/2, 3\pi/2) \setminus E_6$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z with $\arg z = \theta$ and $|z| \geq R_1$,*

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\alpha+\varepsilon}\}. \tag{2.6}$$

Lemma 2.5 ([GCC, p. 30]). *Let $n \geq 1$, and let $P_1, \ldots, P_n$ be nonconstant polynomials of respective degrees $d_1, \ldots, d_n$. Suppose that when $i \neq j$, $\deg(P_i - P_j) = \max\{d_i, d_j\}$. Set $A(z) = \sum_{j=1}^{n} B_j(z) e^{P_j(z)}$, where $B_j(z)$ $(\not\equiv 0)$ are meromorphic functions satisfying $\sigma(B_j) < d_j$. Then $\sigma(A) = \max_{1 \leq j \leq n}\{d_j\}$.*

By induction, we can easily prove the following lemma.

Lemma 2.6. *Let $f(z) = g(z)/d(z)$, where $g(z)$ is a transcendental entire function, and let $d(z)$ be the canonical product* (*or polynomial*) *formed with the nonzero poles of $f(z)$. Then*

$$f^{(n)} = \frac{1}{d}[g^{(n)} + D_{n,n-1} g^{(n-1)} + D_{n,n-2} g^{(n-2)} + \cdots + D_{n,1} g' + D_{n,0} g] \tag{2.7}$$

and

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + D_{n,n-1}\frac{g^{(n-1)}}{g} + D_{n,n-2}\frac{g^{(n-2)}}{g} + \cdots + D_{n,1}\frac{g'}{g} + D_{n,0}, \tag{2.8}$$

where $D_{n,j}$ is a sum of a finite number of terms of the type

$$\sum_{(j_1,\ldots,j_n)} C_{jj_1\cdots j_n}\left(\frac{d'}{d}\right)^{j_1} \cdots \left(\frac{d^{(n)}}{d}\right)^{j_n},$$

$C_{jj_1\cdots j_n}$ are constants, and $j + j_1 + 2j_2 + \cdots + nj_n = n$.

Lemma 2.7 ([C1]). *Let A_0, $A_1, \ldots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F,$$

then $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

The following lemma, due to Gross [Gr], is important in the factorization and uniqueness theory of meromorphic functions, and plays an important role in this paper as well.

LEMMA 2.8 ([Gr], [YY]). *Suppose that $f_1(z), \dots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$.
(ii) *$g_j(z) - g_k(z)$ are not constant for $1 \leq j < k \leq n$.*
(iii) *For $1 \leq j \leq n$ and $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h(z)-g_k(z)})\}$ $(r \to \infty, r \notin E_7)$, where E_7 is a set of finite linear measure.*

Then $f_j(z) \equiv 0$ $(j = 1, \dots, n)$.

LEMMA 2.9 ([XY]). *Suppose that $f_1(z), \dots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}$.
(ii) *If $1 \leq j \leq n+1$ and $1 \leq k \leq n$, then the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1$ and $1 \leq h < k \leq n$, then the order of f_j is less than the order of $e^{g_h - g_k}$.*

Then $f_j(z) \equiv 0$ $(j = 1, \dots, n+1)$.

3. Proofs of the theorems

Proof of Theorem 1.4. First of all we prove that equation (1.3) cannot have a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. Assume there is such a solution f. Rewrite (1.3) as

$$B_1 f' e^{-z} + A_1 f e^{a_1 z} + A_2 f e^{a_2 z} = -\{f^{(k)} + B_{k-1} f^{(k-1)} + \dots + B_2 f''\}. \tag{3.1}$$

For $a_2 \neq -1$, by (3.1) and Lemma 2.5, we have

$$\begin{aligned} 1 &= \sigma\{B_1 f' e^{-z} + A_1 f e^{a_1 z} + A_2 f e^{a_2 z}\} \\ &= \sigma[-\{f^{(k)} + B_{k-1} f^{(k-1)} + \dots + B_2 f''\}] < 1, \end{aligned}$$

a contradiction. For $a_2 = -1$, by (3.1) and Lemma 2.5, we have:

(i) If $B_1 f' + A_2 f \not\equiv 0$, then

$$\begin{aligned} 1 &= \sigma\{(B_1 f' + A_2 f) e^{-z} + A_1 f e^{a_1 z}\} \\ &= \sigma[-\{f^{(k)} + B_{k-1} f^{(k-1)} + \dots + B_2 f''\}] < 1, \end{aligned}$$

a contradiction.

(ii) If $B_1 f' + A_2 f \equiv 0$, then

$$1 = \sigma\{A_1 f e^{a_1 z}\} = \sigma[-\{f^{(k)} + B_{k-1} f^{(k-1)} + \dots + B_2 f''\}] < 1,$$

a contradiction. Consequently, $\sigma(f) \geq 1$.

Now we prove that $\sigma(f) = \infty$. Assume that $f \not\equiv 0$ is a meromorphic solution of (1.3) with $1 \leq \sigma(f) = \sigma < \infty$. From (1.3), we know that the poles of $f(z)$ can occur only at the poles of A_j $(j = 1, 2)$ and B_l $(l = 1, \dots, k-1)$.

Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\sigma(d) = \beta \le \alpha = \max\{\sigma(A_j)\ (j = 1, 2), \sigma(B_l)\ (l = 1, \dots, k-1)\} < 1$, g be an entire function and $1 \le \sigma(g) = \sigma(f) = \sigma < \infty$. Substituting $f = g/d$ into (1.3), by Lemma 2.6 we get

(3.2)
$$\frac{g^{(k)}}{g} + [B_{k-1} + D_{k,k-1}]\frac{g^{(k-1)}}{g} + [B_{k-2} + B_{k-1}D_{k-1,k-2} + D_{k,k-2}]\frac{g^{(k-2)}}{g} + \cdots + \Big[B_2 + D_{k,2} + \sum_{i=3}^{k-1} B_i D_{i,2}\Big]\frac{g''}{g} + \Big[B_1 e^{-z} + D_{k,1} + \sum_{i=2}^{k-1} B_i D_{i,1}\Big]\frac{g'}{g} + B_1 D_{1,0} e^{-z} + \sum_{i=2}^{k-1} B_i D_{i,0} + D_{k,0} + A_1 e^{a_1 z} + A_2 e^{a_2 z} = 0.$$

By Lemma 2.4, for any given ε $(0 < \varepsilon < 1 - \alpha)$, there is a set $E_6 \subset [-\pi/2, 3\pi/2)$ of linear measure zero such that if $\theta \in [-\pi/2, 3\pi/2) \setminus E_6$, then there is a constant $R_1 = R_1(\theta) > 1$ such that for all z with $\arg z = \theta$ and $|z| \ge R_1$,

$$|B_l(z)| \le \exp\{r^{\alpha+\varepsilon}\} \quad (l = 1, \dots, k-1). \tag{3.3}$$

By Lemma 2.1, for $0 < \varepsilon < \min\big\{\frac{|a_2|-|a_1|}{|a_2|+|a_1|}, 1 - \alpha\big\}$, there exists a set $E_1 \subset [-\pi/2, 3\pi/2)$ of linear measure zero such that if $\theta \in [-\pi/2, 3\pi/2) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$ such that for all z with $\arg z = \theta$ and $|z| = r \ge R_0$,

$$\left|\frac{g^{(j)}(z)}{g(z)}\right| \le r^{k(\sigma-1+\varepsilon)}, \quad j = 1, \dots, k, \tag{3.4}$$

$$\left|\frac{d^{(j)}(z)}{d(z)}\right| \le r^{k(\beta-1+\varepsilon)}, \quad j = 1, \dots, k, \tag{3.5}$$

and

$$\begin{aligned} |D_{k,j}| &= \Big|\sum_{(j_1,\dots,j_k)} C_{jj_1\cdots j_k}\Big(\frac{d'}{d}\Big)^{j_1}\Big(\frac{d''}{d}\Big)^{j_2}\cdots\Big(\frac{d^{(k)}}{d}\Big)^{j_k}\Big| \\ &\le \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}|\Big|\frac{d'}{d}\Big|^{j_1}\Big|\frac{d''}{d}\Big|^{j_2}\cdots\Big|\frac{d^{(k)}}{d}\Big|^{j_k} \\ &\le \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}| r^{j_1(\beta-1+\varepsilon)} r^{2j_2(\beta-1+\varepsilon)}\cdots r^{kj_k(\beta-1+\varepsilon)} \\ &= \sum_{(j_1\cdots j_k)} |C_{jj_1\cdots j_k}| r^{(j_1+2j_2+\cdots+kj_k)(\beta-1+\varepsilon)}. \end{aligned} \tag{3.6}$$

From $j_1 + \cdots + kj_k = k - j \le k$ and (3.6), we have

$$|D_{k,j}| \le Mr^{k(\beta-1+\varepsilon)}, \tag{3.7}$$

where $M > 0$ is some constant. Let $z = re^{i\theta}$, $a_1 = |a_1|e^{i\theta_1}$, $a_2 = |a_2|e^{i\theta_2}$, $\theta_1, \theta_2 \in [-\pi/2, 3\pi/2)$.

CASE 1: $\arg a_1 \ne \pi$, that is, $\theta_1 \ne \pi$.

(i) Assume that $\theta_1 \ne \theta_2$. By Lemmas 2.2 and 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2, \pi/2) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ (where E_4 and E_5 are defined as in Lemma 2.3, $E_1 \cup E_4 \cup E_5 \cup E_6$ is of linear measure zero), and either

$$\delta(a_1 z, \theta) > 0, \qquad \delta(a_2 z, \theta) < 0,$$

or

$$\delta(a_1 z, \theta) < 0, \qquad \delta(a_2 z, \theta) > 0.$$

When $\delta(a_1 z, \theta) > 0$ and $\delta(a_2 z, \theta) < 0$, for sufficiently large r we get

$$|A_1 e^{a_1 z}| \ge \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.8}$$

$$|A_2 e^{a_2 z}| \le \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} < 1. \tag{3.9}$$

By (3.8) and (3.9) we have

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\ge |A_1 e^{a_1 z}| - |A_2 e^{a_2 z}| \ge \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} - 1 \\ &\ge (1 - o(1)) \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\}. \end{aligned} \tag{3.10}$$

By (3.2), we get

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| \le{} & \left|\frac{g^{(k)}}{g}\right| + |B_{k-1} + D_{k,k-1}| \left|\frac{g^{(k-1)}}{g}\right| \\ & + |B_{k-2} + B_{k-1} D_{k-1,k-2} + D_{k,k-2}| \left|\frac{g^{(k-2)}}{g}\right| \\ & + \cdots + \left|B_2 + D_{k,2} + \sum_{i=3}^{k-1} B_i D_{i,2}\right| \left|\frac{g''}{g}\right| + \left[|B_1|\,|e^{-z}| + \left|D_{k,1} + \sum_{i=2}^{k-1} B_i D_{i,1}\right|\right] \left|\frac{g'}{g}\right| \\ & + |B_1 D_{1,0}|\,|e^{-z}| + \sum_{i=2}^{k-1} |B_i D_{i,0}| + |D_{k,0}|. \end{aligned} \tag{3.11}$$

Since $\theta \in (-\pi/2, \pi/2)$, it follows that $|e^{-z}| = e^{-r\cos\theta} < 1$. Substituting (3.3), (3.4), (3.7) and (3.10) into (3.11), we obtain

$$(1 - o(1)) \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} \le M_1 r^{M_2} \exp\{r^{\alpha+\varepsilon}\}, \tag{3.12}$$

where $M_1, M_2 > 0$ are some constants. As $\delta(a_1 z, \theta) > 0$ and $\alpha + \varepsilon < 1$ we see that (3.12) is a contradiction. When $\delta(a_1 z, \theta) < 0$ and $\delta(a_2 z, \theta) > 0$, a similar proof also yields a contradiction.

(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2, \pi/2) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ and $\delta(a_1 z, \theta) > 0$.

Since $|a_1| \leq |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, it follows that $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For sufficiently large r, we have, by Lemma 2.2,

$$|A_1 e^{a_1 z}| \leq \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}, \tag{3.13}$$

$$|A_2 e^{a_2 z}| \geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}. \tag{3.14}$$

By (3.13) and (3.14) we get

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\geq |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \\ &\geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} - \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \\ &= \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\}[\exp\{\eta r\} - 1], \end{aligned} \tag{3.15}$$

where

$$\eta = (1-\varepsilon)\delta(a_2 z, \theta) - (1+\varepsilon)\delta(a_1 z, \theta).$$

Since $0 < \varepsilon < \frac{|a_2| - |a_1|}{|a_2| + |a_1|}$, it follows that

$$\begin{aligned} \eta &= (1-\varepsilon)|a_2| \cos(\theta_2 + \theta) - (1+\varepsilon)|a_1| \cos(\theta_1 + \theta) \\ &= (1-\varepsilon)|a_2| \cos(\theta_1 + \theta) - (1+\varepsilon)|a_1| \cos(\theta_1 + \theta) \\ &= [(1-\varepsilon)|a_2| - (1+\varepsilon)|a_1|] \cos(\theta_1 + \theta) \\ &= [|a_2| - |a_1| - \varepsilon(|a_2| + |a_1|)] \cos(\theta_1 + \theta) > 0. \end{aligned}$$

Then, from (3.15), we get

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq (1 - o(1)) \exp\{[(1+\varepsilon)\delta(a_1 z, \theta) + \eta] r\}. \tag{3.16}$$

Since $\theta \in (-\pi/2, \pi/2)$, it follows that $|e^{-z}| = e^{-r \cos\theta} < 1$. Substituting (3.3), (3.4), (3.7) and (3.16) into (3.11), we obtain

$$(1 - o(1)) \exp\{[(1+\varepsilon)\delta(a_1 z, \theta) + \eta] r\} \leq M_1 r^{M_2} \exp\{r^{\alpha+\varepsilon}\}. \tag{3.17}$$

As $\delta(a_1 z, \theta) > 0$, $\eta > 0$ and $\alpha + \varepsilon < 1$ we see that (3.17) is a contradiction.

Case 2: $a_1 < -1$, that is, $\theta_1 = \pi$.

(i) Assume that $\theta_1 \neq \theta_2$; then $\theta_2 \neq \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\pi/2, \pi/2) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos\theta > 0$, we have $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos\theta < 0$. For sufficiently large r, Lemma 2.2 gives

$$|A_1 e^{a_1 z}| \leq \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} < 1, \tag{3.18}$$

$$|A_2 e^{a_2 z}| \geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}. \tag{3.19}$$

By (3.18) and (3.19) we obtain

(3.20)

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| &\geq \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} - 1 \\ &\geq (1 - o(1)) \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\}. \end{aligned}$$

Since $\theta \in (-\pi/2, \pi/2)$, it follows that $|e^{-z}| = e^{-r\cos\theta} < 1$. Substituting (3.3), (3.4), (3.7) and (3.20) into (3.11), we obtain

$$(1 - o(1))\exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} \le M_1 r^{M_2} \exp\{r^{\alpha+\varepsilon}\}. \tag{3.21}$$

As $\delta(a_2 z, \theta) > 0$ and $\alpha + \varepsilon < 1$ we see that (3.21) is a contradiction.

(ii) Assume that $\theta_1 = \theta_2$; then $\theta_1 = \theta_2 = \pi$. By Lemma 2.3, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (\pi/2, 3\pi/2)\setminus(E_1 \cup E_4 \cup E_5 \cup E_6)$. Then $\cos\theta < 0$, $\delta(a_1 z, \theta) = |a_1|\cos(\theta_1 + \theta) = -|a_1|\cos\theta > 0$, $\delta(a_2 z, \theta) = |a_2|\cos(\theta_2 + \theta) = -|a_2|\cos\theta > 0$. Since $|a_1| \le |a_2|$, $a_1 \ne a_2$ and $\theta_1 = \theta_2$, it follows that $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta)$, and for sufficiently large r, we get (3.13), (3.14) and (3.16). Since $\theta \in (\pi/2, 3\pi/2)$, it follows that $|e^{-z}| = e^{-r\cos\theta} > 1$. Substituting (3.3), (3.4), (3.7) and (3.16) into (3.11), we obtain

$$(1 - o(1))\exp\{[(1+\varepsilon)\delta(a_1 z, \theta) + \eta] r\} \le M_1 r^{M_2} \exp\{r^{\alpha+\varepsilon}\} e^{-r\cos\theta}. \tag{3.22}$$

Thus

$$(1 - o(1))\exp\{\gamma r\} \le M_1 r^{M_2} \exp\{r^{\alpha+\varepsilon}\}, \tag{3.23}$$

where $\gamma = (1+\varepsilon)\delta(a_1 z, \theta) + \eta + \cos\theta$. Since $\eta > 0$, $\cos\theta < 0$, $\delta(a_1 z, \theta) = -|a_1|\cos\theta$, $a_1 < -1$, it follows that

$$\begin{aligned}\gamma &= -(1+\varepsilon)|a_1|\cos\theta + \cos\theta + \eta = -[(1+\varepsilon)|a_1| - 1]\cos\theta + \eta \\ &> -[(1+\varepsilon) - 1]\cos\theta + \eta = -\varepsilon\cos\theta + \eta > 0.\end{aligned}$$

As $\alpha + \varepsilon < 1$, we see that (3.23) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = \sigma(g) = \infty$. ■

Proof of Theorem 1.5. Assume f ($\not\equiv 0$) is a meromorphic solution of (1.3); then $\sigma(f) = \infty$ by Theorem 1.4. Set $g_0(z) = f(z) - z$. Then z is a fixed point of $f(z)$ if and only if $g_0(z) = 0$. Now, $g_0(z)$ is a meromorphic function and $\sigma(g_0) = \sigma(f) = \infty$. Substituting $f = g_0 + z$ into (1.3), we have

$$\begin{aligned} g_0^{(k)} + B_{k-1} g_0^{(k-1)} + \cdots + B_2 g_0'' + B_1 e^{-z} g_0' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) g_0 \\ = -[B_1 e^{-z} + z A_1 e^{a_1 z} + z A_2 e^{a_2 z}]. \end{aligned} \tag{3.24}$$

We can rewrite (3.24) in the form

$$g_0^{(k)} + h_{0,k-1} g_0^{(k-1)} + \cdots + h_{0,2} g_0'' + h_{0,1} g_0' + h_{0,0} g_0 = h_0, \tag{3.25}$$

where

$$h_0 = -[h_{0,1} + z h_{0,0}] = -B_1 e^{-z} - z A_1 e^{a_1 z} - z A_2 e^{a_2 z}.$$

We claim $h_0 \not\equiv 0$. Suppose that $-B_1 e^{-z} - z A_1 e^{a_1 z} - z A_2 e^{a_2 z} = 0$; then $z A_1 e^{(a_1+1)z} + z A_2 e^{(a_2+1)z} = -B_1$. Hence, by Lemma 2.9, we have $A_1 \equiv 0$, $A_2 \equiv 0$ and $B_1 \equiv 0$, a contradiction. Here we just consider the meromorphic solutions of infinite order satisfying $g_0(z) = f(z) - z$, and by Lemma 2.7 we conclude that $\overline{\lambda}(g_0) = \overline{\tau}(f) = \infty$.

Now we consider the fixed points of $f'(z)$. Set $g_1(z) = f'(z) - z$. Then z is a fixed point of $f'(z)$ if and only if $g_1(z) = 0$. Now, $g_1(z)$ is a meromorphic function and $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$. Set $R(z) = A_1 e^{a_1 z} + A_2 e^{a_2 z}$. Then $R' = (A_1' + a_1 A_1)e^{a_1 z} + (A_2' + a_2 A_2)e^{a_2 z}$. Differentiating both sides of (1.3), we have

$$\begin{aligned}(3.26)\quad & f^{(k+1)} + B_{k-1} f^{(k)} + (B_{k-1}' + B_{k-2}) f^{(k-1)} + (B_{k-2}' + B_{k-3}) f^{(k-2)} \\ & + \cdots + (B_3' + B_2) f''' + (B_2' + B_1 e^{-z}) f'' + [(B_1 e^{-z})' + R] f' + R' f = 0.\end{aligned}$$

By (1.3),

$$(3.27)\qquad f = -\frac{1}{R}[f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_2 f'' + B_1 e^{-z} f'].$$

Substituting (3.27) into (3.26), we have

$$\begin{aligned}(3.28)\quad & f^{(k+1)} + \left(B_{k-1} - \frac{R'}{R}\right) f^{(k)} + \left(B_{k-1}' + B_{k-2} - B_{k-1}\frac{R'}{R}\right) f^{(k-1)} \\ & + \left(B_{k-2}' + B_{k-3} - B_{k-2}\frac{R'}{R}\right) f^{(k-2)} + \cdots + \left(B_3' + B_2 - B_3 \frac{R'}{R}\right) f''' \\ & + \left(B_2' + B_1 e^{-z} - B_2 \frac{R'}{R}\right) f'' + \left[(B_1 e^{-z})' + R - B_1 e^{-z}\frac{R'}{R}\right] f' = 0.\end{aligned}$$

We can write (3.28) in the form

$$(3.29)\quad f^{(k+1)} + h_{1,k-1} f^{(k)} + h_{1,k-2} f^{(k-1)} + \cdots + h_{1,2} f''' + h_{1,1} f'' + h_{1,0} f' = 0,$$

where $h_{1,j}$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions defined by (3.28). Substituting $f' = g_1 + z$, $f'' = g_1' + 1$, $f^{(j+1)} = g_1^{(j)}$ $(j = 2, \ldots, k)$ into (3.29), we get

$$(3.30)\quad g_1^{(k)} + h_{1,k-1} g_1^{(k-1)} + h_{1,k-2} g_1^{(k-2)} + \cdots + h_{1,2} g_1'' + h_{1,1} g_1' + h_{1,0} g_1 = h_1,$$

where

$$\begin{aligned} h_1 &= -(h_{1,1} + z h_{1,0}) \\ &= -\left[B_2' + B_1 e^{-z} - B_2 \frac{R'}{R} + z(B_1 e^{-z})' + zR - z B_1 e^{-z}\frac{R'}{R}\right] \\ &= -\frac{1}{R}\{B_2' R - B_2 R' + zR^2 + [B_1 R + z(B_1' - B_1) R - z B_1 R'] e^{-z}\}.\end{aligned}$$

We claim $h_1 \not\equiv 0$. Suppose that $h_1 \equiv 0$; then

$$\begin{aligned}(3.31)\quad & [B_1 A_1 + z(B_1' - B_1) A_1 - z B_1 (A_1' + a_1 A_1)] e^{(a_1 - 1)z} \\ & + [B_1 A_2 + z(B_1' - B_1) A_2 - z B_1 (A_2' + a_2 A_2)] e^{(a_2 - 1)z} \\ & + [B_2' A_1 - B_2 (A_1' + a_1 A_1)] e^{a_1 z} + [B_2' A_2 - B_2 (A_2' + a_2 A_2)] e^{a_2 z} \\ & + 2z A_1 A_2 e^{(a_1 + a_2)z} + z A_1^2 e^{2a_1 z} + z A_2^2 e^{2 a_2 z} = 0.\end{aligned}$$

By $a_1 a_2 \neq 0$, $a_1 \neq a_2$, $a_1 < -1$, $|a_1| \leq |a_2|$, we see that:

(i) $2a_1 \neq \beta \in \{a_1 - 1, a_2 - 1, a_1, a_2, a_1 + a_2, 2a_2\} = I_1$, hence we write (3.31) in the form

$$zA_1^2 e^{2a_1 z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0, \tag{3.32}$$

where $\Gamma_1 \subseteq I_1$. By Lemmas 2.8 and 2.9, we get $A_1 \equiv 0$, a contradiction.

(ii) If $2a_1 = a_2 - 1$, then $2a_2 \neq \beta \in \{a_1 - 1, a_1, a_2, a_1 + a_2, 2a_1\} = I_2$, hence we write (3.31) in the form

$$zA_2^2 e^{2a_2 z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0, \tag{3.33}$$

where $\Gamma_2 \subseteq I_2$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, a contradiction.

(iii) If $2a_1 = a_2$, then $2a_2 \neq \beta \in \{a_1 - 1, a_2 - 1, a_1, a_1 + a_2, 2a_1\} = I_3$, hence we write (3.31) in the form

$$zA_2^2 e^{2a_2 z} + \sum_{\beta \in \Gamma_3} \alpha_\beta e^{\beta z} = 0, \tag{3.34}$$

where $\Gamma_3 \subseteq I_3$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, a contradiction. By (3.30) and Lemma 2.7 we know that $\overline{\lambda}(g_1) = \overline{\lambda}(f' - z) = \overline{\tau}(f') = \sigma(g_1) = \sigma(f) = \infty$.

Now we consider the fixed points of $f''(z)$. Set $g_2(z) = f''(z) - z$. Then z is a fixed point of $f''(z)$ if and only if $g_2(z) = 0$. Now, $g_2(z)$ is a meromorphic function and $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$. Set $G(z) = B_1 e^{-z}$. Differentiating both sides of (3.26), we have

(3.35)

$$\begin{aligned} f^{(k+2)} &+ B_{k-1} f^{(k+1)} + (2B'_{k-1} + B_{k-2}) f^{(k)} + (B''_{k-1} + 2B'_{k-2} + B_{k-3}) f^{(k-1)} \\ &+ (B''_{k-2} + 2B'_{k-3} + B_{k-4}) f^{(k-2)} + \cdots + (B''_4 + 2B'_3 + B_2) f^{(4)} \\ &+ (B''_3 + 2B'_2 + G) f''' + (B''_2 + 2G' + R) f'' + (G'' + 2R') f' + R'' f = 0. \end{aligned}$$

By (3.27) and (3.35), we have

$$\begin{aligned} f^{(k+2)} &+ B_{k-1} f^{(k+1)} + \left(2B'_{k-1} + B_{k-2} - \frac{R''}{R}\right) f^{(k)} \\ &+ \left(B''_{k-1} + 2B'_{k-2} + B_{k-3} - B_{k-1}\frac{R''}{R}\right) f^{(k-1)} + \cdots \\ &+ \left(B''_4 + 2B'_3 + B_2 - B_4\frac{R''}{R}\right) f^{(4)} + \left(B''_3 + 2B'_2 + G - B_3\frac{R''}{R}\right) f''' \\ &+ \left(B''_2 + 2G' + R - B_2\frac{R''}{R}\right) f'' + \left(G'' + 2R' - G\frac{R''}{R}\right) f' = 0. \end{aligned} \tag{3.36}$$

Now we prove that $G' + R - GR'/R \not\equiv 0$. Suppose $G' + R - GR'/R \equiv 0$; then

$$f_1e^{(a_1-1)z}+f_2e^{(a_2-1)z}+2A_1A_2e^{(a_1+a_2)z}+A_1^2e^{2a_1z}+A_2^2e^{2a_2z}=0, \tag{3.37}$$

where f_j $(j=1,2)$ are meromorphic functions with $\sigma(f_j)<1$. Set $K=\{a_1-1,a_2-1,a_1+a_2,2a_1,2a_2\}$. By the conditions of Theorem 1.4 ($a_1a_2\neq 0$, $a_1\neq a_2$, $a_1<-1$), it is clear that $2a_1\neq a_1-1,a_1+a_2,2a_2$.

(i) If $2a_1\neq a_2-1$, then we write (3.37) in the form

$$A_1^2e^{2a_1z}+\sum_{\beta\in\Gamma_1}\alpha_\beta e^{\beta z}=0,$$

where $\Gamma_1\subseteq K\setminus\{2a_1\}$. By Lemmas 2.8 and 2.9, we get $A_1\equiv 0$, a contradiction.

(ii) If $2a_1=a_2-1$, then $2a_2\neq a_1-1,a_2-1,a_1+a_2,2a_1$. Hence, we write (3.37) in the form

$$A_2^2e^{2a_2z}+\sum_{\beta\in\Gamma_2}\alpha_\beta e^{\beta z}=0,$$

where $\Gamma_2\subseteq K\setminus\{2a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2\equiv 0$, a contradiction.

Hence, $G'+R-GR'/R\not\equiv 0$ is proved. Set

$$\psi(z)=G'R+R^2-GR' \quad \text{and} \quad \phi(z)=G''R+2R'R-GR''. \tag{3.38}$$

By (3.28) and (3.38), we get

$$\begin{aligned} f'=\frac{-R}{\psi(z)}\Big\{&f^{(k+1)}+\Big(B_{k-1}-\frac{R'}{R}\Big)f^{(k)}+\Big(B'_{k-1}+B_{k-2}-B_{k-1}\frac{R'}{R}\Big)f^{(k-1)}\\ &+\Big(B'_{k-2}+B_{k-3}-B_{k-2}\frac{R'}{R}\Big)f^{(k-2)}+\cdots+\Big(B'_3+B_2-B_3\frac{R'}{R}\Big)f'''\\ &+\Big(B'_2+G-B_2\frac{R'}{R}\Big)f''\Big\}. \end{aligned} \tag{3.39}$$

Substituting (3.38) and (3.39) into (3.36), we obtain

$$\begin{aligned} &f^{(k+2)}+\Big[B_{k-1}-\frac{\phi}{\psi}\Big]f^{(k+1)}+\Big[2B'_{k-1}+B_{k-2}-\frac{R''}{R}-\frac{\phi}{\psi}\Big(B_{k-1}-\frac{R'}{R}\Big)\Big]f^{(k)}\\ &+\Big[B''_{k-1}+2B'_{k-2}+B_{k-3}-B_{k-1}\frac{R''}{R}-\frac{\phi}{\psi}\Big(B'_{k-1}+B_{k-2}-B_{k-1}\frac{R'}{R}\Big)\Big]f^{(k-1)}\\ &+\cdots+\Big[B''_4+2B'_3+B_2-B_4\frac{R''}{R}-\frac{\phi}{\psi}\Big(B'_4+B_3-B_4\frac{R'}{R}\Big)\Big]f^{(4)}\\ &+\Big[B''_3+2B'_2+G-B_3\frac{R''}{R}-\frac{\phi}{\psi}\Big(B'_3+B_2-B_3\frac{R'}{R}\Big)\Big]f'''\\ &+\Big[B''_2+2G'+R-B_2\frac{R''}{R}-\frac{\phi}{\psi}\Big(B'_2+G-B_2\frac{R'}{R}\Big)\Big]f''=0. \end{aligned} \tag{3.40}$$

We can write (3.40) in the form

$$f^{(k+2)} + h_{2,k-1} f^{(k+1)} + h_{2,k-2} f^{(k)} + \cdots + h_{2,1} f''' + h_{2,0} f'' = 0, \tag{3.41}$$

where $h_{2,j}$ $(j = 0, 1, \ldots, k-1)$ are meromorphic functions defined by (3.40). Substituting $f'' = g_2 + z$, $f''' = g_2' + 1$, $f^{(j+2)} = g_2^{(j)}$ $(j = 2, \ldots, k)$ into (3.41) we get

$$g_2^{(k)} + h_{2,k-1} g_2^{(k-1)} + h_{2,k-2} g_2^{(k-2)} + \cdots + h_{2,1} g_2' + h_{2,0} g_2 = h_2, \tag{3.42}$$

where

$$\begin{aligned} -h_2 &= h_{2,1} + z h_{2,0}, \\ h_{2,0} &= B_2'' + 2G' + R - B_2 \frac{R''}{R} - \frac{\phi(z)}{\psi(z)} \left(B_2' + G - B_2 \frac{R'}{R} \right), \\ h_{2,1} &= B_3'' + 2B_2' + G - B_3 \frac{R''}{R} - \frac{\phi(z)}{\psi(z)} \left(B_3' + B_2 - B_3 \frac{R'}{R} \right). \end{aligned}$$

Set $D_1 = B_3'' + 2B_2'$ and $D_2 = B_3' + B_2$. Obviously, D_j $(j = 1, 2)$ are meromorphic functions with $\sigma(D_j) < 1$. We get

$$h_{2,1} = \frac{L_1(z)}{\psi(z)}, \qquad h_{2,0} = \frac{L_0(z)}{\psi(z)}, \tag{3.43}$$

$$\frac{-h_2}{z} = \frac{1}{z} h_{2,1} + h_{2,0}, \tag{3.44}$$

where

(3.45)

$$\begin{aligned} L_1(z) = {} & D_1 G' R + D_1 R^2 - D_1 G R' + G' G R + G R^2 - G^2 R' - B_3 G' R'' \\ & - B_3 R'' R - D_2 G'' R + B_3 G'' R' - 2 D_2 R' R + 2 B_3 R'^2 + D_2 G R'', \end{aligned}$$

(3.46)

$$\begin{aligned} L_0(z) = {} & B_2'' G' R + B_2'' R^2 - B_2'' G R' + 2 G'^2 R + 3 G' R^2 - 2 G G' R' + R^3 \\ & - 3 G R' R - B_2 G' R'' - B_2 R'' R - B_2' G'' R - G'' G R + B_2 G'' R' \\ & - 2 B_2' R' R + 2 B_2 R'^2 + B_2' G R'' + G^2 R''. \end{aligned}$$

Therefore, by (3.43) and (3.44), we have

$$\frac{-h_2}{z} = \frac{1}{\psi(z)} \left[\frac{1}{z} L_1(z) + L_0(z) \right]. \tag{3.47}$$

Now we prove that $h_2 \not\equiv 0$. In fact, if $h_2 \equiv 0$, then by (3.47) we have

$$\frac{1}{z} L_1(z) + L_0(z) = 0. \tag{3.48}$$

By (3.45) and (3.46), we can rewrite (3.48) in the form

(3.49)
$$f_1e^{(a_1-1)z}+f_2e^{(a_2-1)z}+f_3e^{(a_1-2)z}+f_4e^{(a_2-2)z}+f_5e^{2a_1z}+f_6e^{2a_2z}+f_7e^{(a_1+a_2)z} + f_8e^{(2a_1-1)z} + f_9e^{(2a_2-1)z} + f_{10}e^{(a_1+a_2-1)z} + A_1^3e^{3a_1z} + A_2^3e^{3a_2z} + 3A_1^2A_2e^{(2a_1+a_2)z} + 3A_1A_2^2e^{(a_1+2a_2)z} = 0,$$

where f_j $(j = 1,\dots,10)$ are meromorphic functions with $\sigma(f_j) < 1$. Set $J = \{a_1-1, a_2-1, a_1-2, a_2-2, 2a_1, 2a_2, a_1+a_2, 2a_1-1, 2a_2-1, a_1+a_2-1, 3a_1, 3a_2, 2a_1+a_2, a_1+2a_2\}$. By the conditions of Theorem 1.4 ($a_1a_2 \neq 0$, $a_1 \neq a_2$, $a_1 < -1$), it is clear that $3a_1 \neq a_1-1, a_1-2, 2a_1, 2a_1-1, 3a_2, 2a_1+a_2, a_1+2a_2$ and $3a_2 \neq 2a_2, 3a_1, 2a_1+a_2, a_1+2a_2$.

(i) If $3a_1 \neq a_2-1, a_2-2, 2a_2, a_1+a_2, 2a_2-1, a_1+a_2-1$, then we write (3.49) in the form
$$A_1^3e^{3a_1z} + \sum_{\beta\in\Gamma_1} \alpha_\beta e^{\beta z} = 0,$$
where $\Gamma_1 \subseteq J \setminus \{3a_1\}$. By Lemmas 2.8 and 2.9, we get $A_1 \equiv 0$, a contradiction.

(ii) If $3a_1 = \gamma$ such that $\gamma \in\{a_2-1,\ a_2-2,\ 2a_2,\ a_1+a_2,\ 2a_2-1,\ a_1+a_2-1\}$, then $3a_2 \neq \beta$ for all $\beta \in J \setminus \{3a_2\}$. Hence, we write (3.49) in the form
$$A_2^3e^{3a_2z} + \sum_{\beta\in\Gamma_2} \alpha_\beta e^{\beta z} = 0,$$
where $\Gamma_2 \subseteq J \setminus \{3a_2\}$. By Lemmas 2.8 and 2.9, we get $A_2 \equiv 0$, a contradiction.

Hence, $h_2 \not\equiv 0$ is proved. By Lemma 2.7 and (3.42), we have $\overline{\lambda}(g_2) = \overline{\lambda}(f'' - z) = \overline{\tau}(f'') = \sigma(g_2) = \sigma(f) = \infty$. The proof of Theorem 1.5 is complete. ∎

Acknowledgements. The authors would like to thank the referee for his/her helpful remarks and suggestions to improve the paper. This paper is supported by ANDRU (Agence Nationale pour le Développement de la Recherche Universitaire) and University of Mostaganem (UMAB) (PNR Project Code 8/u27/3144).

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Habib Habib, Benharrat Belaïdi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B.P. 227 Mostaganem, Algeria
E-mail: habibhabib2927@yahoo.fr
belaidi@univ-mosta.dz

Received 26.9.2011
and in final form 7.4.2012 (2560)