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Random polynomials and (pluri)potential theory

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Abstract. For certain ensembles of random polynomials we give the expected value of the zero distribution (in one variable) and the expected value of the distribution of common zeros of m polynomials (in m variables).

Introduction. It is a classical result of Hammersley that the zeros of random Kac polynomials concentrate on the unit circle as the degrees of the polynomials increase. Some recent papers ([SZ2], [B2], [BS]) show that the zeros of certain ensembles of random polynomials concentrate on sets described by potential and pluripotential theory, specifically, sets given by the support of the Monge–Ampère operator on pluricomplex Green functions (or, in one variable, by the Laplacian on a Green function). In this paper we will extend the results of the cited papers in the following manner:

In [B2] and [SZ2] the zeros of certain ensembles of random polynomials in one variable with i.i.d. Gaussian coefficients of mean zero and variance one are shown to concentrate at the equilibrium measure of compact sets. In Section 1 we extend the results of [B2] and [SZ2] to ensembles with coefficients random variables which are not necessarily Gaussian (Theorem 1.1). We show how the approach via potential theory can give the results for the disc similar to those of Schmerling–Hochberg [SH] and Hughes–Nikeghbali [HN] (where a result of Erdős–Turán is used (Example 1.2)). We show how the connectivity of $\mathbb{C} \setminus K$ affects the results (Example 1.1).

In [BS] the common zeros of m random polynomials in $\mathbb{C}^m$ with Gaussian coefficients of mean zero and variance one were shown to concentrate at the equilibrium measure (as given in pluripotential theory) of compact sets. In Section 2 we extend the results of [BS] to ensembles where the coefficients are Gaussian but the inner product on polynomials of degree $\leq N$ is with respect

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to $w^{2N}d\mu$ for a "weight" function $w \geq 0$. We give (Theorem 2.1) the weak limit of the expectation of the normalized counting measure of the common zeros of m random polynomials on $\mathbb{C}^m$ in the form $(2\pi)^{-m}(dd^cV_{K,Q})^m$, where $V_{K,Q}$ is a weighted pluricomplex Green function. In the case $m = 1$ this result is used to answer a question of Shiffman and Zelditch [SZ2] on the concentration of zeros of certain ensembles of polynomials on curves in the plane (see Example 2.1).

Recent papers of R. Berman ([Be1], [Be2]) study common zeros of sections of certain holomorphic line bundles. There is some similarity between those results and the results of Section 2 of this paper.

1. Random polynomials.

We let $\mathcal{P}_N$ denote the vector space of polynomials (in one complex variable) of degree $\leq N$. An element of $\mathcal{P}_N$ may be uniquely written in the form

$$f(z) = \sum_{j=0}^{N} b_j z^j \quad \text{with } b_j \in \mathbb{C}. \tag{1.1}$$

If $\mathcal{P}_N$ is endowed with a probability measure the elements of $\mathcal{P}_N$ are referred to as *random polynomials*. For example, considering the b_j's as independent identically distributed (i.i.d.) complex Gaussian random variables with mean 0 and variance 1 (i.e. each b_j has distribution function $\pi^{-1}e^{-|\xi|^2}d\lambda$ for $\xi \in \mathbb{C}$ and $d\lambda$ Lebesgue measure on $\mathbb{C}$) puts a probability measure on $\mathcal{P}_N$. In this case the polynomials are often referred to as *Kac polynomials*.

For $f \in \mathcal{P}_N$ we let

$$Z_f := \sum_{f(z)=0} \delta(z) \tag{1.2}$$

be the counting measure of the zeros of f and

$$\widetilde{Z}_f := \frac{1}{\deg(f)} \sum_{f(z)=0} \delta(z) \tag{1.3}$$

be the normalized counting measure of the zeros of f.

We are interested in asymptotic properties of $\widetilde{Z}_f$. To this end we consider the product probability space

$$\mathcal{P} := \prod_{N=1}^{\infty} \mathcal{P}_N. \tag{1.4}$$

We will consider ensembles of random polynomials, generalizing the Kac polynomials, which were introduced by Shiffman and Zelditch [SZ2] as follows: Let K be a regular (in the sense of potential theory, i.e. regular for the exterior Dirichlet problem) compact set $\subset \mathbb{C}$ and μ a finite Borel measure with $\operatorname{supp}(\mu) = K$. Applying the Gram–Schmidt orthogonalization procedure on the monomials we obtain orthonormal polynomials

$$p_N(z) = \sum_{j=0}^{N} c_j^N z^j. \tag{1.5}$$

For convenience, we assume the total mass of μ is 1 so $p_0(z) \equiv 1$.

Given $f \in \mathcal{P}_N$ we can write it uniquely as

$$f = \sum_{j=0}^{N} a_j^N p_j(z). \tag{1.6}$$

We consider the a_j^N to be complex random variables. For example, the Kac polynomials can be obtained with $K = \{z \mid |z| = 1\}$, $d\mu = d\theta/2\pi$ and the a_j^N i.i.d. Gaussians.

To state the results we will use the following concepts from potential theory. We let V_K denote the Green function of the unbounded component of $\mathbb{C} \setminus K$ with logarithmic pole at ∞. We assume V_K is defined on $\mathbb{C}$ by setting $V_K = 0$ on the bounded components of $\mathbb{C} \setminus K$ and on K. Then, assuming K is regular, V_K is continuous on $\mathbb{C}$ and the equilibrium measure of K is

$$d\mu_{\mathrm{eq}}(K) := \frac{1}{2\pi}\, dd^c V_K. \tag{1.7}$$

Here $d^c = i(\partial - \overline{\partial})$ so dd^c is the Laplacian in the underlying real coordinates of $\mathbb{C}$. We let $\operatorname{cap}(K)$ denote the logarithmic capacity of K.

We also assume that (K, μ) satisfies the Bernstein–Markov (BM) inequality. That is, given $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that for all $f \in \mathcal{P}_N$ we have

$$\|f\|_K \le C(1+\varepsilon)^N \|f\|_{L^2(\mu)}. \tag{1.8}$$

It is known (for example [B2, Proposition 3.4]) that, as a consequence of (1.8),

$$\lim_{N\to\infty} \frac{1}{N} \log |c_N^N| = -\log(\operatorname{cap}(K)). \tag{1.9}$$

Theorem 1.1 below generalizes a result of Shiffman and Zelditch [SZ2] and was proved in [B2, Theorem 4.3] assuming the a_j^N are i.i.d. complex Gaussians of mean 0 and variance 1. Below we show the result is valid under less stringent assumptions on the a_j^N, namely that they have continuous distribution functions φ_j^N satisfying the uniform estimates

$$\text{(i)}\quad |\varphi_j^N| \le T_1, \qquad \text{(ii)}\quad \int\limits_{|z|\ge R} \varphi_j^N \, d\lambda \le \frac{T_2}{R^2}, \tag{1.10}$$

where T_1, T_2 are constants independent of N, j.

THEOREM 1.1. *Let K be a regular compact set $\subset \mathbb{C}$ and μ a measure on K such that (K,μ) satisfies the BM inequality. Suppose the random variables a_j^N satisfy* (1.10). *Then, with probability one in $\mathcal{P}$, we have, for ensembles defined by* (1.6),

$$\lim_N \widetilde{Z}_{f_N} = d\mu_{\text{eq}}(K) \quad \textit{weak}^* \textit{ on } \mathbb{C} \cup \{\infty\}.$$

The proof of Theorem 1.1 is deduced from the following "deterministic" result (see [BSS], also [B2, Theorem 4.2]).

THEOREM 1.2 (Blatt, Saff, Simkani). *Let (K,μ) be as in Theorem* 1.1 *and let $f_N(z) = \sum_{j=0}^N b_j^N z^j$ be a sequence of polynomials satisfying*

- (i) $\overline{\lim}_N \|f_N\|_K^{1/N} \le 1$,
- (ii) $\lim_{N\to\infty} N^{-1} \log |b_N^N| = -\log(\text{cap}(K))$,
- (iii) *for each bounded connected component in $\mathbb{C} \setminus K$ there is a point z_0 such that* $\lim_N |f_N(z_0)|^{1/N} = 1$.

Then $\lim_N \widetilde{Z}_{f_N} = d\mu_{\text{eq}}(K)$ *weak* on* $\mathbb{C} \cup \{\infty\}$.

Proof of Theorem 1.1. We will deduce Theorem 1.1 from Theorem 1.2 by showing that each of the conditions (i)–(iii) holds with probability one in $\mathcal{P}$.

We first note that (i) and (ii) imply that $\lim_N \|f_N\|_K^{1/N} = 1$.

We will use the Borel–Cantelli lemma in the following form: Let $Y_N \subset \mathcal{P}_N$ be a measurable subset for $N = 1, 2, \ldots$ and let $Y := \{\{f_N\} \in \mathcal{P} \mid f_N \in Y_N$ for all but finitely many $N\}$. Then, letting G denote the probability on $\mathcal{P}$ and G_N on $\mathcal{P}_N$, we have

$$G(Y) = 1 \quad \text{if } \sum_{N=1}^{\infty} G_N(Y_N^c) < \infty. \tag{1.11}$$

Now, for condition (i) in Theorem 1.2 we set

$$V_1 := \{\{f_N\} \in \mathcal{P} \mid \overline{\lim} \|f_N\|_K^{1/N} \le 1, \tag{1.12}$$

$$V_1' := \{\{f_N\} \in \mathcal{P} \mid \|f_N\|_{L^2(\mu)} \le N^2(N+1) \quad \text{for all but finitely many } N\}. \tag{1.13}$$

Then

$$\begin{aligned} G_N(\{f_N \in \mathcal{P}_N \mid \|f_N\|_{L^2(\mu)} \ge N^2(N+1)\}) &= \text{Prob}\Big(\Big(\sum_{k=0}^N |a_k^N|^2\Big)^{1/2} \ge N^2(N+1)\Big) \\ &\le \text{Prob}(a_k^N \ge N^2 \text{ for some } 0 \le k \le N) \le T_2 \frac{N+1}{N^4} \end{aligned} \tag{1.14}$$

using (1.10)(ii). Thus $G(V_1') = 1$ and since $V_1' \subset V_1$ we have $G(V_1) = 1$.

For condition (ii) in Theorem 1.2 we set

$$V_2 := \{\{f_N\} \in \mathcal{P} \mid \lim_N |a_N^N|^{1/N} = 1 \text{ for } f_N \text{ in the form (1.6)}\}. \tag{1.15}$$

Note that by (1.9), $\{f_N\} \in V_2$ if and only if $\{f_N\}$ satisfies (ii).

Let

$$V_2' := \{\{f_N\} \in \mathcal{P} \mid \ 1/N \le |a_N^N| \le N \text{ for all but finitely many } N\}. \tag{1.16}$$

Then

$$G_N(\{f_N \in \mathcal{P}_N \mid |a_N^N| \le 1/N \text{ or } |a_N^N| \ge N\}) \le \frac{\pi T_1 + T_2}{N^2} \tag{1.17}$$

by (1.10). Hence using (1.11) we have $G(V_2') = 1$ and since $V_2' \subset V_2$ we conclude that $G(V_2) = 1$.

For condition (iii) in Theorem 1.2 we take a point z_0 in a bounded component of $\mathbb{C} \setminus K$ and let

$$V_3 := \{\{f_N\} \in \mathcal{P} \mid \lim_N |f_N(z_0)|^{1/N} = 1\}, \tag{1.18}$$

$$V_3' := \{\{f_N\} \in \mathcal{P} \mid 1/N \le |f_N(z_0)| \text{ for all but finitely many } N\}. \tag{1.19}$$

Then

$$\begin{aligned} G_N(\{f_N \in \mathcal{P}_N \mid |f_N(z_0)| \le 1/N\}) \\ = \int\limits_{|a_0^N + a_1^N p_1(z_0) + \cdots + a_N^N p_N(z_0)| \le 1/N} \varphi(a_0^N) \cdots \varphi(a_N^N)\, d\lambda_0 \cdots d\lambda_N \end{aligned} \tag{1.20}$$

where $d\lambda_j$ $(0 \le j \le N)$ is Lebesgue measure on a copy of $\mathbb{C}$ determined by a_j^N.

Using Fubini's theorem and first integrating over the copy of $\mathbb{C}$ determined by a_0^N turns the integral into one over a disc of radius $\le 1/N$ so by (1.10)(ii) the value of the integral is $\le \pi T_1/N^2$. Thus, by (1.11), $G(V_3') = 1$. But $V_3 \supset V_1 \cap V_3'$ so $G(V_3) = 1$.

This concludes the proof of Theorem 1.1. ■

EXAMPLE 1.1. Let $K = [-1, 1]$ and $d\mu = dx/2$. It is well-known that

$$d\mu_{\mathrm{eq}}(K) = \frac{1}{\pi} \frac{dx}{\sqrt{1 - x^2}}.$$

Random polynomials (given by (1.6)) are of the form

$$\sum_{J=0}^{N} a_j^N L_j(z)$$

where the $L_j(z)$ are Legendre polynomials normalized to have norm 1 in $L^2(dx/2)$ so their leading coefficients satisfy (1.9).

Condition (iii) in Theorem 1.2 is vacuous so Theorem 1.1 holds if each of (i) and (ii) hold with probability one. That is (instead of (1.10)) Theorem 1.1

holds if we have, in the space of sequences of random variables,

$$\text{Prob}(\lim_N |a_N^N|^{1/N} = 1) = 1 \tag{1.21}$$

and

$$\text{Prob}(\lim_N (\max_{0\le k\le N} |a_k^N|)^{1/N} = 1) = 1. \tag{1.22}$$

Example 1.2. Let $K = \{z \mid |z| = 1\}$ and $d\mu = d\theta/2\pi$. It is well-known that $d\mu_{\text{eq}}(K) = d\theta/2\pi$. The monomials are the orthonormal polynomials and we obtain the ensemble of Kac polynomials. To verify condition (iii) of Theorem 1.2 we use the point $z_0 = 0$ and then since $p_j(z_0) = 0$ for $j = 1, 2, \ldots$ we find that $\lim_N |f_N(z_0)|^{1/N} = 1$ if and only if $\lim_N |a_0^N|^{1/N} = 1$. Thus the conclusion of Theorem 1.1 holds if (1.21), (1.22) hold and

$$\text{Prob}(\lim_N |a_0^N|^{1/N} = 1) = 1. \tag{1.23}$$

Note that conditions (1.21), (1.22) and (1.23) are similar to conditions occurring in the papers [SH] and [HN].

We also remark that in the case of the unit circle, it is a straightforward exercise to see that the condition on weak* convergence in Theorem 1.1 is equivalent to a condition used in [SH] and [HN], namely:

$$\lim_N \widetilde{Z}_{f_N} = \frac{d\theta}{2\pi} \quad \text{weak}^* \text{ on } \mathbb{C}\cup\{\infty\}$$

if and only if for all $\delta > 0$ and all $0 \le \theta_1 < \theta_2 \le 2\pi$,

$$\lim_{n\to\infty} \frac{1}{N} \operatorname{card}(Z_{f_N} \cap \{1-\delta \le |z| \le 1+\delta,\ \theta_1 \le \arg(z) \le \theta_2\}) = \frac{\theta_2-\theta_1}{2\pi}.$$

2. The weighted case.

We will give the expected distribution of the common zeros of m polynomials in $\mathbb{C}^m$ in certain ensembles (defined below). The case $m = 1$ will be used to answer a question of Shiffman and Zelditch ([SZ2, p. 32]) concerning the distribution of zeros of certain ensembles on curves in the plane (see Example 2.1).

We let $\mathcal{P}_N(\mathbb{C}^m)$ denote the vector space of polynomials on $\mathbb{C}^m$ of total degree $\le N$. For $f \in \mathcal{P}_N(\mathbb{C}^m)$ we can write

$$f = \sum_{|\alpha|\le N} b_\alpha z^\alpha \tag{2.1}$$

where $b_\alpha \in \mathbb{C}$ and α is a multiindex. The space $\mathcal{P}_N(\mathbb{C}^m)$ is of dimension $d(N) := \binom{n+m}{m}$.

We will study the following ensemble of random polynomials on $\mathbb{C}^m$, that is, we will put a probability measure on $\mathcal{P}_N(\mathbb{C}^m)$ as follows:

Let K be a locally regular (in the sense of pluripotential theory—see [Si] for the definition) compact set in $\mathbb{C}^m$. Let $w \ge 0$ be a continuous function on K (called the weight function) with the property that $\{z \in K \mid w > 0\}$ is non-pluripolar (such weights are called *admissible*—see [SaT, Appendix B]). Let μ be a finite positive Borel measure on K with $\operatorname{supp}(\mu) = K$.

For each $N \in \mathbb{N}$, the monomials are linearly independent in $L^2(w^{2N}\mu)$. We order the monomials via the lexicographic ordering of their exponents and apply the Gram–Schmidt orthogonalization procedure. For each multiindex $\alpha \in \mathbb{N}^m$ we obtain a polynomial $p_\alpha^N(z)$. These polynomials are of the form

$$p_\alpha^N(z) = c_\alpha^N z^\alpha + (\text{monomials of lower lexicographic order}). \tag{2.2}$$

They are orthonormal, that is, they satisfy

$$\int_{\mathbb{C}^m} p_\alpha^N(z) \overline{p_\beta^N(z)} w^{2N} d\mu = \delta_{\alpha,\beta} \tag{2.3}$$

for all multiindices α, β.

Any $f \in \mathcal{P}_N(\mathbb{C}^m)$ may be written uniquely as

$$f(z) = \sum_{|\alpha| \le N} a_\alpha^N p_\alpha^N(z). \tag{2.4}$$

We obtain an ensemble of random polynomials by considering the a_α^N to be random variables. Note that in (2.4), the expansion depends on N, so a fixed polynomial f has different expansions, depending on N.

The a_α^N will, in fact, be assumed to be i.i.d. complex Gaussians with mean zero and variance one. Given m such random polynomials $F = (f_1, \ldots, f_m)$, their common zero set is, with probability one, a discrete subset of $\mathbb{C}^m$ consisting of N^m points. This is because, by Bertini's theorem [GH], a generic F has isolated common zeros and by Bézout's theorem [GH] the common zero set must consist of precisely N^m points. We let

$$Z_F := \sum_{F(z)=0} \delta(z) \tag{2.5}$$

be the counting measure of the zeros and

$$\widetilde{Z}_F := \frac{1}{N^m} Z_F \tag{2.6}$$

be the normalized counting measure of the zeros. We will give results on the asymptotics of $\widetilde{Z}_F$ but first we need some concepts from pluripotential theory (see [K], [Si], [SaT, Appendix B]). We let

$$Q := -\log w \tag{2.7}$$

and define the weighted pluricomplex Green function by

$$V_{K,Q}(z) := \sup\{u(z) \mid u \le Q \text{ on } K \text{ and } u \in \mathcal{L}\} \tag{2.8}$$

where $\mathcal{L}$ is the Lelong class of plurisubharmonic (p.s.h.) functions

$$\mathcal{L} = \{u \mid u \text{ is p.s.h. on } \mathbb{C}^m \text{ and } u \leq \log^+(z) + C\}. \tag{2.9}$$

It is known that, under the assumptions that K is locally regular and w is continuous,

(2.10) (i) $V_{K,Q}$ is continuous,
(ii) $V_{K,Q}$ is a locally bounded p.s.h. function,
(iii) $(dd^c V_{K,Q})^m$ is a Borel measure with support in K and total mass $(2\pi)^m$.

Here $(dd^c)^m$ denotes the Monge–Ampère operator.

We also assume that the triple (K, w, μ) satisfies the weighted Bernstein–Markov inequality (see [B3] for conditions that this inequality hold). That is, for all $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that for all $f \in \mathcal{P}_N(\mathbb{C}^m)$ we have

$$\|w^N f\|_K \leq C(1+\varepsilon)^N \|w^N f\|_{L^2(d\mu)}. \tag{2.11}$$

Then we have, letting E_N denote expectation over $\mathcal{P}_N(\mathbb{C}^m)$:

Theorem 2.1.

$$\lim_N E_N(\widetilde{Z}_F) = \left(\frac{1}{2\pi}\right)^m (dd^c V_{K,Q})^m \qquad \textit{weak}^*.$$

Proof. The proof is analogous to that of Theorem 3.1 in [BS]. We will therefore only give the details to the proof of Lemma 2.2 below. Theorem 2.1 will follow from Lemmas 2.1–2.3.

Lemma 2.1. *Let*

$$\phi_N(z) = \sup\{|f(z)| \mid f \in \mathcal{P}_N,\ \|w^N f\|_K \leq 1\}. \tag{2.12}$$

Then $\lim_N N^{-1} \log \phi_N(z) = V_{K,Q}$ *uniformly on compact subsets of* $\mathbb{C}^m$*.*

Lemma 2.2. *Let*

$$S_N(z,\xi) := \sum_{|\alpha| \leq N} p_\alpha^N(z)\, \overline{p_\alpha^N(\xi)}. \tag{2.13}$$

Then for all $\varepsilon > 0$ *there is a constant* $C = C(\varepsilon) > 0$ *such that*

$$\frac{1}{d(N)} \leq \frac{S_N(z,z)}{\phi_N(z)^2} \leq C^2 (1+\varepsilon)^{2N} d(N). \tag{2.14}$$

Proof of Lemma 2.2. Let $f \in \mathcal{P}_N(\mathbb{C}^m)$ with $\|w^N f\|_K \leq 1$. Then

$$\begin{aligned} |w^N f(z)| &= \Big| \int_K S_N(z,\xi) f(\xi) w(\xi)^{2N}\, d\mu(\xi) \Big| \\ &\leq \int_K |S_N(z,\xi)| w(\xi)^N\, d\mu(\xi) \end{aligned} \tag{2.15}$$

$$\begin{aligned}
&\le \int_K S_N(z,z)^{1/2} S_N(\xi,\xi)^{1/2} w(\xi)^N \, d\mu(\xi) \\
&= S_N(z,z)^{1/2} \int_K S_N(\xi,\xi)^{1/2} w(\xi)^N \, d\mu(\xi) \\
&\le S_N(z,z)^{1/2} \|1\|_{L^2(\mu)} \|S_N(\xi,\xi)\|_{L^2(w^{2N}\, d\mu)} \\
&\le S_N(z,z)^{1/2}\, d(N)^{1/2}
\end{aligned}$$

where the total mass of μ is normalized to be one. Taking the sup over $f \in \mathcal{P}_N$ with $\|w^N f\|_K \le 1$ we obtain the left inequality in (2.14).

By the weighted BM inequality we have

$$\|w^N p_\alpha^N\|_K \le C(1+\varepsilon)^N. \tag{2.16}$$

But $p_\alpha^N / \|w^N p_\alpha^N\|_K$ is in the family of functions defining ϕ_N (see (2.12)), so

$$\frac{|p_\alpha^N(z)|}{\|w^N p_\alpha^N\|_K} \le \phi_N(z) \tag{2.17}$$

and so

$$|p_\alpha^N(z)| \le C(1+\varepsilon)^N \phi_N(z). \tag{2.18}$$

Thus

$$S_N(z,z) = \sum_{|\alpha|\le N} |p_\alpha^N(z)|^2 \le C^2(1+\varepsilon)^{2N} \phi_N(z)^2 d(N), \tag{2.19}$$

which gives the right inequality in (2.14).

Lemma 2.3. *We have*

$$\lim_N \frac{1}{2N} \log S_N(z,z) = V_{K,Q}(z)$$

uniformly on compact subsets of $\mathbb{C}^m$.

Proof of Theorem 2.1. We use the probabilistic Poincaré–Lelong formula (see [BS]), which in this situation gives

$$E_N(\widetilde{Z}_F) = \left(\frac{1}{4\pi N} dd^c \log S_N(z,z)\right)^m. \tag{2.20}$$

Theorem 2.1 now follows from Lemma 2.3 and the fact that the Monge–Ampère operator is continuous under uniform limits.

Example 2.1. Let $K = \partial\Omega$ be the boundary of an open set $\Omega \subset \mathbb{C}$. We assume $\partial\Omega$ is of class C^1. Then K is locally regular. The measure $d\mu = |dz|$ satisfies the BM inequality (condition Λ^* in [StT, Theorem 4.2.3] is satisfied which is sufficient for the BM inequality—in [StT] the term "Bernstein–Markov inequality" is not used). Also $\operatorname{supp}(\mu) = K$. Thus by [StT, Theorem 3.2.1(vi)] the weighted BM inequality is satisfied with $w^2 = \varrho$ where ϱ

is a continuous positive function on $\partial\Omega$. Thus, by Theorem 2.1,

$$\lim_N E_N(\widetilde{Z}_f) = \frac{1}{2\pi}\, dd^c V_{K,Q} \quad \text{weak}^*. \tag{2.21}$$

By (2.10)(iii) we have $\operatorname{supp}(dd^c V_{K,Q}) \subset K$. In other words, the zeros concentrate on $\partial\Omega$. (This question was raised in [SZ2, p. 32].)

Specific information on $dd^c V_{K,Q}$, in particular conditions under which it is absolutely continuous with respect to $|dz|$, may be found in [SaT, Section IV 2].

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