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The gradient lemma

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Abstract. We show that if a decreasing sequence of subharmonic functions converges to a function in $W^{1,2}_{\mathrm{loc}}$ then the convergence is in $W^{1,2}_{\mathrm{loc}}$.

1. Introduction. This paper is based on a talk I gave in Kraków on April 30, 2003 and is in part motivated by Błocki's paper [1].

Proposition 1.1. *Denote by* SH^- *the negative subharmonic functions defined on some domain in* $\mathbb{C}^n$, *and by* $W^{1,2}_{\mathrm{loc}}$ *the usual Sobolev class. Then* $u \in \mathrm{SH}^- \cap W^{1,2}_{\mathrm{loc}}$ *if and only if* $u \in \mathrm{SH}^- \cap L^1_{\mathrm{loc}}(\Delta u)$.

Using Proposition 1.1, we prove the *gradient lemma*:

Lemma 1.2. *If* u_j *is a decreasing sequence of functions in* SH^- *with limit* $u \in W^{1,2}_{\mathrm{loc}}$, *then* $u_j \in W^{1,2}_{\mathrm{loc}}$ *and* $u_j \to u$ *in* $W^{1,2}_{\mathrm{loc}}$ *as* $j \to \infty$.

In the last section, we use the gradient lemma in connection with the class $\mathcal{E}$.

2. Proof of Proposition 1.1. The problem is local, so we can assume that $u \in \mathrm{SH}^-(B)$ where B is the unit ball in $\mathbb{C}^n, 0 < r < s < 1$. Define $\widetilde{u} = \sup\{\varphi \in \mathrm{SH}^-(B);\ \varphi|_{rB} \le u|_{rB}\}$. Then $0 \ge \widetilde{u} \ge u, \widetilde{u} \in \mathrm{SH}^-(B), \widetilde{u} = u$ on rB and $\widetilde{u}$ is harmonic on $B \setminus rB$ and $\widetilde{u}(x) = \int_{sB} g(x,y)\Delta\widetilde{u}(y)$ where g is the Green function for B.

The smallest harmonic majorant of u on sB can be estimated from below on rB by $c\int u\,dv$ where c is a positive constant (depending on s and t) and dv is the Lebesgue measure on B. It follows that

$$\int_{sB} g(x,y)\Delta u(y) + c\int_{B} u\,dv \le u \quad \text{ on } rB.$$

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For $x \in B$, we have

$$0 \geq \widetilde{u}(x) \geq \int_{sB} g(x,y)\Delta u(y) + c\,\frac{|x|^2-1}{r^2-1}\int_B u\,dv =: \overline{u}(x).$$

Since

$$\int_B (1-|x|^2)\Delta\widetilde{u} = \int_B -\widetilde{u}\,dv$$

we get

$$\begin{aligned}\int_B |\operatorname{grad}\widetilde{u}|^2\,dv &= \int_B -\widetilde{u}\Delta\widetilde{u} \leq \int_B -\overline{u}\Delta\widetilde{u}\\ &= \int_B \Big\{\int_{sB} -g(x,y)\Delta u(y)\Big\}\Delta\widetilde{u} + \frac{c}{r^2-1}\int_B u\,dv\int_B (1-|x|^2)\Delta\widetilde{u}\\ &\leq \int_{sB} -\widetilde{u}\Delta u + \frac{c}{1-r^2}\Big(\int_B u\,dv\Big)^2 \leq \int_{sB} -u\Delta u + \frac{c}{1-r^2}\Big(\int_B u\,dv\Big)^2\end{aligned}$$

so if $\int_{sB} -u\Delta u < \infty$, then

$$(*)\qquad \int_{rB} |\operatorname{grad} u|^2 \leq \int_B |\operatorname{grad}\widetilde{u}|^2 \leq \int_{sB} -u\Delta u + \frac{c}{1-r^2}\Big(\int_B u\,dv\Big)^2$$

and we have proved the first half of Proposition 1.1.

Assume now that $u \in \mathrm{SH}^- \cap W^{1,2}_{\mathrm{loc}}$. We prove that then $\int_{rB} -u\Delta u < \infty$. Let $0 \leq t \in C_0^\infty(B)$, $t = 1$ on sB. Then

$$\int_{rB} -u\Delta u \leq \int_B -tu\Delta u = \int_B dtu \wedge d^c u \wedge (dd^c|z|^2)^{n-1}$$

$$\leq \Big[\int_{\operatorname{supp} t} dtu \wedge d^c tu \wedge (dd^c|z|^2)^{n-1}\Big]^{1/2}\Big[\int_{\operatorname{supp} t} du \wedge d^c u \wedge (dd^c|z|^2)^{n-1}\Big]^{1/2} < \infty,$$

which completes the proof of Proposition 1.1.

3. Proof of Lemma 1.2. If $u_j \geq u$ then $\widetilde{u}_j \geq \widetilde{u}$ so by $(*)$, $\int_B -\widetilde{u}_j\Delta\widetilde{u}_j \leq \int_B -\widetilde{u}\Delta\widetilde{u} < \infty$ and

$$\begin{aligned}\int_{rB} d(u_j-u)\wedge d^c(u_j-u)\wedge(dd^c|z|^2)^{n-1}\\ &\leq \int_B d(\widetilde{u}_j-\widetilde{u})\wedge d^c(\widetilde{u}_j-\widetilde{u})\wedge(dd^c|z|^2)^{n-1}\\ &= -\int_B (\widetilde{u}_j-\widetilde{u})dd^c(\widetilde{u}_j-\widetilde{u})\wedge(dd^c|z|^2)^{n-1}\\ &\leq \int_B (\widetilde{u}_j-\widetilde{u})dd^c\widetilde{u}\wedge(dd^c|z|^2)^{n-1}.\end{aligned}$$

The last term tends to zero as j tends to infinity and the proof is complete.

4. The class $\mathcal{E}$. We denote by $\mathrm{PSH}^-(\Omega)$ the class of negative plurisubharmonic functions defined on the domain Ω in $\mathbb{C}^n$.

A domain Ω in $\mathbb{C}^n$ is called *hyperconvex* if there is a negative exhaustion function for Ω, i.e. a function $\psi \in \mathrm{PSH}^-(\Omega)$ such that

$$\{z \in \Omega;\, \psi(z) < c\} \subset\subset \Omega, \qquad \forall c < 0.$$

We say that a function $v \in \mathrm{PSH}^-(\Omega)$ is in $\mathcal{F}(\Omega)$ if there is a decreasing sequence of functions $v_j \in \mathcal{E}_0(\Omega)$ with $\lim v_j = v$ and $\sup \int (dd^c v_j)^n < \infty$. Here $\mathcal{E}_0(\Omega)$ is the class of bounded plurisubharmonic functions u such that $\lim_{z\to\xi} u(z) = 0$ for all $\xi \in \partial\Omega$ and $\int_\Omega (dd^c u)^n < \infty$. Finally, $u \in \mathcal{E}(\Omega)$ if for every $\omega \subset\subset \Omega$ there is a function $u \le u_\omega \in \mathcal{F}(\Omega)$ with equality on ω. See [C1, C2] for further properties of this and related classes.

THEOREM 4.1. *Suppose Ω is a hyperconvex subset of $\mathbb{C}^n$. Then there is a constant c, depending on Ω only, such that if $u \in \mathcal{F}(\Omega)$ then*

$$\int_{\{u<-1\}} |\mathrm{grad}\, u|^2\, dv \le c \int_\Omega (dd^c u)^n.$$

Proof. By the gradient lemma and Theorem 2.1 in [3], we can assume that $u \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$. Then $\{u < -1\} \subset\subset \Omega$. We can choose m and t such that $-1 < m(|z|^2 - t) < 0$ on Ω.

Integration by parts gives

$$\begin{aligned}
\int_{\{u<-1\}} |\mathrm{grad}\, u|^2\, dv &= \frac{1}{m^{n-1}} \int_{\{u<-1\}} du \wedge d^c u \wedge (dd^c m(|z|^2 - t))^{n-1} \\
&\le \frac{1}{m^{n-1}} \int_\Omega du \wedge d^c u \wedge (dd^c \max(m(|z|^2 - t), u))^{n-1} \\
&= \frac{1}{m^{n-1}} \int_\Omega -\max(m(|z|^2 - t), u)(dd^c u)^2 \wedge (dd^c \max(m(|z|^2 - t), u))^{n-2} \\
&\le \frac{1}{m^{n-1}} \int_\Omega m(t - |z|^2)(dd^c u)^n \le \frac{1}{m^{n-1}} \int_\Omega (dd^c u)^n. \ \blacksquare
\end{aligned}$$

COROLLARY 4.2. *If $u \in \mathcal{E}$, then $u \in \mathrm{PSH}^- \cap W^{1,2}_{\mathrm{loc}}$.*

COROLLARY 4.3. *Suppose Ω is a hyperconvex domain in $\mathbb{C}^2$. Then there is a constant c, depending on Ω only, such that if $u \in \mathcal{F}(\Omega)$ then*

$$\int_\Omega |\mathrm{grad}\, u|^2\, dv \le c \int_\Omega (dd^c u)^2.$$

THEOREM 4.4 (Błocki [1]). *Suppose Ω is a hyperconvex subset of $\mathbb{C}^2$. Then $u \in \mathcal{E}$ if and only if $u \in \mathrm{PSH}^- \cap W^{1,2}_{\mathrm{loc}}$.*

Proof. If $u \in \mathcal{E}$, then $u \in \mathrm{PSH}^- \cap W^{1,2}_{\mathrm{loc}}$ by Corollary 4.2. Conversely, if $u \in \mathrm{PSH}^- \cap W^{1,2}_{\mathrm{loc}}$, then $u \in L^2_{\mathrm{loc}}$ and $|\mathrm{grad}\, u|^2 \in L^1_{\mathrm{loc}}$. Therefore, since $dd^c u^2 = 2du \wedge d^c u + 2udd^c u$, it follows that $dd^c(udd^c u)$ is a well defined positive measure so u is in $\mathcal{E}$. ∎

Remark. For $u \in \mathcal{F}(\mathbb{B})$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^n$, $n > 1$, we have

$$\int_{\mathbb{B}} |\mathrm{grad}\, u|^2 \, dv \le c_n^{(n-2)/n} \Big[\int_{\mathbb{B}} (1 - |z|^2)(dd^c u)^n \Big]^{2/n}$$

where c_n is the volume of $\mathbb{B}$.

Remark. Let $u, w \in \mathcal{F}(\Omega)$ with Ω hyperconvex. Then, using integration by parts and Theorem 5.5 in [3], we have

$$\int_{\Omega} du \wedge d^c u \wedge (dd^c w)^{n-1} \le \Big[\int_{\Omega} -w(dd^c u)^n \Big]^{2/n} \Big[\int_{\Omega} -w(dd^c w)^n \Big]^{(n-2)/n}.$$

Choosing w to be a strictly plurisubharmonic function (see e.g. [4]), we get local estimates for $|\mathrm{grad}\, u|^2$.

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